E. Garcia and P.J. Antsaklis, “Model-Based Control of Networked Distributed Systems with Multirate State Feedback Updates,” International J Control, pp. 1503-1517, Vol.86, Issue 9, 2013. DOI:10.1080/00207179.2013.787649

Model-Based Control of Networked Distributed Systems with Multirate State Feedback Updates

Abstract. This paper presents a model-based multirate control technique for stabilization of uncertain discrete-time systems that transmit information through a limited bandwidth communication network. This model-based multirate approach is applied to two networked architectures. First, we discuss the implementation of a centralized control system with distributed sensing capabilities and, second, we address the problem of stabilization of networks of coupled subsystems with distributed sensors and controllers. In both cases we provide necessary and sufficient conditions for stability of the uncertain system with multirate model updates. Furthermore, we show that, in general, an important reduction of network bandwidth can be obtained using the multirate approach with respect to the single rate implementations.

Keywords: multirate systems; networked control systems; distributed control; lifting techniques.

1. Introduction

In recent years there has been a strong interest in the analysis, development, and controller synthesis for networks of interconnected systems. Examples of such systems can be found in a wide variety of applications such as: power networks, multiagent robotic systems and coordination of autonomous vehicles, large chemical processes comprised of several subsystems interacting one with each other, and also in areas that consider economic and/or social systems. In addition, the availability of cheap, fast, embedded sensor and controller subsystems that are capable to communicate via a shared digital network allow for the different subsystems to share their local information with other (possibly the rest of) subsystems so it can be used to achieve a common objective in a more efficient way (Camponogara, *et.al.* 2002; Dunbar 2007; Sinopoli, *et.al.* 2003).

The support of the National Science Foundation under Grant No. CNS-1035655 is gratefully acknowledged.

However, digital communication networks have limited bandwidth and not all agents can communicate at a given time instant. It becomes necessary to be able to schedule the broadcast of information by the different nodes in such a way that bandwidth constraints are not violated (Lian, Moyne, and Tilbury 2001; Antsaklis, and Baillieul 2007). Different approaches that reduce the necessary network bandwidth have been investigated such as minimum bit rate stabilization (Elia and Mitter 2001), packetized control (Georgiev and Tilbury 2004), and model-based control (Montestruque and Antsaklis, 2003).

Multirate sampling has been mainly used to reduce computation effort in sampled data systems (Yu and Yu 2007). The work in (Yu and Yu 2007) provides experimental results of a Model Predictive Control (MPC) multirate controller in which a multivariable plant is approximated by 3 single-output models which are modeled and sampled at different rates. The main purpose in this work is to sample the output representing the slow dynamics of the system at a lower rate in order to reduce the dimension and complexity of the optimization problem at those sampling instants when only the fast dynamics are sampled. Other references concerning the implementation of multirate MPC algorithms are (Scattolini and Schiavoni 1995) and (Lee, Gelormino, and Morari 1992).

The multirate implementation generalizes the dual-rate approach frequently used in sampled-data and networked systems (Li, Shah, and Chen 2002; Ding and Chen 2004; Ding and Chen 2005; Tange 2005) where two different rates are used in the control system. These two rates correspond each one to the actuator (fast rate) and the sensor (slow rate). In the dual-rate approach it is assumed that the entire output vector is measured and sent trough the network at the same time instant.

Multirate systems have also been studied using different approaches. The authors of (Sezer and Siljak 1990) provide stabilizability conditions for continuous-time Linear Time Invariant (LTI) systems using decentralized controllers and sampling the system inputs using different sampling rates. Vadigepalli and Doyle (2003) developed a multirate version of the Distributed and Decentralized Estimation and Control (DDEC) algorithm in (Mutambara and Durrant-Whyte 2000) for large scale process based on model distribution and internodal communication. The model distribution provides a definition of the states of interest to be estimated locally by each node using measurements that are sampled periodically using different intervals at each node. Communication between nodes is used in order to share information due to the interactions between local subsystems resulting from the model decomposition overlapping states.

The present paper extends initial results described in (Garcia and Antsaklis 2010) to consider two important architectures in control systems that communicate their measurements using a limited bandwidth communication channel. The first problem considers the control of large scale uncertain systems with multiple outputs and spatially distributed sensing implementation. In this type of implementation the sensors that measure different elements of the state vector can be located at distant positions. A limited bandwidth communication network is used to send all sensor measurements to the controller. In order to schedule the sampling and sharing of information and to increase the sampling intervals at each sensor node as much as possible we implement a multirate sampling scheme. The results concerning this problem provide a simple approach to address robustness to parameter uncertainties using multirate sampling in contrast to (Yu and Yu 2007; Scattolini and Schiavoni 1995; Lee, Gelormino, and Morari 1992; Li, Shah, and Chen 2002; Tange 2005; Sezer and Siljak 1990; Vadigepalli and Doyle 2003) where it is assumed that the plant and model parameters are the same.

The second problem that is analyzed in the present paper corresponds to a decentralized architecture in which it is necessary to stabilize a set of coupled, uncertain, and unstable dynamical systems. In this architecture a Local Control Unit (LCU) is implemented within each subsystem and it is assumed that each subsystem is able to measure its own state at all times but it is allowed to broadcast its measurements to other subsystems only at certain instants according to its assigned update rate. Similar to the problem described above, we do not restrict the subsystems to use the same update rates but those rates can be all different in general.

In order to reduce the necessary network bandwidth for stabilization we follow the Model-Based Networked Control Systems (MB-NCS) approach in Montestruque and Antsaklis (2003; 2004; 2005), Lunze and Lehmann (2010), Garcia and Antsaklis (2012), Garcia, Vitaioli, and Antsaklis (2011). In MB-NCS a nominal model of the system is implemented in the controller node to provide estimates of the state of the system for the intervals of time that feedback measurements are not transmitted by the sensor nodes.

Similar MB-NCS implementations have been used for distributed systems by Sun and El-Farra (2008; 2012a; 2012b), El-Farra and Sun (2009). In (Sun and El-Farra 2008) the authors study the stabilization of coupled continuous-time systems employing the MB-NCS in (Montestruque and Antsaklis 2003) and using a single rate approach which forces all agents to send measurement updates through the network all at the same time instants. That approach was extended in (El-Farra and Sun 2009) and in (Sun and El-Farra 2012a; 2012b) to the case when a schedule for the updates is pre-assigned but all subsystems still use the same update rate, i.e. a single rate approach in which the agents send updates at different time instants. The approach in our paper offers more flexibility and provides, in general, further reduction of network communication. Each subsystem is allowed to send measurements using its own update rate.

Subsystems with slower dynamics or represented by more accurate models are able to use lower update rates and they are not restricted to use the same rates as the subsystems with faster dynamics.

The paper is organized as follows. Section 2 provides a background analysis of MB-NCS using single rate. Section 3 considers the centralized control problem using multirate sensor measurements. Section 4 and 5 address the decentralized architecture using single rate and multirate updates, respectively. Section 6 provides an illustrative example for estimating admissible model uncertainties and section 7 concludes the paper.

2. Preliminaries and MB-NCS with single-rate updates

We recall a simple result concerning discrete-time LTI systems:

Theorem 1 (Antsaklis and Michel 1997). The equilibrium x=0 of system $x(k+1)=Fx(k)$ is asymptotically stable if and only if all eigenvalues of F are within the unit circle of the complex plane, i.e. if $\lambda_1 ... \lambda_n$ denote the eigenvalues of F, then $\left|\lambda_j\right| < 1,\ j = 1,...,n$.■

Let us consider a discrete-time system and model given, respectively, by:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k) \tag{2}$$

with $A, \hat{A} \in \mathbb{R}^{n\times n}$ and $B, \hat{B} \in \mathbb{R}^{n\times m}$. The control input is given by $u(k) = K\hat{x}(k)$. In MB-NCS measurements of the system which are used to update the model of the system at the controller node. The entire state of the system is sampled periodically. The sampling term refers to the time event in which the sensor measures the state and sends this measurement through the network in order to update the corresponding state of the model. The sampling period used by the sensor is greater or equal than the original operating period of the discrete-time plant. By using this approach we try to reduce the number of samples that are transmitted from the sensor to the model/controller. We search for larger sampling periods, which are integer multiples of the operating period of the system, i.e. if the original period of the system is denoted by T time units then the sampling periods are allowed take values T, $2T$, $3T$, … In this paper we refer to the sampling periods that the sensors use to measure and update the model as update periods in order to differentiate them from the underlying period of the plant T. The associated rates to the update periods are called therefore update rates. Assume without loss of generality that the underlying period of the plant is T=1.

The MB-NCS was studied in (Garcia and Antsaklis 2010) using a lifting technique. The lifting process has the purpose of extending the input and output spaces properly in order to obtain a Linear Time Invariant (LTI) system description for sampled-data, multirate, or linear time-varying periodic systems. Since the lifted system is a LTI system, the available tools and results for LTI systems are applicable to the lifted system as well. For details on the discrete-time lifting technique used in this work refer to Chen and Francis (1995) and to Bittanti and Colaneri (2000) and Kahane, Mirkin, and Palmor (1999) for alternative approaches.

A MB-NCS with periodic updates is clearly a linear time-varying periodic system by considering an output $\underline{x}$ that is equal to $\hat{x}$ when measurements are not transmitted and equal to x when we have an update.

Denote the input and output for the lifted system by $\underline{u}$ and $\underline{x}$ respectively and they are given by the following

$$\underline{u}(kh) = \begin{bmatrix} u(kh) \\ u(kh+1) \\ \vdots \\ u((kh+h-1) \end{bmatrix} = \begin{bmatrix} Kx(kh) \\ K\hat{x}(kh+1) \\ \vdots \\ K\hat{x}((kh+h-1) \end{bmatrix} \tag{3}$$

$$\underline{x}(kh) = \begin{bmatrix} I \\ \hat{A} \\ \hat{A}^2 \\ \vdots \\ \hat{A}^{h-1} \end{bmatrix} x(kh) + \begin{bmatrix} 0 & 0 & \dots & 0 \\ \hat{B} & 0 & \dots & 0 \\ \hat{A}\hat{B} & \hat{B} & \dots & 0 \\ \vdots & & & \vdots \\ \hat{A}^{h-2}\hat{B} & \hat{A}^{h-3}\hat{B} & \dots & 0 \end{bmatrix} \underline{u}(kh). \tag{4}$$

The dimension of the state is preserved and the state equation expressed in terms of the lifted input is given by:

$$x((k+1)h) = A^h x(kh) + \begin{bmatrix} A^{h-1}B & A^{h-2}B & \dots & B \end{bmatrix} \underline{u}(kh). \tag{5}$$

Theorem 2. The lifted system is asymptotically stable if and only if the eigenvalues of

$$A^h + \sum_{j=0}^{h-1} A^{h-1-j} BK(\hat{A} + \hat{B}K)^j \tag{6}$$

are within the unit circle of the complex plane.

Proof. To prove this theorem we note that (5) is the same as the state equation that characterizes the autonomous linear time invariant system:

$$x((k+1)h) = (A^h + \sum_{j=0}^{h-1} A^{h-1-j} BK(\hat{A} + \hat{B}K)^j) x(kh). \tag{7}$$

Equation (7) can be obtained by directly substituting (3) in (5), and then substituting the value of each individual output by its equivalent in terms of the state *x(kh)*, i.e.

$$\begin{aligned} \hat{x}(kh+1) &= \hat{A}x(kh) + \hat{B}u(kh) = (\hat{A} + \hat{B}K)x(kh) \\ \hat{x}(kh+2) &= (\hat{A} + \hat{B}K)^2 x(kh) \\ &\vdots \end{aligned} \tag{8}$$

The resulting equation can be simply expressed as (7). The lifted system is an LTI system and by Theorem 1 it is asymptotically stable if and only if the eigenvalues of (6) have magnitude less than one.■

3. Spatially distributed systems with multirate sensor measurements

In this section we use the MB-NCS approach and the lifting procedure to establish stability conditions for discrete-time spatially distributed systems when their states are sampled periodically but using different sampling rates. This multirate approach is necessary especially when different sensors are used to measure different elements of the system output, see Figure 1.

When the sensors use the same network to transmit their measurements to the controller node then the multirate approach brings important benefits to the operation of the overall networked system. By allowing the sensors to transmit their measurements using different sampling periods we avoid packet collisions and networked induced delays compared to the case when all of the sensors need to sample and transmit at the same instants. Additionally, we will show that in many cases a further reduction in network communication can be obtained by using different update rates for each sensor compared to the single rate case shown in Garcia and Antsaklis (2010). Although the multirate sampling case requires a more complex analysis, the same lifting approach can be used in order to find a system representation for the LTI equivalent (lifted) system.

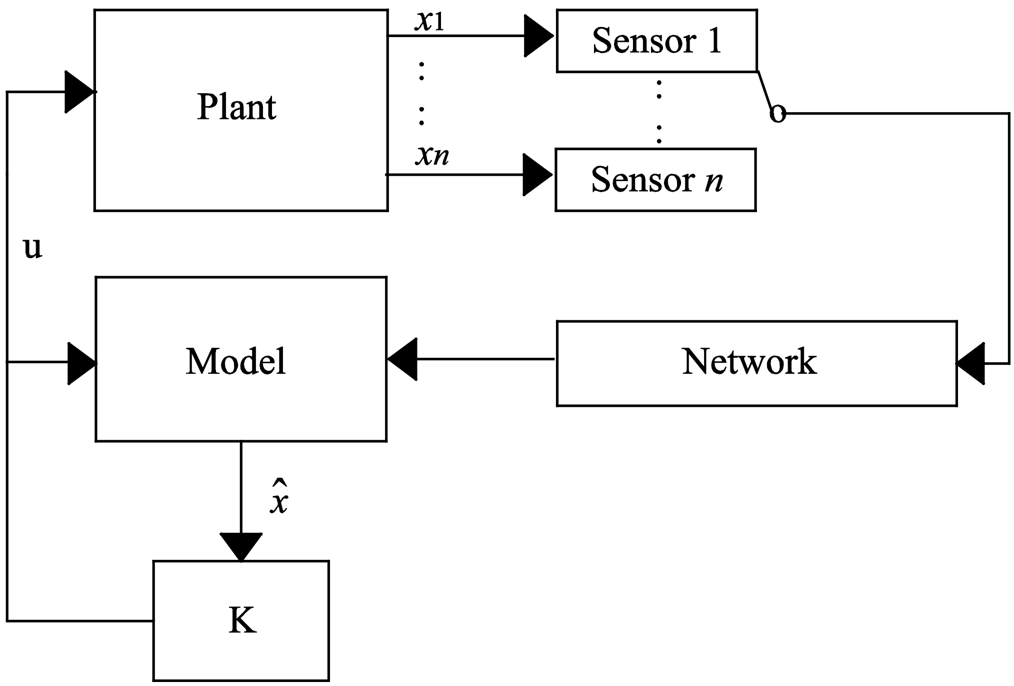

Figure 1. Networked control system with centralized controller and distributed multirate sensor measurements.

Consider a multi-output system depicted in Figure 1. In this case we do not assume that a single sensor measures all states at the same time; instead we consider a spatially distributed system for which different sensors measure different elements of the state and send this information to the centralized controller at different rates. The state of the model is partially updated according to the information that is received at any given update instant.

In what follows we will provide details on how the state of the model is partially updated and how to obtain the response of the corresponding lifted system. The lifted system state equation directly provides necessary and sufficient conditions for the stability of the multirate system.

We consider an N-partition of the state of the system (1), and the model (2) likewise, according to the number of sensors that are used to obtain measurements:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_N \end{bmatrix} \quad (9)$$

where $x_i, \hat{x}_i \in \mathbb{R}^{ni}$, $\sum_{i=1}^{N} ni = n$. In general, each subset of the system state may have different dimensions. Let s_i represent the update period that sensor *i* uses to send measurements in order to update the corresponding part of the model state $\hat{x}_i$. Let *s* represent the minimum common multiple of all s_i.

In order to obtain the response of the multirate system using the model-based control input with partial updates we define all the update instants within a period *s* by arranging the periods s_i and its multiples up to before *s* in increasing order as follows:

$$\begin{aligned} & s_1, 2s_1, \ldots, (r_1 - 1)s_1 \\ & s_2, 2s_2, \ldots, (r_2 - 1)s_2 \\ & \vdots \\ & s_N, 2s_N, \ldots, (r_N - 1)s_N \end{aligned} \quad (10)$$

where $r_i = \frac{s}{s_i}$ for *i*=1…*N* represents the relative update rate compared to the update rate given by the period *s*. Let h_i for *i*=1…*p*-1 represent the update instants in increasing order, where *p* is the total number of update instants within a period *s* including the update at time *s*. Note that at any given instant one or more sets of states $\hat{x}_i$ can be updated. This procedure can be better shown trough a simple example.

Consider the state of a system that is partitioned into three subsets x_1, x_2, x_3 with corresponding periods $s_1 = 3, s_2 = 4, s_3 = 6$. Then we proceed to define all update instants within a period s=12, as follows:

$$\begin{aligned} h_1 &= 3 = s_1 \\ h_2 &= 4 = s_2 \\ h_3 &= 6 = 2s_1 = s_3 \\ h_4 &= 8 = 2s_2 \\ h_5 &= 9 = 3s_1 \end{aligned} \tag{11}$$

Note that at the time instant h_3 we have two partial updates for this example.

Let us define, in general, the partial update matrices:

$$I_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{ni} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

that is, the ith partial update matrix $I_i \in \mathbb{R}^{n \times n}$ contains an identity matrix at the position corresponding to x_i and zeros elsewhere. Define

$$I_{h_i} = I_i + I_j + I_k \ldots \tag{13}$$

The matrices I_{h_i} represent all updates that happen at time instant h_i.

Theorem 3. The uncertain system with distributed sensors as shown in Figure 1 with model-based control input and with partial multirate model updates is asymptotically stable for given update periods s_i if and only if the eigenvalues of

$$A^s + \sum_{i=1}^{p} A^{h_p - h_i} \Xi_{h_i - h_{i-1}} U_{h_i - 1} \tag{14}$$

are within the unit circle of the complex plane, where

$$\Xi_{h_i-h_{i-1}} = \sum_{j=0}^{h_i-h_{i-1}-1} A^{h_i-h_{i-1}-1-j} BK(\hat{A}+\hat{B}K)^j$$
$$U_{h_i} = (I-I_{h_i})(\hat{A}+\hat{B}K)^{h_i-h_{i-1}} U_{h_i-1} + I_{h_i}(A^{h_i} + \sum_{q=1}^{i} A^{h_i-h_q} \Xi_{h_q-h_{q-1}} U_{h_q-1}) \tag{15}$$

with $h_0 = 0, U_{h_0} = I, h_p = s$.

Proof. Let us consider the beginning of a period *s*. At this time instant all sensors send measurements and we have that $\hat{x}(ks) = x(ks)$. At the time of the first update after time *ks*, that is, at time $ks + h_1$ we have:

$$x(ks+h_1) = (A^{h_1} + \sum_{j=0}^{h_1-1} A^{h_1-1-j} BK(\hat{A}+\hat{B}K)^j) x(k) = (A^{h_1} + \Xi_{h_1}) x(ks) \tag{16}$$

and the model state after the update has taken place is given by:

$$\hat{x}(ks+h_1) = ((I-I_{h_1})(\hat{A}+\hat{B}K)^{h_1} + I_{h_1}(A^{h_1} + \Xi_{h_1})) x(k) = U_{h_1} x(ks). \tag{17}$$

Following a similar analysis we can obtain the response of both the system and the model at time $ks + h_2$ as a function of $x(ks+h_1)$ and $\hat{x}(ks+h_1)$:

$$x(ks+h_2) = A^{h_2-h_1} x(ks+h_1) + \sum_{j=0}^{h_2-h_1-1} A^{h_2-h_1-1-j} BK(\hat{A}+\hat{B}K)^j \hat{x}(ks+h_1) \tag{18}$$

$$\hat{x}(ks+h_2) = (I-I_{h_2})(\hat{A}+\hat{B}K)^{h_2-h_1} \hat{x}(ks+h_1) + I_{h_2} x(ks+h_2) \tag{19}$$

but, since both $x(ks+h_1)$ and $\hat{x}(ks+h_1)$ can be expressed in terms of *x*(*ks*), we obtain the following:

$$x(ks+h_2) = (A^{h_2} + A^{h_2-h_1}\Xi_{h_1} + \Xi_{h_2-h_1} U_{h_1}) x(ks) \tag{20}$$

$$\hat{x}(ks+h_2) = ((I-I_{h_2})(\hat{A}+\hat{B}K)^{h_2-h_1} U_{h_1} + I_{h_2}(A^{h_2} + A^{h_2-h_1}\Xi_{h_1} + \Xi_{h_2-h_1} U_{h_1})) x(ks) = U_{h_2} x(ks). \tag{21}$$

By following the same analysis for each update instant h_i up to $h_p = s$ we obtain

$$x((k+1)s) = (A^s + \sum_{i=1}^{p} A^{h_p - h_i} \Xi_{h_i - h_{i-1}} U_{h_i - 1}) x(ks). \tag{22}$$

Since (22) represents an LTI system then the networked system is asymptotically stable when the eigenvalues of (14) lie inside the unit circle. ■

Example 1. Consider the 5th order unstable system given by:

$$A = \begin{bmatrix} -1.05 & 0.35 & 0.02 & 0.35 & 0.24 \\ 0.35 & 1.1 & 0.1 & 0.035 & 0.035 \\ 0.03 & 0.32 & 0.21 & -0.4 & 0.6 \\ 0.06 & 0.03 & 0.3 & -0.7 & 0.55 \\ 0.035 & 0.03 & 0.6 & 0.3 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Its nominal model is given by:

$$\hat{A} = \begin{bmatrix} -1 & 0.35 & 0.02 & 0.35 & 0.24 \\ 0.35 & 1 & 0.05 & 0.035 & 0.035 \\ 0.03 & 0.35 & 0.15 & -0.45 & 0.6 \\ 0.06 & 0.035 & 0.34 & -0.75 & 0.6 \\ 0.035 & 0.035 & 0.5 & 0.3 & 0.4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

If it were possible to implement a single sensor and send periodic updates, that is, to send the whole state at the same time instants we would need to use a period h<10 according to Theorem 2. Figure 2 shows the response of the system and the model for the initial conditions shown below and using the update period h=9, the largest stabilizing single rate period. Note that the entire state of the model is updated at the same time instants.

$$x_0 = [1 \quad 0.5 \quad 0.6 \quad 0.3 \quad -0.5]^T$$

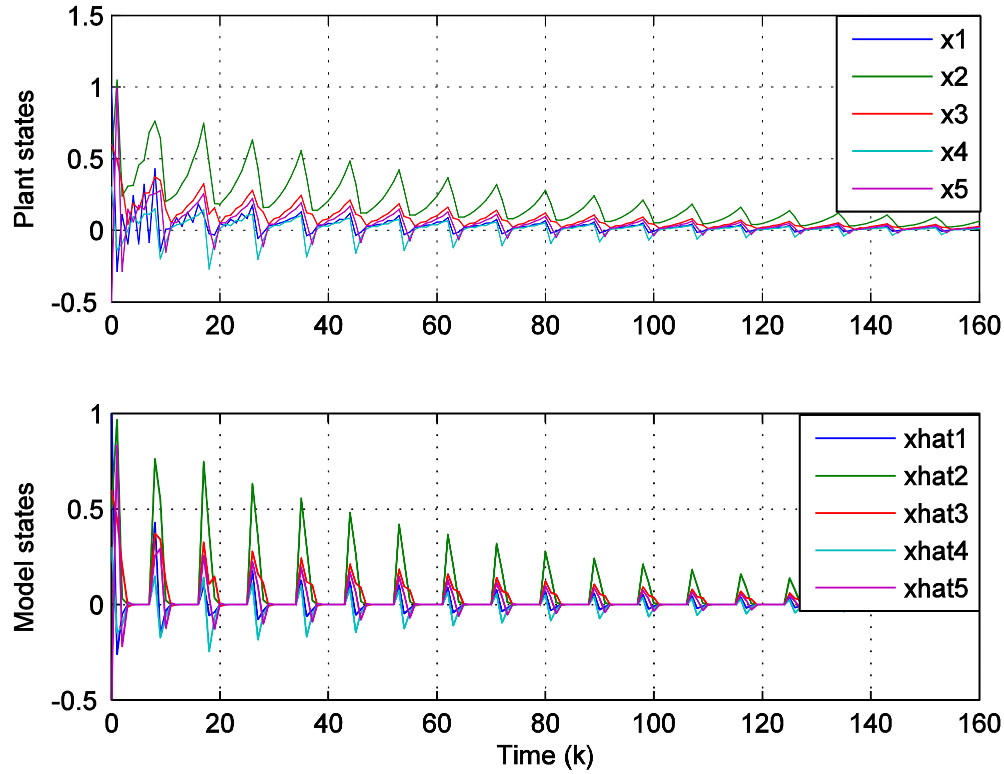

Figure 2. Response of the plant and model for the single rate system in example 1 for *h*=9.

Now suppose that each state of the system can be measured by a different sensor node. By using the results in this section it is possible to show the existence of multirate stabilizing update periods. The combination of stabilizing periods is quite large including those that involve lower sampling periods. In many cases it is possible to decrease a few of the individual sampling periods while increasing the rest of them. For instance, if we select the periods $s_1 = 10, s_2 = 6, s_3 = 15, s_4 = 10, s_5 = 30,$ the eigenvalues of (14) are all inside the unit circle, which means that the multirate networked system is asymptotically stable.

Figure 3 shows the response of the plant and the model for this particular selection and using the same initial conditions as in Figure 2. It is important to note that many update period combinations result in a stable system and the one selected is one that provides for significant reduction of network communication compared to the more conservative single-rate implementation in which all sensors would need to use a sampling period $h<10$.

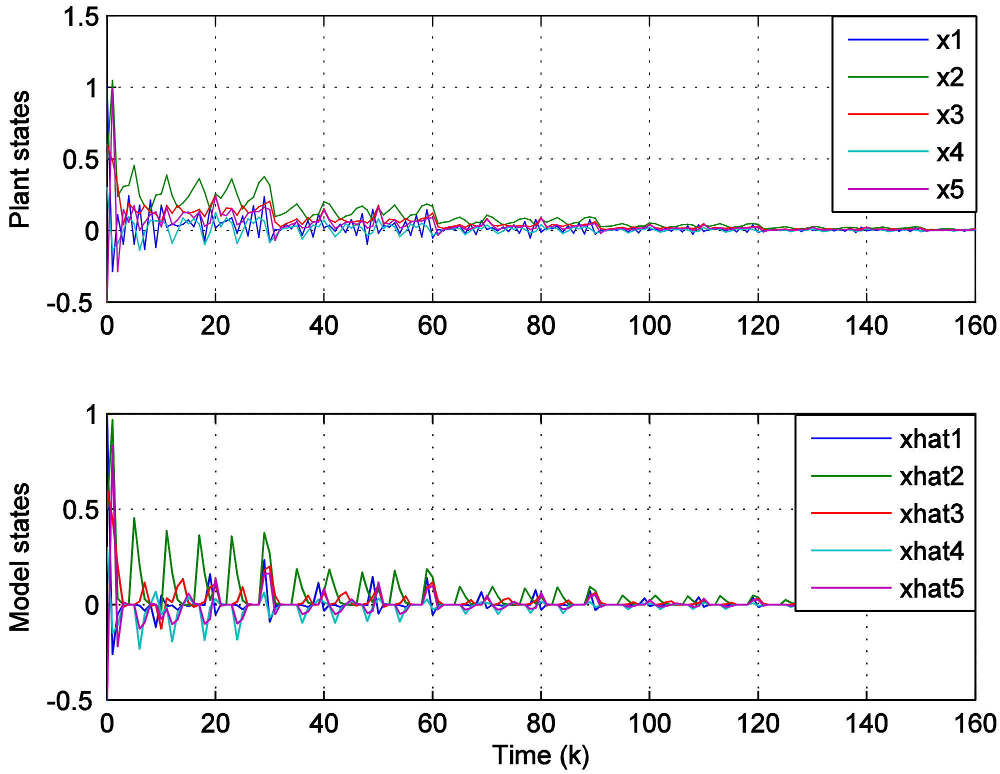

Figure 3. Response of the multirate system for the following choice of update periods: $s_1 = 10,\ s_2 = 6,\ s_3 = 15,\ s_4 = 10,\ s_5 = 30,$

4. Stabilization of coupled systems using single-rate updates

In this section we consider a different networked architecture which is shown in Figure 4. This figure represents a network of coupled interconnected subsystems where the measurements can be transmitted over a network and the dark arrows represent the physical interconnections or coupling of the subsystems.

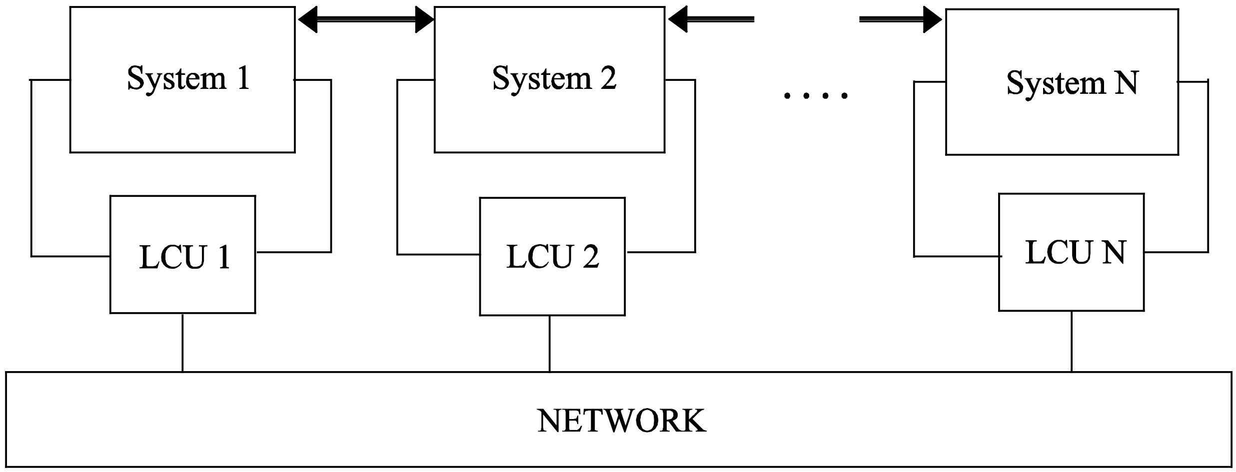

Figure 4. Model-based networked interconnected systems.

To simplify the analysis and to clearly explain the model-based implementation we restrict our attention to the case in which all subsystems send measurements to update the models in other subsystems using the same sampling period. Next section will generalize this approach to the multirate case.

We consider a network of N interconnected agents or subsystems as seen in Figure 4. Each subsystem can be described by a discrete-time state-space representation as follows:

$$
\begin{aligned}
x_1(k+1) &= A_1 x_1(k) + B_1 u_1(k) + \sum_{j=2}^{N} A_{1j} x_j(k) \\
x_2(k+1) &= A_2 x_2(k) + B_2 u_2(k) + \sum_{j=1, j\neq 2}^{N} A_{2j} x_j(k) \\
&\vdots \\
x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + \sum_{j=1, j\neq i}^{N} A_{ij} x_j(k) \\
&\vdots \\
x_N(k+1) &= A_N x_N(k) + B_N u_N(k) + \sum_{j=1}^{N-1} A_{Nj} x_j(k)
\end{aligned}
\tag{23}
$$

In this framework each Local Control Unit (LCU) contains copies of the models of *all subsystems* including the model corresponding to its own local dynamics in order to generate estimates of the states of all subsystems in the network. The model of each subsystem is represented by:

$$\hat{x}_i(k+1) = \hat{A}_i \hat{x}_i(k) + \hat{B}_i \hat{u}_i(k) + \sum_{j=1, j \neq i}^{N} \hat{A}_{ij} \hat{x}_j(k) \tag{24}$$

for each $i \in \mathrm{N}$, N denotes the set $\{1, 2, \ldots N\}$ of N integers where $x_i, \hat{x}_i \in \mathbb{R}^{ni}$ represent respectively the real state of the *i-th* unit and the state of the corresponding model, $u_i, \hat{u}_i \in \mathbb{R}^{mi}$ represent the local input for subsystem *i* and for model *i*, respectively. The matrices A_i, A_{ij}, B_i, $\hat{A}_i, \hat{A}_{ij}, \hat{B}_i$ are of appropriate dimensions. Note that the subsystems could have different dynamics and different dimensions, the dimensions *mi* and *ni* could be all different in general. Note also that each LCU*i* has access to its local state x_i at all times which is used to compute the local subsystem control input:

$$u_i(k) = K_i x_i(k) + \sum_{j=1, j \neq i}^{N} K_{ij} \hat{x}_j(k) \tag{25}$$

where K_i and K_{ij} are the stabilizing control gains to be designed.

We define the model control inputs as

$$\hat{u}_i(k) = K_i \hat{x}_i(k) + \sum_{j=1, j \neq i}^{N} K_{ij} \hat{x}_j(k) \tag{26}$$

These control inputs are applied to all models in all LCUs whereas (25) is applied to each local subsystem. It is clear now that although LCU*i* computes an estimate $\hat{x}_i$ of x_i, this estimated state is not used to control subsystem *i* since we have the real state available. At LCU*i* we use $\hat{x}_i$ as input for the models ensuring that the same model equations with the same model control inputs are implemented at all LCUs.

Define the augmented plant and model state vectors:

$$\begin{aligned} x &= [x_1^{\,T}\ x_2^{\,T}\ \ldots x_n^{\,T}]^T \\ \hat{x} &= [\hat{x}_1^{\,T}\ \hat{x}_2^{\,T}\ \ldots \hat{x}_n^{\,T}]^T \end{aligned} \tag{27}$$

The dynamics of the overall system and model can be represented by:

$$x(k+1) = Ax(k) + Bu(k) \tag{28}$$

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k). \tag{29}$$

The forms of the matrices $A, \hat{A} \in \mathbb{R}^{n \times n}$ and $B, \hat{B} \in \mathbb{R}^{n \times m}$ where $n = \sum_{i=1}^{N} ni$ and $m = \sum_{i=1}^{N} mi$ for this type of implementation are as follows:

$$A = \begin{bmatrix} A_1 & A_{12} & \ldots & A_{1n} \\ A_{21} & A_2 & \ldots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \ldots & A_n \end{bmatrix} \quad B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & B_n \end{bmatrix}$$
$$\hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_{12} & \ldots & \hat{A}_{1n} \\ \hat{A}_{21} & \hat{A}_2 & \ldots & \hat{A}_{2n} \\ \vdots & & \ddots & \vdots \\ \hat{A}_{n1} & \hat{A}_{n2} & \ldots & \hat{A}_n \end{bmatrix} \quad \hat{B} = \begin{bmatrix} \hat{B}_1 & 0 & \cdots & 0 \\ 0 & \hat{B}_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \hat{B}_n \end{bmatrix}. \tag{30}$$

We can describe the dynamics of the overall system as given in the next proposition.

Proposition 4. Assume $(\hat{A}, \hat{B})$ is stabilizable. The dynamics of the overall distributed system can be represented by:

$$x(k+1) = (A + BK_{diag})x + BK_{off}\hat{x} \tag{31}$$

where $K_{off} = K - K_{diag}$. $K_{diag} = diag(K_i)$ is a matrix containing the controller gains K_i as main diagonal sub matrices. The controller *K* is a stabilizing controller for the overall model dynamics, i.e. $\hat{A} + \hat{B}K$ is Hurwitz.

Proof. We rewrite (28) in the next form

$$x(k+1) = Ax + Bu = Ax + B(K_{diag}x(k) + (K - K_{diag})\hat{x}(k)) \tag{32}$$

where *u* is the augmented vector containing each agent local subsystem control inputs

$$u = [u_1^{\ T}\ u_2^{\ T} \ldots u_n^{\ T}]^T. \tag{33}$$

Equation (32) can be simple rewritten as (31). ■

Theorem 5. System (28) with control input $u(k) = K_{diag}x(k) + K_{off}\hat{x}(k)$ and single-rate periodic updates of the states of the models is asymptotically stable if only if the eigenvalues of

$$(A + BK_{diag})^h + \sum_{j=0}^{h-1}(A + BK_{diag})^{h-1-j} BK_{off}(\hat{A} + \hat{B}K)^j \tag{34}$$

are within the unit circle of the complex plane, where *h* is the update period.

Proof. By Proposition 4 we have that the overall networked system can be expressed as a function of the state of the system and the state of the model as in (31), which is of the form (1) with state matrix $A + BK_{diag}$, controller gain K_{off}, and single-rate updates. Then Theorem 2 can be applied directly using these modified matrices. Similarly, the lifted system

$$x((k+1)h) = ((A + BK_{diag})^h + \sum_{j=0}^{h-1}(A + BK_{diag})^{h-1-j} BK_{off}(\hat{A} + \hat{B}K)^j)x(kh) \tag{35}$$

is an LTI system and it is asymptotically stable if and only if the eigenvalues of (34) lie strictly inside the unit circle. ■

Example 2. We consider a network of *N*=6 subsystems represented as in equation (23) all with different dynamics. The dimensions of the systems vary from 1 to 3 as well. The models for all different parameters represent an uncertainty as follows: 12% alteration in the A_i matrices, 10% in A_{ij}, and 6% in B_i. Some of the systems are unstable and every agent is coupled to all other agents including those with different dimensions by corresponding coupling matrices A_{ij}. By evaluating the eigenvalues of (34) for different values of *h*, we can find that for $h \le 11$ the overall networked system is asymptotically stable. Figures 5 and 6 show the response of all subsystems for the largest stabilizing rate *h*=11 and for initial conditions

$$x_1(0) = 1, \quad x_2(0) = 0.5$$
$$x_3(0) = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}, \quad x_4(0) = \begin{bmatrix} -0.5 \\ -0.4 \end{bmatrix}$$
$$x_5(0) = \begin{bmatrix} 1 \\ 0.5 \\ 0.6 \end{bmatrix}, \quad x_6(0) = \begin{bmatrix} 0.3 \\ -0.5 \\ -0.4 \end{bmatrix}$$

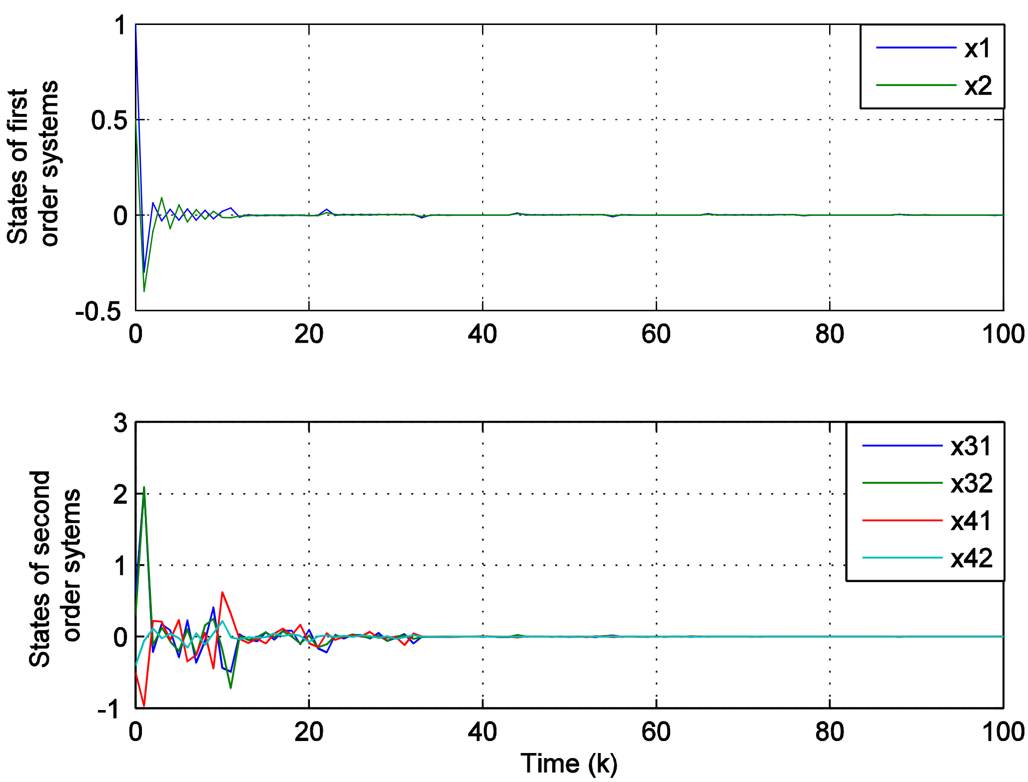

Figure 5. Response of the first order systems (top) and the second order systems (bottom) for *h*=11.

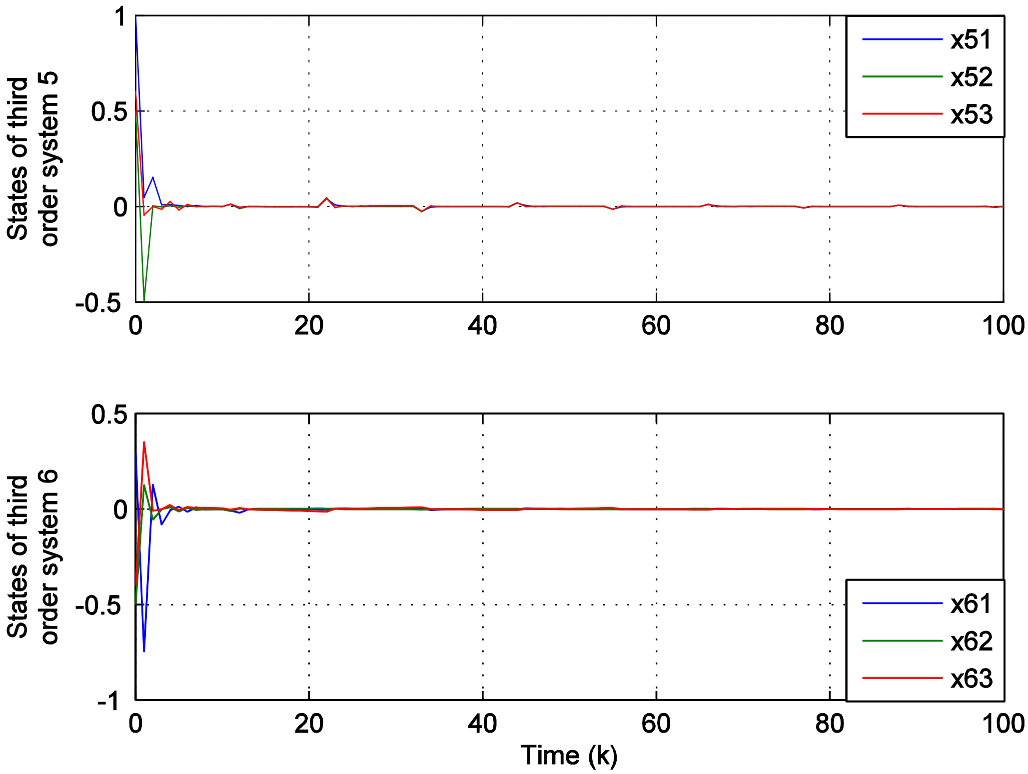

Figure 6. Response of the third order system 1 (top) and third order system 2 (bottom) for h=11.

5. Stabilization of coupled systems using multirate updates

In this section we consider the same problem and approach as in the previous section but we do not restrict all updates to take place at the same time instants. Instead, we allow every subsystem to use a different sampling period. This approach results in a better usage of network resources. By allowing the sensors to transmit their measurements using different sampling periods we reduce the probability of packet collisions and the size of networked induced delays compared to the case when all of the sensors need to sample and transmit at the same instants. Additionally, we will show that in many cases a further reduction in network communication can be obtained by using different update rates for each sensor compared to the single rate case shown in previous section.

Consider a set of N subsystems or agents as represented in Figure 4. Each subsystem can be represented by (23) and the models by equation (24), for each $i \in \mathrm{N}$, N denotes the set $\{1,2,...N\}$ of N integers where $x_i, \hat{x}_i \in \mathbb{R}^{ni}$ represent respectively the real state of the *i-th* unit and the state of the corresponding model, $u_i, \hat{u}_i \in \mathbb{R}^{mi}$ represent the local input for subsystem i and for model i, respectively. The matrices A_i, A_{ij}, B_i, $\hat{A}_i, \hat{A}_{ij}, \hat{B}_i$ are of appropriate dimensions. Note that the subsystems could have different dynamics and different dimensions, the dimensions *mi* and *ni* could be all different in general. Note also that each LCU*i* has access to its local state x_i at all times which is used to compute the local subsystem control input. The system control inputs u_i and the model control inputs $\hat{u}_i$ are given by (25) and (26), respectively. Likewise, the overall networked system can be represented by equations (28) and (29), where $A, \hat{A} \in \mathbb{R}^{n \times n}$ and $B, \hat{B} \in \mathbb{R}^{n \times m}$ with $n = \sum_{i=1}^{N} ni$ and $m = \sum_{i=1}^{N} mi$. The system and model matrices are given by (30).

Let s_i represent the sampling period that sensor i uses to send measurements in order to update the states of all models corresponding to subsystem i, that is, a measurement x_i to update all models $\hat{x}_i$. Let s represent the minimum common multiple of all s_i.

In order to obtain the response of the overall networked system with multirate samplings and using the model-based control inputs we define all the update instants within a period s by arranging the periods s_i and its multiples up to before s in increasing order as in (10), where $r_i = \frac{s}{s_i}$ for i=1…N represents the relative sampling rate compared to the sampling rate given by the period s. Let h_i for i=1…p-1 represent the update instants in increasing order, where p is the total number of update instants within a period *s* including the update at time *s*. Note that at any given instant one or more sets of states $\hat{x}_i$ can be updated. Let the update matrix I_i corresponding to subsystem *i* be given by (12). The matrices I_{h_i} represent all updates that take place at time instant h_i and are given by (13).

Theorem 6. The overall networked system (28) with model-based control inputs and with *multirate model updates* is asymptotically stable for given inter-sample periods s_i if and only if the eigenvalues of

$$(A+BK_{diag})^s + \sum_{i=1}^{p}(A+BK_{diag})^{h_p-h_i}\,\Xi_{h_i-h_{i-1}}U_{h_i-1} \tag{36}$$

lie inside the unit circle, where

$$\begin{aligned}\Xi_{h_i-h_{i-1}} &= \sum_{j=0}^{h_i-h_{i-1}-1}(A+BK_{diag})^{h_i-h_{i-1}-1-j}BK_{off}(\hat{A}+\hat{B}K)^j \\ U_{h_i} &= (I-I_{h_i})(\hat{A}+\hat{B}K)^{h_i-h_{i-1}}U_{h_i-1} + I_{h_i}((A+BK_{diag})^{h_i} + \sum_{q=1}^{i}(A+BK_{diag})^{h_i-h_q}\,\Xi_{h_q-h_{q-1}}U_{h_q-1})\end{aligned} \tag{37}$$

with $h_0 = 0, U_{h_0} = I, h_p = s$.

Proof. Let us consider the beginning of a period *s*. At this time instant all sensors send measurements and we have that $\hat{x}(ks) = x(ks)$. At the time of the first update after time *ks*, that is, at time $ks + h_1$ we have:

$$\begin{aligned}x(ks+h_1) &= ((A+BK_{diag})^{h_1} + \sum_{j=0}^{h_1-1}(A+BK_{diag})^{h_1-1-j}BK_{off}(\hat{A}+\hat{B}K)^j)x(ks) \\ &= ((A+BK_{diag})^{h_1} + \Xi_{h_1})x(ks)\end{aligned} \tag{38}$$

and the model state after the update has taken place is given by:

$$\begin{aligned}\hat{x}(ks+h_1) &= ((I-I_{h_1})(\hat{A}+\hat{B}K)^{h_1} + I_{h_1}((A+BK_{diag})^{h_1} + \Xi_{h_1}))x(ks) \\ &= U_{h_1}x(k)\end{aligned} \tag{39}$$

Following a similar analysis we can obtain the response of both the system and the model at time $ks+h_2$ as a function of $x(ks+h_1)$ and $\hat{x}(ks+h_1)$:

$$x(ks+h_2)=(A+BK_{diag})^{h_2-h_1}x(ks+h_1)+\sum_{j=0}^{h_2-h_1-1}(A+BK_{diag})^{h_2-h_1-1-j}BK_{off}(\hat{A}+\hat{B}K)^j\hat{x}(ks+h_1) \tag{40}$$

$$\hat{x}(ks+h_2)=(I-I_{h_2})(\hat{A}+\hat{B}K)^{h_2-h_1}\hat{x}(ks+h_1)+I_{h_2}x(ks+h_2) \tag{41}$$

but, since both $x(ks+h_1)$ and $\hat{x}(ks+h_1)$ can be expressed in terms of $x(ks)$, we get:

$$x(ks+h_2)=((A+BK_{diag})^{h_2}+(A+BK_{diag})^{h_2-h_1}\Xi_{h_1}+\Xi_{h_2-h_1}U_{h_1})x(ks) \tag{42}$$

$$\begin{aligned}\hat{x}(ks+h_2)&=((I-I_{h_2})(\hat{A}+\hat{B}K)^{h_2-h_1}U_{h_1}+I_{h_2}((A+BK_{diag})^{h_2}+(A+BK_{diag})^{h_2-h_1}\Xi_{h_1}+\Xi_{h_2-h_1}U_{h_1}))x(ks)\\&=U_{h_2}x(ks)\end{aligned} \tag{43}$$

Following the same analysis for each update instant h_i up to $h_p=s$ we obtain

$$x((k+1)s)=((A+BK_{diag})^s+\sum_{i=1}^{p}(A+BK_{diag})^{h_p-h_i}\Xi_{h_i-h_{i-1}}U_{h_i-1})x(ks) \tag{44}$$

Since (44) represents an LTI system then the overall networked system is asymptotically stable when the eigenvalues of (36) lie strictly inside the unit circle. ■

Example 3. Consider the same set of systems and models as in example 2 with the same initial conditions. By allowing the agents to broadcast their states using different rates we are able to asymptotically stabilize the overall system and to further reduce the communication between agents. There exist many combinations of periods s_i that result in a stable system. The next selection that significantly increases the sampling periods and reduces network traffic compared to h=11 in example 2 was used in the simulation shown in Figures 7 and 8,

$$s_1=60,\ s_2=15,\ s_3=10,\ s_4=10,\ s_5=30,\ s_6=12.$$

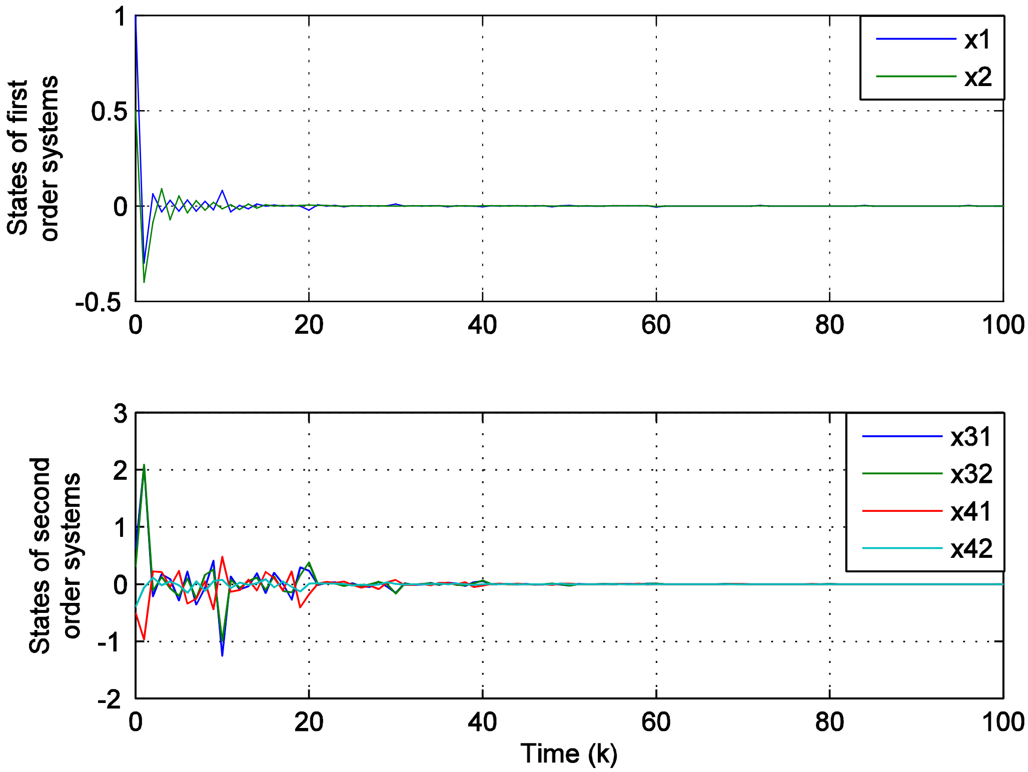

Figure 7. Response of the first order systems (top) and the second order systems (bottom) for $s_1 = 60, s_2 = 15, s_3 = 10, s_4 = 10$.

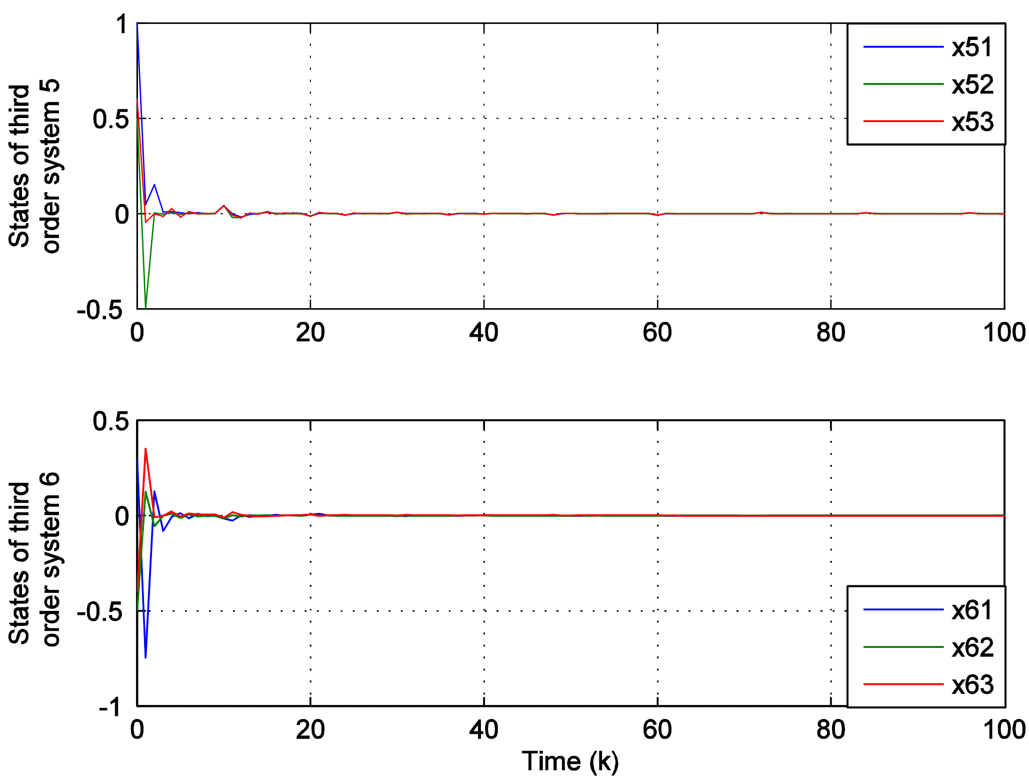

Figure 8. Response of the third order system 1 (top) and third order system 2 (bottom) for $s_5 = 30, s_6 = 12$.

6. Applicability of multirate MB-NCS

Sections 3 and 5 provided necessary and sufficient conditions for stability of MB-NCS using multirate model updates. It was also shown that the multirate approach may significantly increase the update periods with respect to the single rate update counterparts addressed in sections 2 and 4. Since we are considering model uncertainties, the necessary and sufficient stability conditions are based on the model parameters and the plant parameters as well. The exact value of the plant parameters is generally unknown. Next, we provide a simple but illustrative example in which we fixed the updates rates and we use the results of the paper in order to estimate admissible model uncertainties, i.e. possible uncertainties that result in an overall stable system using the predefined update rates.

Example 4. Consider a set of 3 first order unstable, coupled nominal systems of the form (23). The known nominal models (24) are given by the following parameters:

$$\begin{aligned}
&\hat{A}_1 = -1.5, \quad \hat{A}_2 = 2.1, \quad \hat{A}_3 = -1.6 \\
&\hat{B}_1 = 1, \quad \hat{B}_2 = -1, \quad \hat{B}_3 = 2 \\
&\hat{A}_{12} = \hat{A}_{21} = -0.38, \quad \hat{A}_{13} = \hat{A}_{31} = 0.2, \quad \hat{A}_{23} = \hat{A}_{32} = 0.5.
\end{aligned}$$

We select the following periods $s_1 = 15, s_2 = 6, s_3 = 10$. Note that the same procedure can be repeated for any selection of periods s_i. Assume that the plant parameters are given by:

$$\begin{aligned}
&A_1 = \hat{A}_1 + \tilde{A}_1, \quad A_2 = \hat{A}_2 + \tilde{A}_2, \quad A_3 = \hat{A}_3 + \tilde{A}_3 \\
&B_i = \hat{B}_i, \quad A_{ij} = \hat{A}_{ij}
\end{aligned}$$

where the parameters $\tilde{A}_i$ represent the unknown model uncertainties. By fixing the update periods we are able to evaluate (36) for different values of $\tilde{A}_i$. The contour plots of Figure 9 show the maximum eigenvalues of (36) for 6 different values of $\tilde{A}_3$ and for different combinations of $\tilde{A}_1$ and $\tilde{A}_2$ in increments of 0.1. By following this search it is possible to evaluate all possible combinations of model uncertainties that result in an overall stable multirate MB-NCS. In these illustrations the admissible uncertainty combinations are those that result in a maximum eigenvalues within the contour labeled by 1. Similar searches are possible for larger number of uncertain parameters and also including the different update rates.

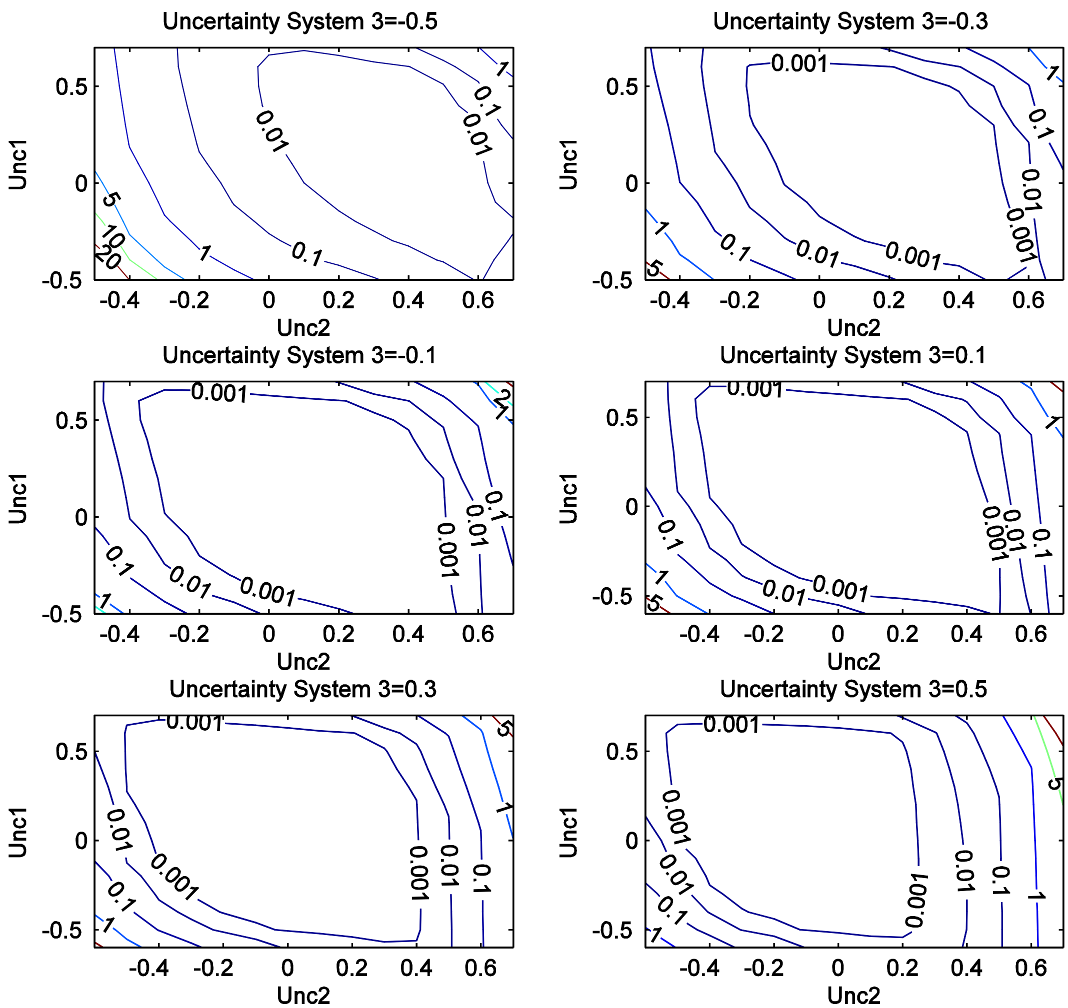

Figure 9. Maximum eigenvalues of (36). Model uncertainties that preserve stability inside contour labeled as 1 for update periods $s_1 = 15, s_2 = 6, s_3 = 10$.

7. Conclusions

The single-rate MB-NCS approach has been extended in this paper in order to control networked systems using multirate sampling. The use of multirate model updates results in an important reduction of network traffic when applied to high-order systems with centralized controller and distributed sensors. This is achieved by assigning low update rates to sensors corresponding to slow dynamics. The model-based multirate approach was also applied for control of groups of coupled systems that exchange information through a shared network. Similarly, the slow dynamics subsystems are able to operate using a low update rate. In these two cases, the model based multirate framework provides greater flexibility for scheduling measurement updates while maintaining stability of the network interconnected systems.

References

Antsaklis, P.J., and Baillieul, J. (2007), 'Special issue on technology of networked control systems', *Proceedings of the IEEE*, 95, 5-8.

Antsaklis, P.J., and Michel, A.N. (1997), *Linear systems,* New York: McGraw Hill, 1st Edition.

Bittanti, B., and Colaneri, P. (2000), 'Invariant representations of discrete-time periodic systems', *Automatica*, 36, 1777-1793.

Camponogara, E., Jia, D., Krogh, B.H., and Talukdar, S., (2002) 'Distributed model predictive control', *IEEE Control Systems Magazine*, 22, 44-52.

Chen, T., and Francis, B. (1995), *Optimal sampled data control systems*, Springer.

Ding, F., and Chen, T. (2005), 'Hierarchical identification of lifted state-space models for general dual-rate systems', *IEEE Transactions on Circuits and Systems*, 52, 1179-1187.

Ding, F., and Chen, T. (2004), 'Combined parameter and output estimation of dual-rate systems using an auxiliary model', *Automatica*, 40, 1739-1748.

Dunbar, W.B. (2007), 'A distributed receding horizon control algorithm for dynamically coupled nonlinear systems', *IEEE Transactions on Automatic Control*, 52, pp. 1249-1263.

Elia, N., and Mitter, S.K. (2001), 'Stabilization of linear systems with limited information', *IEEE Transactions on Automatic Control*, 46, 1384-1400.

El-Farra, N.H., and Sun, Y. (2009), 'Quasi-decentralized control of process systems using wireless sensor networks with scheduled sensor transmissions', in *Proceedings of the American Control Conference*, pp. 3390-3396.

Garcia, E., and Antsaklis, P.J. (2010), 'Model-based control using a lifting approach', *in Proceedings of the 18th Mediterranean Conference on Control and Automation,* pp. 105-110.

Garcia, E., and Antsaklis, P.J. (2012), 'Decentralized model-based event-triggered control of networked systems', *in Proceedings of the American Control Conference,* pp. 6485-6490.

Garcia, E., Vitaioli, G., and Antsaklis, P.J. (2011), 'Model-based tracking control over networks', in *Proceedings of the IEEE International Conference on Control Applications*, pp. 1226-1231.

Georgiev, D., and Tilbury D.M. (2004), 'Packet-Based control', in *Proceedings of the American Control Conference*, pp. 329 – 336.

Kahane, C.A., Mirkin, L.B., and Palmor, J.Z. (1999), 'Discrete-time lifting via implicit descriptor systems', in *Proceedings of the European Control Conference*.

Lee, J.H., Gelormino, M.S., and Morari, M. (1992), 'Model predictive control of multirate sample-data systems: a state approach', *International Journal of Control*, 55, 153-191.

Li, D., Shah, S.L., and Chen, T. (2002), 'Analysis of dual-rate inferential control systems', *Automatica*, 38, 1053-1059.

Lian, F.L., Moyne, J.R., and Tilbury, D.M. (2001), 'Performance evaluation of control networks: Ethernet, Controlnet, and Devicenet', *IEEE Control Systems Magazine*, 21, 66 – 83.

Lunze, J., and Lehmann, D. (2010), 'A state-feedback approach to event-based control', *Automatica*, 46, 211- 215.

Montestruque, L.A., and Antsaklis, P.J. (2002), 'State and output feedback control in model-based networked control systems', in *Proceedings of the IEEE Conference on Decision and Control*, pp. 1620-1625.

Montestruque, L.A., and Antsaklis, P.J. (2003), 'On the model-based control of networked systems', *Automatica*, 39, 1837 – 1843.

Montestruque, L.A., and Antsaklis, P.J. (2004), 'Stability of model-based control of networked systems with time varying transmission times', *IEEE Transactions on Automatic Control*, 49, 1562-1572.

Mutambara, A.G.O., and Durrant-Whyte, H.E. (2000), 'Estimation and control for a modular wheeled mobile robot', *IEEE Transactions on Control Systems Technology*, 8, 35-46.

Scattolini, R., and Schiavoni, N. (1995), 'A multirate model-based predictive controller', *IEEE Transactions on Automatic Control*, 40, 1093-1097.

Sezer, M.E., and Siljak, D.D. (1990), 'Decentralized multirate control', *IEEE Transactions on Automatic Control*, 35, 60-65.

Sinopoli, B., Sharp, C., Schenato, L., Schaffert, S., and Sastry, S. (2003), 'Distributed control applications within sensor networks', *Proceedings of the IEEE,* 91, 1235-1246.

Sun, Y., and El-Farra, N.H. (2008), 'Quasi-decentralized model-based networked control of process systems', *Computers and Chemical Engineering*, 32, 2016-2029.

Sun, Y., and El-Farra, N.H. (2012a), 'Integrating control and scheduling of distributed energy resources over networks', *Chemical Engineering Science,* 67, 80-91.

Sun, Y., and El-Farra, N.H. (2012b), 'Resource aware quasi-decentralized control of networked process systems over wireless sensor networks', *Chemical Engineering Science,* 69, 93-106.

Tange, Y. (2005), 'State space parameterization of stabilizing multirate controllers for MIMO linear time-invariant plants', in *Proceedings of the IEEE Conference on Decision and Control-European Control Conference*, pp. 1583-1588.

Vadigepalli, R., and Doyle, F.J. (2003), 'A distributed state estimation and control algorithm for plantwide processes', *IEEE Transactions on Control Systems Technology*, 11, 119-127.

Yu, D.W., and Yu, D.L. (2007), 'Multirate model predictive control of a chemical reactor based on three neural models', *Biochemical Engineering Journal*, 37, 86-97.