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Control of Networked Switched Systems using Passivity and Dissipativity

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Passivität und Dissipativität sind wichtige bei der Untersuchung von Cyber-physischen Systemen (CPS), die typischerweise aus zeit- und ereignisgesteuerter Dynamik bestehen. Dieser Beitrag stellt Bedingungen die die Stabilität vernetzter schaltender Systeme garantieren, die zur Modellierung von CPS verwendet werden können. Es werden Ergebnisse präsentiert, um Stabilität sicherzustellen, wenn Systeme über eine Rückführung miteinander verbunden werden. Außerdem Methoden zur Sicherstellung der Stabilität von geschalteten Systemen, die über ein Netzwerk mit Totzeiten verbunden sind, behandelt.

Passivity and dissipativity are important in the study of Cyber Physical Systems (CPS) where systems with time and event-driven dynamics are connected. This paper provides conditions which guarantee stability of interconnected switched systems which may be used to model CPS. Results are presented for maintaining stability when switched systems are connected in feedback. Then methods for maintaining stability of passive switched systems that are connected over a network with delays are presented.

Schlagwörter:

Keywords: Switched systems, dissipative systems, networked control systems

1 Introduction

As computing and communication devices become ever-smaller and ever-cheaper, they are increasingly embedded in objects and structures that interact directly with the physical environment and extend human capabilities. Multiple sensing and actuation units that gather, process, exchange, and use information as a team are the cyber-world of computing and communications with the physical world are called Cyber-Physical Systems (CPS), see e.g. [1, 2]. CPS span engineered, physical, and biological systems and create new applications with enormous societal impact and economic benefit.

CPS arise when physical systems interact closely with cyber systems. Physical systems have dynamics that follow physical laws that arise in a variety of application areas including kinematics, fluid dynamics, and electrical circuits. These dynamics are modeled using differential/difference equations and algebraic equations that depend strongly on time. The cyber systems evolve based on the occurrence of events, both physical and in software, and typically have little or no dependence on time. These include computational systems, communication systems, or any discrete-state based system. Combining these vastly different components results in system models that are governed by hybrid dynamics. Often in CPS, these hybrid systems may be interconnected over a network.

This paper provides an overview of some promising approaches for analysis and synthesis of CPS. While some of the results in this paper have appeared previously (citations included), additional details on the approaches are provided with an original example. The remainder of this introduction summarizes the major components of this analysis approach.

Switched Systems
An important class of hybrid systems is switched systems. These systems are modeled by a finite set of dynamics with a rule that determines switching among the

individual subsystems. At each time, a single subsystem is active and the system evolves according to the dynamics of this subsystem. In this paper, we consider subsystems that are nonlinear and time-invariant. We also assume the switching rule is allowed to be arbitrary. Although results that assume arbitrary switching have the most restrictive stability conditions, this assumption allows for stability that is robust to variations in switching. This is an important consideration when systems are interconnected; while the switching signal may be fixed when a system operates in isolation, switching is often unknown when systems are interconnected. For more on switched systems, refer to the following surveys and the references therein [3–5].

Passivity and Dissipativity Approach
Another facet of CPS is that these systems are often complex and made up of several interacting, often heterogeneous, components. Traditional approaches for analyzing large scale systems have focused on energy-based concepts [6]. While the main approach for stability of a single system has been Lyapunov theory, for large scale systems passivity and dissipativity theory offer significant advantages [7,8]. For example, stability of the feedback interconnection of two passive systems is guaranteed, and the stability for dissipative systems in feedback can be verified by checking a simple condition. When considering a quadratic supply rate, as in QSR-dissipativity, a general test may be applied to determine whether a feedback interconnection is stable [9,10]. This result is a generalization of both the passivity theorem and the small gain theorem and may be applied to a large class of interconnected systems, for example, feedback interconnections that contain an unstable system.

Dissipativity theory for switched systems is a relatively new topic. Dissipativity theory has been considered for continuous time switched systems [11–13] and discrete time switched systems [14–16]. Most of these papers focus on a notion of decomposable dissipativity that breaks the energy supply rate for each subsystem into an active rate and an inactive rate. The special case of passivity indices for switched systems was addressed in [13].

Control Over Networks
There are many applications where systems are controlled over a network. The use of existing wired or wireless networks typically reduces costs and allows for the network to be reconfigured. However, the use of existing networks often adds delay to the communication channel and, at times, data may be dropped entirely. One solution to the delay problem, introduced in [17] and [18], is to use passivity theory and the wave variable transformation [19]. The wave variable transformation is used to map the generalized power variables, the ones used to show passivity, to wave variables. After being transformed to wave variables, the energy exchanged with the network is decoupled between waves going out over the network and waves coming in from the network. The decoupling makes the delayed channel lossless so no energy is added or removed by the network. This approach was extended by several recent papers including [20–24]. A more generalized approach was considered in [25] where switched systems were considered in this framework.

Structure of Paper
The remainder of this paper is organized as follows. Section 2 provides background material for the paper. First, the classical results of dissipativity and passivity are reviewed. Then notions of dissipativity and passivity for switched systems are covered. The problem of networking passive systems over delayed networks is also discussed. The main results of the paper are presented in Sections 3 and 4. Section 3 covers results on interconnected dissipative switched systems without delay. An example of this method is included. The problem of connecting passive switched systems in feedback over a network with delay is considered in Section 4. Finally, concluding remarks are given in Section 5.

2 Background Material

2.1 Mathematical Preliminaries

A real valued vector x of dimension n will be denoted $x \in \mathbb{R}^n$. In this paper, nonlinear non-switched systems of interest are of the form ,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. It is assumed that the vector field f is Lipschitz with respect to x. It can be assumed without a loss of generality that $f(0,0) = 0$ and $h(0,0) = 0$.

The notion of input-output stability used in this paper is $\mathcal{L}_2$ stability [27]. This definition requires notation for signal truncation. Truncating the signal $x(t)$ at time T, denoted $x_T(t)$, is given by

$$x_T(t) = \begin{cases} x(t), & \text{for } t \leq T \\ 0, & \text{for } t > T \end{cases}. \tag{2}$$

The $\mathcal{L}_2$ norm of a signal $x(t)$ is given by

$$||x(t)||_2 \triangleq \sqrt{\int_0^\infty x^T(\tau)x(\tau)d\tau}. \tag{3}$$

Definition 1. *[27] A nonlinear system (1) is finite-gain $\mathcal{L}_2$ stable if there exist constants γ and β to satisfy,*

$$||y_T(t)||_2 \leq \gamma\, ||u_T(t)||_2 + \beta. \tag{4}$$

Additionally, the smallest γ such that there exists a β to satisfy the inequality is referred to as the $\mathcal{L}_2$ gain of the system.

In addition to $\mathcal{L}_2$ stability, Lyapunov stability will be used in this paper. The definition is omitted but may be found in [27].

2.2 Dissipativity Theory

Dissipativity is a characterization of system behavior based on a generalized notion of energy [7]. The property captures the behavior of a system that only stores and dissipates energy without generating its own. The property defines externally supplied energy in order to bound internally stored energy.

Dissipativity for state space systems is typically shown by finding a storage function $V : \mathbb{R}^n \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive real line. As a notion of internally stored energy, V is non-negative and assumed to be zero at the equilibrium, $V(0) = 0$. Dissipativity is shown with respect to an energy supply rate $\omega : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$.

Definition 2. *[7] A nonlinear system (1) is dissipative with respect to an energy supply rate $\omega(u, y)$ if there exists a non-negative storage function $V(x)$ such that*

$$\int_{t_1}^{t_2} \omega(u, y)dt \geq V(x(t_2)) - V(x(t_1)). \quad (5)$$

While this definition allows the supply rate to be any function of input and output, it may be difficult to find an appropriate energy supply rate in practice. A more tractable form of dissipativity is the quadratic form known as QSR dissipativity [9, 10].

Definition 3. *[9] A nonlinear system (1) is QSR dissipative if it is dissipative with respect to the supply rate*

$$\omega(u, y) \triangleq \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = y^T Q y + 2y^T S u + u^T R u. \quad (6)$$

Systems that are QSR dissipative are $\mathcal{L}_2$ stable when $Q < 0$. QSR dissipativity provides stability results for the feedback interconnection (Fig. 1) of two systems. Specifically, the feedback of two QSR dissipative systems forms a new system that is QSR dissipative [10]. Stability may be assessed by the new Q in the QSR parameters of the interconnection. Using the signals $r(t) = [r_1^T(t)\ r_2^T(t)]^T$ and $y(t) = [y_1^T(t)\ y_2^T(t)]^T$, the feedback interconnection ($r \to y$) is $\mathcal{L}_2$ stable when there exists a constant α such that following matrix is negative definite,

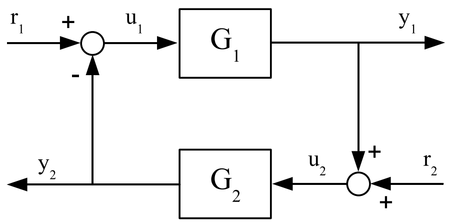

Bild 1: The feedback interconnection of two systems G_1 and G_2.

$$\hat{Q} = \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^T \\ -S_1^T + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix} < 0.$$

While the restriction to QSR dissipative systems reduces the number of dissipative systems that can be considered, the class of dissipative systems captured by QSR dissipativity includes all passive and $\mathcal{L}_2$ stable systems as well as many unstable and non-minimum phase systems.

Passive systems form a special class of dissipative systems. Passivity theory has its origins in electrical circuit theory where feedback interconnections of passive circuit components formed stable interconnections because the components only dissipated energy without generating their own energy.

Definition 4. *A nonlinear system (1) is passive if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \epsilon y^T y$ for $\epsilon \geq 0$. Furthermore, if $\epsilon > 0$ the system is said to be output strictly passive (OSP).*

This definition uses the system input u and output y. In this paper, these will be referred to as power variables even when their product is not a traditional notion of power.

Passive systems are Lyapunov stable when they are detectable. In addition, when two passive systems are interconnected in negative feedback (Fig. 1) the resulting system is passive. This provides a guarantee that any two passive systems can be combined in feedback to form a stable feedback interconnection. There is no such guarantee for combining two stable systems in feedback. This property of passivity, feedback invariance, makes it a strong tool for the analysis and synthesis of interconnected systems. Both passivity and dissipativity provide results for feedback stability and feedback invariance. These properties can be used for the analysis and synthesis of large scale systems [6]. Passivity has the added property that it is preserved when systems are combined in parallel as in Fig. 2. For more detail on passivity, refer to [27] and [28].

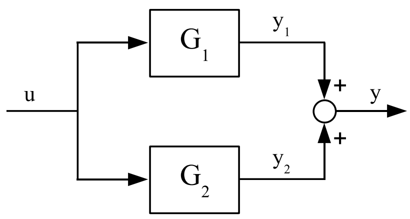

Bild 2: The parallel interconnection of two systems.

The subset of passive systems that are OSP is important because these systems are $\mathcal{L}_2$ stable. They also form feedback interconnections that are $\mathcal{L}_2$ stable. These results can be extended to internal stability with an appropriate detectability assumption for nonlinear systems. Systems that are $\mathcal{L}_2$ stable can be shown to be asymptotically stable if they are also zero-state detectable.

Definition 5. *[29] An autonomous ($u(t) = 0$) nonlinear system,*

$$\begin{aligned} \dot{x} &= f(x, 0) \\ y &= h(x, 0), \end{aligned} \tag{7}$$

is zero-state detectable (ZSD) if $y(t) = h(x(t), 0) = 0$ $\forall t \geq t_0$ implies $\lim_{t\to\infty} x(t) = 0$.

This definition is a weaker condition than zero-state observability where $y(t) = h(x(t), 0) = 0$ $\forall t \geq t_0$ implies $x(t) = 0$ $\forall t \geq t_0$.

2.3 Dissipative Switched Systems

The concept of dissipativity has been extended to switched systems using multiple storage functions. The definition given in [11] focuses on two main conditions. The primary condition is that each subsystem is dissipative when it is active. The energy supply rate may be different for every subsystem. The second condition is that each subsystem is dissipative, with respect to a different rate, when it is inactive. The notion of cross supply rate is used to capture energy transfer from each active subsystem to each inactive subsystem. The cross supply rates can be different for each subsystem and each active system. For a switched system with M modes, the system may have M different energy storage functions, M energy supply rates, and $M(M-1)$ cross supply rates.

In the present paper, dissipativity is applied to switched systems of the form,

$$\begin{aligned} \dot{x} &= f_\sigma(x, u) \\ y &= h_\sigma(x, u), \end{aligned} \tag{8}$$

where the switching signal $\sigma(t)$ indicates the current active subsystem out of the set $\Sigma = \{1, ..., M\}$, i.e. $\sigma : \mathbb{R}^+ \to \Sigma$. It is assumed that for each subsystem $i \in \Sigma$, f_i is Lipschitz with respect to x, $f_i(0, 0) = 0$, and $h_i(0, 0) = 0$. A single switching instant is denoted by t_{i_k}, which is the k^{th} time that the i^{th} subsystem becomes active. This system becomes inactive at time $t_{(i_k+1)}$ and becomes active again at time $t_{i_{(k+1)}}$. The values of i are a subset of $\mathbb{Z}^+$ (the positive integers) from 1 to M, and k take on values in $\mathbb{Z}^+$ that is allowed to be infinite. To avoid Zeno behavior, it is assumed that on any finite time interval, t_0 to arbitrary time T, the system switches a finite number of times K, where K may depend on the time T chosen. To avoid trivial asymptotic analysis, it is assumed that the system switches an infinite number of times on the infinite time horizon.

The following definition uses the notion of class-K functions. A function $\alpha(x)$ is class-K if it is defined for $x \in [0, \infty)$, $\alpha(0) = 0$, and it is strictly increasing [27].

Definition 6. *[11] A switched system (8) is dissipative if there exist storage functions $V_i(x)$ bounded by class-K functions*

$$\underline{\alpha}_i(||x||) \leq V_i(x) \leq \overline{\alpha}_i(||x||), \tag{9}$$

energy supply rates $\omega_i(u, y)$, and cross supply rates $\omega_j^i(u, y, x, t)$ such that the following conditions hold.

1. *Each subsystem i is dissipative with respect to $\omega_i(u, y)$ while active, i.e. for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$ and $\forall i, k$,*

$$\int_{t_1}^{t_2} \omega_i(u, y)dt \geq V_i(x(t_2)) - V_i(x(t_1)). \tag{10}$$

2. *Each subsystem j is dissipative with respect to $\omega_j^i(u, y, x, t)$ when it is inactive, i.e. for each active subsystem i, $\forall j \neq i$, and for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$,*

$$\int_{t_1}^{t_2} \omega_j^i(u, y, x, t)dt \geq V_j(x(t_2)) - V_j(x(t_1)). \tag{11}$$

3. *For all i and j there exist absolutely integrable functions $\phi_j^i(t)$ and some input $u^*(t)$ such that $\forall t \geq t_0$ $f_i(0, u^*) = 0$, $\omega_i(u^*, y) \leq 0$, and*

$$\omega_j^i(u^*, y, x, t) \leq \phi_j^i(t), \forall j \neq i. \tag{12}$$

Along with the definition of dissipativity for switched systems, [11] also covered stability conditions for a dissipative switched system. The special case of passivity was also considered.

Definition 7. *A switched system (8) is passive if it is dissipative with respect to the energy supply rates $\omega_i(u, y) = u^T y - \epsilon_i y^T y$ where $\epsilon_i \geq 0, \forall i$.*

A switched system is considered output strictly passive (OSP) if it is passive with $\epsilon_i > 0$ for all i. Passive switched systems are Lyapunov stable when all subsystems are ZSD. Asymptotic stability can be shown when negative output feedback is applied or when the system is OSP.

Theorem 1. *Consider a switched system that is output strictly passive. If all subsystems are ZSD, then the switched system is asymptotically stable.*

By itself, this result is only an indirect method of showing asymptotic stability. There are more direct methods of showing asymptotic stability in the literature (for example, see [3–5] and the references therein). However, when using Theorem 1 in conjunction with Theorem 2, open-loop conditions for asymptotic stability of the feedback interconnection of two switched systems are derived.

Theorem 2. *The negative feedback interconnection of two output strictly passive switched systems is again an output strictly passive switched system.*

The proof of this result can be found in [30]. These two results can be applied to the feedback interconnection of two switched systems. The switched systems must be OSP and have all subsystems be asymptotically zero-state detectable. When each of these switched systems meets the two open-loop conditions, the resulting interconnected system is OSP and asymptotically stable.

These open-loop conditions for closed loop stability will be applied to networked control systems later in this paper.

2.4 Network Structure

The network control structure used in this paper is given in Fig. 3. Switched system G_1 is the mapping $e_1 \to y_1$ and switched system G_2 is the mapping $e_2 \to y_2$. Typically one of these systems is a given plant and the other is the designed passive controller. The delays in the network are assumed to be constant, but the two delays T_1 and T_2 may be different and potentially unknown. The

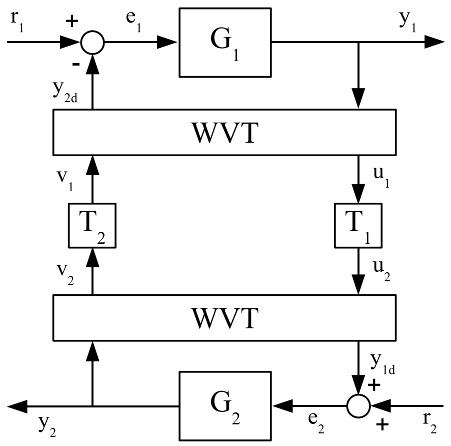

Bild 3: The network control structure for passive plants makes use of the wave variable transformation (WVT blocks) on either side of the network. The blocks T_1 and T_2 are the time delays in the network.

signal relationships are given as,

$$e_1 = r_1 - y_{2d} \tag{13}$$
$$e_2 = r_2 + y_{1d}. \tag{14}$$

The network is modeled as a constant delay in each direction,

$$u_2(t) = u_1(t - T_1) \tag{15}$$
$$v_1(t) = v_2(t - T_2). \tag{16}$$

The wave variable transformation (WVT) is defined as in [19]. The linear transformation to wave variables is

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} bI & I \\ bI & -I \end{bmatrix} \begin{bmatrix} y_1 \\ y_{2d} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} bI & I \\ bI & -I \end{bmatrix} \begin{bmatrix} y_{1d} \\ y_2 \end{bmatrix}, \tag{18}$$

where b is the impedance of the channel and can be chosen in the synthesis of a controller. The energy stored in the network is the sum of the energy going into the network minus the energy coming out of the network.

$$V_N = \frac{1}{2} \int_{t_0}^{t} (u_1^T u_1 + v_2^T v_2 - u_2^T u_2 - v_1^T v_1) d\tau. \tag{19}$$

When the system delays are constant, this expression can be simplified to show that the energy in the network is positive.

$$V_N = \frac{1}{2} \int_{t-T_1}^{t} u_1^T u_1 d\tau + \frac{1}{2} \int_{t-T_2}^{t} v_2^T v_2 d\tau \geq 0 \tag{20}$$

The quantity V_N is always nonnegative. By the definition of energy stored in the network (19), it can be seen that the energy on the G_1 side of the network bounds the energy on the G_2 side.

$$\frac{1}{2} \int_{t_0}^{T} (u_1^T u_1 - v_1^T v_1) d\tau \geq \frac{1}{2} \int_{t_0}^{T} (v_2^T v_2 - u_2^T u_2) d\tau \tag{21}$$

$$\implies \int_{t_0}^{T} y_1^T y_{2d} d\tau \geq \int_{t_0}^{T} y_2^T y_{1d} d\tau \tag{22}$$

This fact can be used to show stability of the overall system [17].

Theorem 3. *Consider two nonlinear (1) passive systems that are interconnected over a delayed network using the wave variable transformation (Fig. 3). If the delays in the network are constant, the interconnected system is $\mathcal{L}_2$ stable.*

This result gives stability of the networked control system. Additionally, if the two systems are zero-state detectable, the overall system is asymptotically stable for $r(t) = 0$. More details on this result may be found in [20–24].

This result assumes that the delays are constant. However, the extension to communication channels with time varying delays is straightforward, see e.g. [23]. The remainder of this paper assumes the delays are constant, but the time varying approach can be applied without difficulty.

3 Interconnecting Dissipative Switched Systems

This section focuses on the problem of interconnecting dissipative switched systems. The first part covers a definition of QSR dissipative switched systems. Results are given that guarantee stability for dissipative systems and for the feedback interconnection of two dissipative switched systems. The second part provides an example demonstrating how this method can be applied to synthesize a controller.

3.1 Main Results on Stability of Interconnections

Dissipativity theory can be used to show stability of systems in feedback. Consider the feedback interconnection of two switched systems G_1 and G_2 (Fig. 1). This interconnection forms a new switched system G which is a mapping from $r \to y$. The subsystems of the new system depend on the subsystems of both G_1 and G_2. This causes the number of subsystems of the combined system to grow to as many as $M = M_1 M_2$ where G_1 and G_2 consist of M_1 and M_2 subsystems, respectively. Whenever either system G_1 or G_2 switches, the overall system G switches. This means that the set of switching instants of the new system G is the union of the sets of switching instants of the two individual systems.

Stability of the feedback interconnection is considered here for QSR dissipative switched systems. This is a special case of dissipativity (Def. 6) where the active energy supply rates have a quadratic form.

Definition 8. *A dissipative switched system (8) is QSR dissipative if for each subsystem i the subsystem is dissipative with respect to the energy supply rate*

$$\omega_i(u,y) \triangleq \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{23}$$

for $i \in \{1, ..., M\}$.

The following theorem considers stability of the feedback interconnection of two dissipative switched systems. System G_1 has supply rates $\omega_i^{(1)}$ parametrized by $\{Q_i, S_i, R_i\}$ and G_2 has supply rates $\omega_{\hat{i}}^{(2)}$ with $\{Q_{\hat{i}}, S_{\hat{i}}, R_{\hat{i}}\}$. The result considers the active supply rates and inactive supply rates to establish a bound on the storage functions. Finally, a bound on the system state is inferred from the bound on the storage functions.

Theorem 4. *Consider the feedback interconnection of two QSR dissipative switched systems G_1 and G_2. If there exists a constant α to satisfy*

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + \alpha R_{\hat{i}} & -S_i + \alpha S_{\hat{i}}^T \\ -S_i^T + \alpha S_{\hat{i}} & R_i + \alpha Q_{\hat{i}} \end{bmatrix} \leq 0, \tag{24}$$

$$\forall i \in \{1, 2, ..., M_1\}, \forall \hat{i} = \{1, 2, ..., M_2\}, \tag{25}$$

and each subsystem is ZSD (Def. 5) the autonomous ($r(t) = 0$) feedback interconnection G is Lyapunov stable.

Proof. Since both systems are QSR dissipative, there exists $V_i^{(1)}$ for G_1 and $V_{\hat{i}}^{(2)}$ for G_2. The following summed storage functions can be defined

$$V^{(1)} = \sum_{i=1}^{M_1} V_i^{(1)} \quad \text{and} \quad V^{(2)} = \sum_{\hat{i}=1}^{M_2} V_{\hat{i}}^{(1)}.$$

Define $V(x(t)) = V^{(1)}(x_1(t)) + V^{(2)}(x_2(t))$ where $x = [x_1^T \ x_2^T]^T$. By (9) there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $V(x(t)) \leq \rho(||x||)$. Using conditions (11-12), $\forall \epsilon$ there exists a time T such that, $\forall t \geq T$,

$$V(x(t)) - V(x(T)) \leq \frac{1}{2}\rho(\epsilon). \tag{26}$$

This inequality follows directly from the definition of dissipativity. The details have been omitted because the derivation follows similarly to the development in Section 4 that sets up Theorem 8. Using the dissipative relationships we can find a bound on $V(x(T))$ based on previous switching instants,

$$V(x(T)) \leq V(x(t_K)) + \int_{t_K}^{T} \left[\omega_i^{(1)}(u,y) + \omega_{\hat{i}}^{(2)}(u,y)\right] dt + \frac{1}{2}(\rho(\epsilon)).$$

The term inside the integral, $\omega_i^{(1)}(u,y) + \omega_{\hat{i}}^{(2)}(u,y)$, can be written out

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}^T \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q_{\hat{i}} & S_{\hat{i}} \\ S_{\hat{i}}^T & R_{\hat{i}} \end{bmatrix} \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}.$$

The signal relationships can be substituted and the resulting expression simplified to arrive at

$$y^T \hat{Q}_{i\hat{i}} y + 2y^T \hat{S}_{i\hat{i}} r + r^T \hat{R}_{i\hat{i}} r$$

where

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + \alpha R_{\hat{i}} & -S_i + \alpha S_{\hat{i}}^T \\ -S_i^T + \alpha S_{\hat{i}} & R_i + \alpha Q_{\hat{i}} \end{bmatrix},$$

$$\hat{S}_{i\hat{i}} = \begin{bmatrix} S_i & \alpha R_{\hat{i}} \\ -R_i & \alpha S_{\hat{i}} \end{bmatrix} \quad \text{and} \quad \hat{R}_{i\hat{i}} = \begin{bmatrix} R_i & 0 \\ 0 & \alpha R_{\hat{i}} \end{bmatrix}.$$

By assumption, $\hat{Q}_{i\hat{i}} \leq 0, \forall i$ and $\forall \hat{i}$. This implies that the autonomous system ($r(t) = 0$) satisfies $V_i(T) \leq V_i(t_K)$ for all i.

Eqn. (11) implies that a δ_{K-1} can be chosen such that $V(t_{K-1}) \leq \rho(\delta_{K-1})$ implies that $V(t_K) \leq \frac{1}{2}\rho(\epsilon)$. This process can be repeated to define a sequence δ_0, δ_1, ... , δ_K so that V can be bounded at all switching times t_1 to t_K. The result is that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that when $||x(t_0)|| \leq \delta$ then $V(t_0) \leq \rho(\delta)$ and at time t_K, $V(t_K) \leq \frac{1}{2}\rho(\epsilon)$. The bound on $||x(t_0)||$ ultimately implies $V(T) \leq \frac{1}{2}\rho(\epsilon)$. This statement along with (26) shows that $V(t) \leq \rho(\epsilon), \forall t$ which implies that $||x(t)|| \leq \epsilon, \forall t$ whenever $||x(t_0)|| \leq \delta$. Since there always exists such a δ for each ϵ, the feedback interconnection is Lyapunov stable. □

This result gives stability conditions for the feedback of two dissipative systems with no input. However, we are often interested in analyzing systems that have additional systems interconnected as is the case in the study of large scale systems. Alternatively, we might be interested in the input-output properties of a single feedback interconnection. These problems may be addressed by considering the dissipative properties of the feedback system.

Corollary 1. *The feedback interconnection of two QSR dissipative switched systems is QSR dissipative with respect to the supply rate parametrized by any real-valued α,*

$$\omega_i(u,y) = y^T \hat{Q}_{i\hat{i}} y + 2y^T \hat{S}_{i\hat{i}} r + r^T \hat{R}_{i\hat{i}} r$$

where

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + \alpha R_{\hat{i}} & -S_i + \alpha S_{\hat{i}}^T \\ -S_i^T + \alpha S_{\hat{i}} & R_i + \alpha Q_{\hat{i}} \end{bmatrix},$$

$$\hat{S}_{i\hat{i}} = \begin{bmatrix} S_i & \alpha R_{\hat{i}} \\ -R_i & \alpha S_{\hat{i}} \end{bmatrix} \quad \text{and} \quad \hat{R}_{i\hat{i}} = \begin{bmatrix} R_i & 0 \\ 0 & \alpha R_{\hat{i}} \end{bmatrix}$$

for $i = \{1, ..., M_1\}$ *and* $\hat{i} \in \{1, ..., M_2\}$.

This result was derived as part of the proof of the previous theorem so it will not be repeated. Beyond simply assessing stability, this result allows us to assess the dissipative rate of the feedback interconnection. This assessment can be used as additional systems are added in feedback. As long as the dissipative rate of each loop is considered, stability of the overall interconnection can be given if the final connection satisfies the conditions of Theorem 4.

Some results will now be stated for the special case of passivity for switched systems.

Theorem 5. *A passive switched system with all subsystems ZSD is stable for zero input ($u(t) = 0$).*

The passivity property can be used when considering interconnections of systems. The following results shows stability of the feedback and parallel interconnections of two passive systems.

Theorem 6. *The feedback interconnection (Fig. 1) of two passive switched systems G_1 and G_2 forms a passive switched system.*

Theorem 7. *The parallel interconnection (Fig. 2) of two passive switched systems G_1 and G_2 forms a passive switched system.*

These results provide the expected generalizations of passivity theory from nonlinear systems to nonlinear switched systems. More details on these results with proofs can be found in [30]. An example of this is provided in Section 3.2.

3.2 Example

The following example demonstrates the analysis approach for the feedback of two QSR dissipative switched systems. Specifically, this example will illustrate how the analysis methods can be used to synthesize a stabilizing controller for a given switched plant. The example was chosen to be linear to be easy to follow, but the results in this paper apply to nonlinear switched systems.

The switched plant to be controlled has two subsystems given by linear dynamics,

$$\dot{x} = A_i x + B_i u$$
$$y = C_i x.$$

The dynamics are defined for subsystem 1 by

$$A_1 = \begin{bmatrix} -0.5 & 1 \\ 0 & -1.3 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$

and for subsystem 2 by

$$A_2 = \begin{bmatrix} -0.2 & 0.1 \\ 1.5 & -0.7 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1.5 & 1 \end{bmatrix}.$$

Subsystem 1 is stable and passive while subsystem 2 is unstable. The system is QSR dissipative with respect to $Q_{11} = -0.05$, $S_{11} = 0.5$, and $R_{11} = 0$ for subsystem 1 and $Q_{12} = 0.11$, $S_{12} = 0.5$, and $R_{12} = 0$ for subsystem 2. This can be shown by storage functions

$$V_1(x) = 0.466x_1^2 + 0.068x_1x_2 + 0.216x_2^2$$
$$V_1(x) = 0.998x_1^2 - 0.496x_1x_2 + 0.748x_2^2.$$

Cross supply rates can be found to show dissipativity for this switched system. One such set is given by

$$\omega_2^1(u, y, x, t) = u^T y - u^T u$$
$$\omega_1^2(u, y, x, t) = u^T y + 0.22y^T y + 0.25u^T u.$$

A controller can be designed to stabilize this plant. In general, the controller can have an arbitrary number of subsystems. For this example, there exists stabilizing controllers with only one mode. One set of conditions for such a stabilizing controller is that it has QSR parameters $Q_2 < 0$, $S_2 = 0.5$, and $R_2 < -0.11$. An example of a controller that satisfies this condition is given by

$$\dot{x} = -0.1x + u$$
$$y = x + 0.2u$$

that has parameters $Q_2 = -0.05$, $S_2 = 0.5$, and $R_2 = -0.15$. It can be verified that this controller satisfies the conditions of the stability theorem (25). When this controller is placed in feedback with the switched plant, stability is maintained for arbitrary switching.

4 Networking Passive Switched Systems

This section focuses on networking passive switched systems. Consider two switched systems (8) that are both output strictly passive. They are networked in the structure defined in Fig. 3. This interconnection forms a larger switched system from input r to output y. This section incrementally shows that this networked system is asymptotically stable. First it is shown that the output strictly passive property of the subsystems of G_1 and G_2 is preserved when the two systems are connected using the wave variable transformation. Next it is shown that the energy added due to switching is bounded. Putting these two together implies that, for the mapping $r \to y$, zero input implies that $y(t)$ converges to zero asymptotically. With the addition of the condition that the systems are zero-state observable, asymptotic stability is shown.

The first result to be shown is how the wave variable transform and the network interconnections preserve the output strictly passive nature of the active subsystems.

Each subsystem has storage functions $V_i^{(1)}$ for G_1 and $V_{\hat{i}}^{(2)}$ for G_2. Note that the loop signals can be stacked to generate the vectors $e = [e_1^T \ e_2^T]^T$, $r = [r_1^T \ r_2^T]^T$, and $y = [y_1^T \ y_2^T]^T$.

Consider the architecture in Fig. 3 with constant time delays and the wave variable transformation (17-18). Each system G_1 and G_2 being an OSP switched system implies $V_i^{(1)}$ and $V_{\hat{i}}^{(2)}$ exist for G_1 and G_2, respectively, that satisfy

$$\int_{t_1}^{t_2} e_1^T y_1 - \epsilon_i^{(1)} y_1^T y_1 dt \geq V_i^{(1)}(x_1(t_2)) - V_i^{(1)}(x_1(t_1)) \tag{27}$$

$$\int_{t_1}^{t_2} e_2^T y_2 - \epsilon_{\hat{i}}^{(2)} y_2^T y_2 dt \geq V_{\hat{i}}^{(2)}(x_2(t_2)) - V_{\hat{i}}^{(2)}(x_2(t_1)), \tag{28}$$

for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$, $\forall i, k$ and all $\epsilon_i^{(1)}, \epsilon_{\hat{i}}^{(2)} > 0$. Using the wave variable transformation (17-18) and the signal relations in the loop (13-14), the following derivation holds.

$$\begin{aligned}\int_{t_0}^{T} y_1^T y_{2d} dt &\geq \int_{t_0}^{T} y_2^T y_{1d} dt \\ \int_{t_0}^{T} y_1^T (r_1 - e_1) dt &\geq \int_{t_0}^{T} y_2^T (e_2 - r_2) dt \\ \int_{t_0}^{T} (y_1^T r_1 + y_2^T r_2) dt &\geq \int_{t_0}^{T} (y_1^T e_1 + y_2^T e_2) dt.\end{aligned}$$

Define a new energy storage function $V_{i\hat{i}}(x) = V_i^{(1)}(x_1) + V_{\hat{i}}^{(2)}(x_2)$. Applying (27) and (28) to the above inequality gives the following result.

$$\begin{aligned}\int_{t_0}^{T} y^T r dt \geq& \int_{t_0}^{T} y^T e dt \\ \geq& \epsilon_{i\hat{i}} \int_{t_0}^{T} y^T y dt + V_{i\hat{i}}(x(T)) - V_{i\hat{i}}(x(t_0)).\end{aligned}$$

where $\epsilon_{i\hat{i}} = min\{\epsilon_i^{(1)}, \epsilon_{\hat{i}}^{(2)}\}$. This shows that each pair of active subsystems $(i, \hat{i})$ of the mapping $r \to y$ is OSP with storage function $V_{i\hat{i}}$. As mentioned in Section 3.1, the feedback of two switched systems forms a new switched system that with as many subsystems as $M = M_1 M_2$. In order to simplify the notation, instead of tracking both subsystems $(i, \hat{i})$ of the interconnection, only a single index i is used to track the subsystems of the interconnection.

The next step to show is that the energy added to the system, due to switching, is bounded. This can be done by studying the energy added at switching instants t_{i_k} from initial time t_0 to arbitrary time T for a particular subsystem i. It can be assumed that K switches occur on this interval where K may depend on T. Denote the number of times that subsystem i is active on this interval by K_i.

$$\begin{aligned}&\sum_{i_k=1}^{K-1} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \\ &= \sum_{i=1}^{m} \sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] + \\ &\quad \sum_{i=1}^{m} \left[V_{i_1}(x(t_{i_1})) - V_{i_{K_i}}(x(t_{i_{K_i}}))\right] \\ &\leq \sum_{i=1}^{m} \sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] + \sum_{i=1}^{m} V_{i_1}(x(t_{i_1}))\end{aligned}$$

By the definition of passivity, there exist absolutely integrable functions ϕ_j^i to bound the energy accumulated by the j subsystem while the i subsystem is active. For a particular switching sequence, a set of piecewise continuous functions can be defined to indicate the function ϕ_j^i that is valid at each time for the j^{th} inactive subsystem,

$$\phi_j(t) = \begin{cases} \phi_j^i(t) & \forall i \neq j \\ 0 & i = j \end{cases} \tag{29}$$

Since each ϕ_j^i is absolutely integrable, then each ϕ_j is also absolutely integrable. The energy accumulated by each subsystem i can be bounded.

$$\sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] \leq \sum_{k=1}^{K_i} \int_{t_{i_k+1}}^{t_{i_{k+1}}} \phi_i(t) dt$$

This leads to a bound on the energy added due to switching.

$$\begin{aligned}&\sum_{i_k=1}^{K-1} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \\ &\quad \leq \sum_{i=1}^{m} \left[\int_{t_0}^{\infty} \phi_i(t) dt + V_{i_1}(x(t_{i_1}))\right] < \infty\end{aligned}$$

Each of the terms in this finite sum is a finite quantity so the energy is bounded. This upper bound is independent of the choice of T. Taking the limit as $T \to \infty$ shows that the energy is bounded for all time.

This result can be used to show that the squared output is bounded by the following bound.

$$\begin{aligned}\int_{t_0}^{T} y^T y dt =& \sum_{i_k=1}^{K} \int_{t_{i_k-1}}^{t_{i_k}} y^T y dt + \int_{t_K}^{T} y^T y dt \\ \leq& \frac{1}{\epsilon} \sum_{i_k=0}^{K-1} [V_{i_k+1}(x(t_{i_k+1})) - V_{i_k}(x(t_{i_k+1}))] + \\ & \frac{1}{\epsilon} \sum_{i=1}^{m} V_{i_1}(x(t_{i_1}))\end{aligned}$$

In the equation above, there are two summations. The second summation is the sum of the initially stored energy across all subsystems. Since initially stored energy is finite, this sum is finite. The first summation is the energy added due to switching. Again, this bound is independent of the time T chosen earlier. As we take the limit as $T \to \infty$, the bound still holds. This shows that the $\mathcal{L}_2$ norm of the output is finite and bounded above by the sum of the initially stored energy and the energy added due to the switching sequence.

The development in this section leads to the following result. It assumes that each system in feedback is an output strictly passive switched system. It employs the wave variable transformation to guarantee stability despite time delays. The proof is based on the details presented in this section up to this point so will be omitted.

Theorem 8. *Consider two systems, G_1 and G_2, each an OSP switched system with all subsystems zero-state detectable. These two systems are interconnected over a network with time delays using the modified wave variable transformation as in Fig. 3. Then this system is asymptotically stable.*

More details on this result can be found in [25]. The theorem shows how the proposed architecture (Fig. 3) can be used to guarantee stability for an interconnection of two output strictly passive switched systems.

This theorem is applicable as a synthesis tool. This approach assumes that a given plant is an output strictly passive (switched or non-switched) system. The controller must be designed to be an output strictly passive system with asymptotically zero-state detectable subsystems. It is allowed to be switched or non-switched as long as it meets the definition of an output strictly passive switched system given in this paper. The resulting interconnection is an asymptotically stable system despite time delays in the network.

5 Conclusions

This paper focused on the problem of maintaining stability when interconnecting switched systems. Background material was presented on dissipativity and passivity for nonlinear systems as well as for switched systems. The wave variable transformation was introduced as a method of compensating for delays in networked switched systems. While this approach is well established for nonlinear systems, it is a new approach for switched systems. The main results of this paper included providing conditions under which the feedback interconnection of two QSR dissipative switched systems is stable. The special case of passivity for switched systems was also covered. As expected, the property implies stability and is preserved when passive systems are combined in feedback or in parallel. The other main result of this paper provided stability conditions on networked passive switched systems where the wave variable transformation can be used to compensate for delays. Any two output strictly passive switched systems can be connected over a network and stability can be maintained.

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