

On Passivity of a Class of Discrete-Time Switched Nonlinear Systems

Yue Wang¹, Vijay Gupta², and Panos J. Antsaklis², *Fellow, IEEE*

¹Mechanical Engineering Department, Clemson University, Clemson, SC 29634 USA

²Electrical Engineering Department, University of Notre Dame, Notre Dame, IN 46556 USA

This paper analyzes the passivity and feedback passivity of discrete-time switched nonlinear systems with both passive and nonpassive modes that are affine in the control input. When a nonpassive mode is active, the increase in storage function is not necessarily bounded by the energy supplied to the switched system at every time step. Therefore, a switched system with at least one nonpassive mode is defined to be nonpassive in the classical passivity theory. In this paper, we propose a framework to analyze the passivity of such switched systems in a more general sense. We consider switched nonlinear systems which are affine in the control input and may consist of passive, feedback passive modes and modes which can not be rendered passive using feedback. In the proposed framework, we prove that a switched nonlinear system is locally feedback passive if and only if its zero dynamics are locally passive. A lower bound on the ratio of total activation time between (feedback) passive and non-feedback passive modes is obtained to guarantee passive zero dynamics. Finally, we prove that two important properties of classical passivity still hold for the proposed passivity definition. That is, 1) output feedback control can be used to stabilize the switched system, and 2) parallel and negative feedback interconnections of two such passive systems are also passive.

***Index Terms*—Generalized Passivity; Feedback Passivity; Switched Systems; Zero Dynamics; Discrete-Time Systems; Nonlinear Systems**

I. INTRODUCTION

PASSIVITY is an important property of dynamical systems. Passive systems exhibit many desirable properties, e.g., i) the free dynamics and zero dynamics of passive systems are Lyapunov stable, ii) the parallel and negative feedback interconnections of passive systems remain passive, and iii) a passive system can achieve stability using output feedback [1], [2]. The classical definition of passivity [3]–[7] requires that the increase in a suitably defined storage function is bounded by the energy supplied to the system at every time step.

However, in many systems, this condition may be too restrictive. In [8], we considered a feedback passive system controlled across a communication network. In case of packet dropouts, there will be time steps at which the system evolves in open loop, and hence, the traditional passivity definition may be violated at those time steps. Thus, no matter how infrequently the packet drops occur, such a system will be termed nonpassive. In this circumstance, an integral version of the traditional definition of passivity may be more appropriate since this notion can allow the increase in storage function to be greater than the supplied energy if such instances happen with sufficiently low frequency. It was shown in [8] that the

switched system is locally feedback passive if and only if its zero dynamics are locally passive under this notion of passivity.

However, the analysis in [8] relies on the system switching between two modes, with one mode being the open loop version of the other one. In this paper, we develop this concept further and consider switched systems with an arbitrary number of passive, feedback passive modes and modes which can not be rendered passive using feedback (non-feedback passive modes). For such systems, we provide a definition of passivity that reduces to the integral version of the traditional definition of passivity when the system has only one mode. However, for discrete-time switched nonlinear systems, this definition allows for modes that are not passive, provided that such modes are active infrequently enough.

First we summarize some related work on classical passivity and stability theory for nonlinear systems. The Willems-Hill-Moylan conditions provide necessary and sufficient conditions for continuous-time nonlinear systems to be passive [3], [5], [9]. Based on these conditions, the feedback passivity equivalence (i.e. conditions under which a nonpassive system can be rendered passive using feedback) for a continuous-time system is solved in [6]. The Willems-Hill-Moylan conditions specialize to the Kalman-Yacubovitch-Popov (KYP) conditions for continuous-time linear systems. In [7], [10], [11], the conditions have been extended to discrete-time nonlinear systems. The feedback losslessness equivalence and feedback passivity equivalence for discrete-time nonlinear systems has been proposed in [12] and [13], respectively.

There has also been considerable attention on the passivity/dissipativity of switched systems, however, most such works consider systems in which every mode is passive/dissipative. The works related to the present paper are in two primary categories. One line of work uses a common storage function for all modes to characterize the passivity/dissipativity of switched systems [14]–[17]. To get less conservative results, the use of multiple storage functions for different modes of a switched system has been proposed via piecewise quadratic storage functions [18], multiple storage functions and a common supply rate [19], and multiple storage functions and multiple supply rates [20]–[22]. For more general switched systems in which some modes can be nonpassive, only results that guarantee stability are available [23], [24]. Unlike these works, we are interested in system passivity in the presence of nonpassive modes.

The passivity definition that we develop can be considered to be analogous to the generalized asymptotic stability of

discrete-time nonlinear time-varying systems where the Lyapunov function is non-increasing not at every time step, but only on certain unbounded sets [25]. However, unlike [25], the passivity analysis is complicated by the fact that passivity is an input-output property and both the inputs and the outputs of the system are time varying. Due to this difficulty, we analyze the feedback passivity properties of a switched system based on its zero dynamics ([1], [6], [26], and especially [13]). Zero dynamics are the internal dynamics of a system that are consistent with constraining the system output to zero. Under the proposed definition, we prove that a switched nonlinear system is locally feedback passive if and only if its zero dynamics are locally passive. A lower bound on the ratio of total activation time between (feedback) passive and non-feedback passive modes is then derived to guarantee passive zero dynamics.

The paper is organized as follows. Section II reviews the related background on classical passivity theory. In Section III, we introduce the switched nonlinear systems considered in this paper. In Section IV, we propose a generalized passivity definition and develop necessary and sufficient conditions for the passivity of the switched nonlinear systems. In Section V, we apply this definition to switched systems with passive, feedback passive, and non-feedback passive modes. The switched system is proved to be locally feedback passive if and only if its zero dynamics are passive. In Section VI, a lower bound on the total activation time of the (feedback) passive modes versus that of the non-feedback passive modes is derived to guarantee passive zero dynamics. In Section VII, we show that the classical properties of output feedback stabilizability and passivity preservation under parallel and feedback interconnections continue to hold under the new definitions. Two examples are provided in Section VIII. We conclude the paper in Section IX.

II. PRELIMINARY RESULTS

Table I summarizes the main notations used in this paper.

| | |
|-----------------|--|
| \mathbb{R}_+ | set of positive real numbers |
| \mathbb{Z}_+ | set of positive integers |
| \mathbb{R}^n | n -dimensional real vector |
| $\mathbf{0}$ | vector zero |
| 0 | scalar zero |
| \mathcal{C}^n | n _{th} order differentiable |

TABLE I
LIST OF NOTATIONS USED IN THE PAPER.

Consider a system of the form

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \mathbf{u}(k)), \end{cases} \quad (1)$$

where $\mathbf{x}(k) \in \mathbf{X} \subset \mathbb{R}^n$, $\mathbf{u}(k) \in \mathbf{U} \subset \mathbb{R}^m$ and $\mathbf{y}(k) \in \mathbf{Y} \subset \mathbb{R}^m$ are the state, input, and output variables, respectively. \mathbf{X} , \mathbf{U} and \mathbf{Y} are the state, input, and output spaces, respectively. The time index $k \in \{0\} \cup \mathbb{Z}_+$ and f, h are \mathcal{C}^∞ functions. All considerations are restricted to an open set $\mathbf{X} \times \mathbf{U}$ containing

the equilibrium point $(\mathbf{x}^*, \mathbf{u}^*)$ with $\mathbf{x}^* = f(\mathbf{x}^*, \mathbf{u}^*)$. For simplicity, assume that $(\mathbf{x}^*, \mathbf{u}^*) = (\mathbf{0}, \mathbf{0})$ and $h(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Definition 2.1: [12] A system of the form (1) is said to be *locally passive* with respect to the supply rate $\mathbf{u}^\top(k)\mathbf{y}(k)$ if there exists a positive definite function $V(\mathbf{x})$, called the *storage function*, such that the following inequality holds

$$\begin{aligned} V(f(\mathbf{x}(k), \mathbf{u}(k))) - V(\mathbf{x}(k)) &\leq \mathbf{u}^\top(k)\mathbf{y}(k), \\ \forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, k \in \{0\} \cup \mathbb{Z}_+ \end{aligned} \quad (2)$$

with $\mathbf{X} \times \mathbf{U}$ being a neighborhood of the equilibrium point $(\mathbf{0}, \mathbf{0})$.

For our purpose, a condition equivalent to (2) will be more useful.

Definition 2.2: [13, Definition 3 and Theorem 5] A system of the form (1) is said to be locally passive with respect to the supply rate $\mathbf{u}^\top(k)\mathbf{y}(k)$ if there exists a positive definite storage function $V(\mathbf{x})$ such that the following equation holds

$$\begin{aligned} V(f(\mathbf{x}(k), \mathbf{u}(k))) - V(\mathbf{x}(k)) &= \mathbf{u}^\top(k)\mathbf{y}(k) \\ &- (l(\mathbf{x}(k)) + e(\mathbf{x}(k))\mathbf{u}(k))^\top (l(\mathbf{x}(k)) + e(\mathbf{x}(k))\mathbf{u}(k)) \\ &- m^\top(\mathbf{x}(k))m(\mathbf{x}(k)), \quad \forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, k, \end{aligned} \quad (3)$$

where $\mathbf{X} \times \mathbf{U}$ is a neighborhood of the equilibrium point $(\mathbf{0}, \mathbf{0})$ and $l(\mathbf{x}(k)), e(\mathbf{x}(k)), m(\mathbf{x}(k))$ are real functions that are equal to zero if and only if $\mathbf{x}(k) = \mathbf{0}$.

Remark 2.1: Definitions 2.1 and 2.2 are equivalent because the term $-(l + e\mathbf{u})^\top(l + e\mathbf{u}) - m^\top m$ is nonpositive. On the other hand, if there does not exist any storage function such that equation (3) holds, the system is nonpassive. Hence, we may associate a storage function with a nonpassive system and model its increase by the equation

$$\begin{aligned} V(f(\mathbf{x}(k), \mathbf{u}(k))) - V(\mathbf{x}(k)) &= \mathbf{u}^\top(k)\mathbf{y}(k) \\ &+ (l(\mathbf{x}(k)) + e(\mathbf{x}(k))\mathbf{u}(k))^\top (l(\mathbf{x}(k)) + e(\mathbf{x}(k))\mathbf{u}(k)) \\ &- m^\top(\mathbf{x}(k))m(\mathbf{x}(k)), \quad \forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, k, \end{aligned} \quad (4)$$

where the term $(l + e\mathbf{u})^\top(l + e\mathbf{u}) - m^\top m$ can be either positive, negative, or zero.

Let $\mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))$ denote a nonlinear feedback control law. If $\mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))$ is locally regular, i.e., $\frac{\partial \mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))}{\partial \mathbf{v}(k)} \neq \mathbf{0}$ for all $\mathbf{x}(k), \mathbf{v}(k)$, the system

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))) \\ \quad \triangleq \bar{f}(\mathbf{x}(k), \mathbf{v}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))) = \mathbf{v}(k) \end{cases} \quad (5)$$

is referred to as the feedback transformed system of system (1). This transformed system can be used to study the passivity of system (1) via the following results.

Proposition 2.1: [26, Theorem 7.3] A system of the form (1) is said to be *locally feedback passive* if there exist a positive definite storage function $V(\mathbf{x})$ and a regular state feedback control law $\mathbf{u}(\mathbf{x}(k), \mathbf{v}(k))$ with $\mathbf{v}(k)$ as the new input such that $\forall \mathbf{x}, \mathbf{v}, k$,

$$V(\bar{f}(\mathbf{x}(k), \mathbf{v}(k))) - V(\mathbf{x}(k)) \leq \mathbf{v}^\top(k)\mathbf{y}(k).$$

The zero dynamics ([1], [12]) of system (1) are given by constraining the system output to zero using control

$\mathbf{u}^*(\mathbf{x}(k), \mathbf{v}^*(k) = \mathbf{0})$, i.e., by the equations

$$\begin{cases} \mathbf{x}(k+1) = \bar{f}(\mathbf{x}(k), \mathbf{0}) \\ \mathbf{y}(k) = \mathbf{0} \end{cases}. \quad (6)$$

Definition 2.3: [10], [13], [26] The zero dynamics (6) of the system (1) are passive if $\forall \mathbf{x} \in \mathbf{X}, k$,

$$V(\bar{f}(\mathbf{x}(k), \mathbf{0})) - V(\mathbf{x}(k)) \leq 0. \quad (7)$$

The utility of zero dynamics follows from the following results.

Proposition 2.2: ([6, Remark 2.5], [26, Remark 7.6]) A passive zero dynamics is equivalent to a Lyapunov stable system.

Theorem 2.1: [26, Theorem 7.1] If system (1) is locally passive and has relative degree zero at the neighborhood of $\mathbf{x}^* = \mathbf{0}$, then its zero dynamics (6) are also locally passive.

For the particular case of a discrete-time nonlinear system that is affine in the control input and has local relative degree zero, i.e., for a system of the form

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k)) + g(\mathbf{x}(k))\mathbf{u}(k) \\ \mathbf{y}(k) = h(\mathbf{x}(k)) + J(\mathbf{x}(k))\mathbf{u}(k), \end{cases} \quad (8)$$

the converse result also holds.

Theorem 2.2: [26, Theorem 7.3] Suppose $h(\mathbf{0}) = \mathbf{0}$, and there exists a \mathcal{C}^2 storage function, which is positive definite and $V(f(\mathbf{x}(k)) + g(\mathbf{x}(k))\mathbf{u}(k))$ is quadratic in $\mathbf{u}(k)$, $\forall f, g$. System (8) is locally feedback passive if and only if its zero dynamics are locally passive in a neighborhood of $\mathbf{x}^* = \mathbf{0}$.

III. PROBLEM FORMULATION

In this paper, we focus on a discrete-time switched nonlinear system, which is affine in the control input. At time k , let the system evolve as

$$\begin{cases} \mathbf{x}(k+1) = f_{\sigma(k)}(\mathbf{x}(k)) + g_{\sigma(k)}(\mathbf{x}(k))\mathbf{u}(k) \\ \mathbf{y}(k) = h_{\sigma(k)}(\mathbf{x}(k)) + J_{\sigma(k)}(\mathbf{x}(k))\mathbf{u}(k), \end{cases} \quad (9)$$

where $\sigma(k) \in \{1, 2, \dots, N\}$ denotes the mode active at time k , $f_{\sigma(k)}$, $g_{\sigma(k)}$, $h_{\sigma(k)}$ and $J_{\sigma(k)}$ are \mathcal{C}^∞ functions. All considerations are restricted to an open set $\mathbf{X} \times \mathbf{U}$ that is a neighborhood of the origin $(\mathbf{0}, \mathbf{0})$. We make the following assumption in the sequel. For every mode $\sigma(k) \in \{1, 2, \dots, N\}$, the dynamics of (9) are such that

Assumption 1: The point $(\mathbf{x}^*, \mathbf{u}^*) = (\mathbf{0}, \mathbf{0})$ is an equilibrium point.

Assumption 2: $h_{\sigma(k)}(\mathbf{0}) = \mathbf{0}$.

Assumption 3: The dynamics have local relative degree zero.

Assumption 4: $J_{\sigma(k)}$ is locally invertible.

Assumption 5: Each mode is affine in control and the control $\mathbf{u}(k)$ is locally regular.

The dynamics for the various modes may or may not be passive. Let \mathcal{S}_1 , \mathcal{S}_2 denote the set of nonpassive and passive modes, respectively. Furthermore, let \mathcal{S}_1^* denote the set of feedback passive modes. Therefore, the set of non-feedback passive modes is denoted by $\mathcal{S}_1 \setminus \mathcal{S}_1^*$. We assume that the system starts at time $k = 0$ in one of the passive or feedback passive modes. According to the classical definition of passivity, this system is nonpassive because the increase in storage function is not necessarily bounded by the energy

supplied to it when a nonpassive mode is activated. In this paper, we generalize the classical passivity definition to such systems and investigate necessary and sufficiency conditions for passivity and feedback passivity.

Remark 3.1: The assumption of local relative degree zero and locally invertible $J_{\sigma(k)}$ is reasonable because as shown in [12] and [26], respectively, a discrete-time nonlinear system can be rendered lossless/passive if and only if it has relative degree zero and lossless/passive zero dynamics. Therefore, in this paper, we will not study passivity of discrete-time systems with outputs that are independent of inputs. Note that recent work in [27] relaxes this assumption by using the coupled differential/difference representation (DDR) of the system. This, however, requires the existence of a control \mathbf{u} such that $f(\mathbf{x}, \mathbf{u})$ is invertible.

If we choose the feedback control law

$$\mathbf{u}(k) = -J_{\sigma(k)}^{-1}h_{\sigma(k)} + J_{\sigma(k)}^{-1}\mathbf{v}(k) \quad (10)$$

where $\mathbf{v}(k)$ is an auxiliary input to be defined later, the transformed system dynamics become

$$\begin{cases} \mathbf{x}(k+1) = f_{\sigma(k)}^*(\mathbf{x}(k)) + g_{\sigma(k)}^*(\mathbf{x}(k))\mathbf{v}(k) \\ \mathbf{y}(k) = \mathbf{v}(k), \end{cases} \quad (11)$$

where $f_{\sigma(k)}^* \triangleq f_{\sigma(k)} - g_{\sigma(k)}J_{\sigma(k)}^{-1}h_{\sigma(k)}$ and $g_{\sigma(k)}^* \triangleq g_{\sigma(k)}J_{\sigma(k)}^{-1}$. The zero dynamics of (9) are

$$\begin{cases} \mathbf{x}(k+1) = f_{\sigma(k)}^*(\mathbf{x}(k)) \\ \mathbf{y}(k) = \mathbf{0}. \end{cases} \quad (12)$$

Denote by Σ the switching sequence $\{\sigma(0), \sigma(1), \dots\}$. In this paper, we are interested in the following problems.

- i. Since the dynamics of some of the modes are not passive, by classical definition (e.g., [17]), system (9) is not passive. However, if the non passive modes are active infrequently enough, the system must intuitively still be passive. In Section IV, we formalize this intuition by extending the traditional passivity definition. We also provide necessary and sufficient conditions for this definition to hold.
- ii. In Sections V and VI, we use these conditions to prove that a nonpassive system can be rendered locally passive using feedback control laws if and only if its zero dynamics are locally passive.
- iii. In Section VII, we prove that the proposed definition is consistent with the traditional definition of passivity in the sense that useful properties such as stability and compositionality are preserved.

IV. GENERALIZED PASSIVITY DEFINITION

In this section, we define a notion of passivity for system (9) that generalizes the existing definition of passivity for switched systems. We then derive necessary and sufficient conditions for the system to be passive under this definition.

Definition 4.1: The switched system (9) is said to be *locally passive* for a given switching sequence Σ if there exists a positive definite storage function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for the state evolution under the switching sequence Σ , the

following passivity inequality holds,

$$V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) \leq \sum_{k=0}^T \mathbf{u}^T(k) \mathbf{y}(k),$$

$$\forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, T \in \{0\} \cup \mathbb{Z}_+, \quad (13)$$

with $\mathbf{X} \times \mathbf{U}$ being a neighborhood of the equilibrium point $(\mathbf{0}, \mathbf{0})$.

Remark 4.1: There are two changes in this definition as compared to the classical definition given by the inequality (2). The first change is the consideration of the integral version of the change in storage function. This allows system (9) to have nonpassive modes as long as the increase in storage function over a finite horizon T is bounded by the total energy supplied to the system in the period $[0, T]$. The second change is to define the passivity for a particular switching sequence. Clearly, if a switched system is passive for any arbitrary switching sequence, every mode of the system must be passive. To allow for nonpassive modes, we must restrict the switching sequence. As we shall show in Section VI, Definition 4.1 allows us to prove the passivity of a switched system over wide sets of switching sequences.

We now extend the necessary and sufficient conditions for discrete-time nonlinear systems to be passive ([7], [10], [11]) to discrete-time switched nonlinear systems (9) under Definition 4.1. Before that, we summarize the assumptions for the storage function V which will be used in the following results.

Assumption 6: V is positive definite.

Assumption 7: V is \mathcal{C}^2 and quadratic with respect to control.

Theorem 4.1: Suppose there exists a storage function V , which is positive definite and \mathcal{C}^2 . Furthermore, let $V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k))$ be quadratic in \mathbf{u} . Then the switched system (9) is passive for a given switching sequence $\Sigma = \{\sigma(0), \sigma(1), \dots\}$ with storage function V if and only if there exist real functions $l_{\sigma(k)}(\mathbf{x}(k))$, $m_{\sigma(k)}(\mathbf{x}(k))$, and $e_{\sigma(k)}(\mathbf{x}(k))$ such that $\forall T \in \{0\} \cup \mathbb{Z}_+$, $k = 0, 1, \dots, T$,

If $\sigma(k) \in \mathcal{S}_1$,

$$V(f_{\sigma(k)}) - V(\mathbf{x}(k)) = l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)} \quad (14a)$$

$$\left. \frac{\partial V(z)}{\partial z} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} = h_{\sigma(k)}^T + 2l_{\sigma(k)}^T e_{\sigma(k)} \quad (14b)$$

$$g_{\sigma(k)}^T \left. \frac{\partial^2 V(z)}{\partial z^2} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} = J_{\sigma(k)}^T + J_{\sigma(k)} + 2e_{\sigma(k)}^T e_{\sigma(k)} \quad (14c)$$

If $\sigma(k) \in \mathcal{S}_2$,

$$V(f_{\sigma(k)}) - V(\mathbf{x}(k)) = -l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)} \quad (15a)$$

$$\left. \frac{\partial V(z)}{\partial z} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} = h_{\sigma(k)}^T - 2l_{\sigma(k)}^T e_{\sigma(k)} \quad (15b)$$

$$g_{\sigma(k)}^T \left. \frac{\partial^2 V(z)}{\partial z^2} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} = J_{\sigma(k)}^T + J_{\sigma(k)} - 2e_{\sigma(k)}^T e_{\sigma(k)} \quad (15c)$$

$$\sum_{\substack{k:\sigma(k) \in \mathcal{S}_1 \\ k \leq T}} (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))^T (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))$$

$$- \sum_{\substack{k:\sigma(k) \in \mathcal{S}_2 \\ k \leq T}} (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))^T (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))$$

$$\leq \sum_{k=0}^T m_{\sigma(k)}^T m_{\sigma(k)} \quad (16)$$

Proof. The switched system (9) is composed of both passive and nonpassive modes. If the system is in a passive mode, i.e., $\sigma(k) \in \mathcal{S}_2$, according to equation (3), there exists a storage function such that $\forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, k$,

$$V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k)) - V(\mathbf{x}(k)) = \mathbf{u}^T(k) \mathbf{y}(k) - (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))^T \cdot (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k)) - m_{\sigma(k)}^T m_{\sigma(k)}. \quad (17)$$

If the system is in a nonpassive mode, i.e., $\sigma(k) \in \mathcal{S}_1$, according to equation (4), for any storage function V we have $\forall \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}, k$,

$$V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k)) - V(\mathbf{x}(k)) = \mathbf{u}^T(k) \mathbf{y}(k) + (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k))^T \cdot (l_{\sigma(k)} + e_{\sigma(k)} \mathbf{u}(k)) - m_{\sigma(k)}^T m_{\sigma(k)}. \quad (18)$$

(Necessity) The proof has three parts.

- i) We first show that equation (18) yields conditions (14a)-(14c) for nonpassive modes $\sigma(k) \in \mathcal{S}_1$.
- ii) Similarly, we show that equation (17) yields conditions (15a)-(15c) for passive modes $\sigma(k) \in \mathcal{S}_2$.
- iii) Based on the results from part i) and ii), we prove that the inequality (16) must hold if the switched system (9) is passive.

i) Because $V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k))$ is quadratic in \mathbf{u} , the Taylor series expansion of $V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k))$ at the equilibrium $\mathbf{u}(k) = \mathbf{0}$ is given by

$$V(f_{\sigma(k)} + g_{\sigma(k)} \mathbf{u}(k)) = V(f_{\sigma(k)}) + \left. \frac{\partial V(z)}{\partial z} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} \mathbf{u}(k) + \frac{1}{2} \mathbf{u}^T(k) g_{\sigma(k)}^T \left. \frac{\partial^2 V(z)}{\partial z^2} \right|_{z=f_{\sigma(k)}} g_{\sigma(k)} \mathbf{u}(k).$$

Therefore, it follows that

$$\begin{aligned}
& V(f_{\sigma(k)} + g_{\sigma(k)}\mathbf{u}(k)) - V(\mathbf{x}(k)) \\
&= V(f_{\sigma(k)}) - V(\mathbf{x}(k)) \\
&\quad + \frac{\partial V(z)}{\partial z} \Big|_{z=f_{\sigma(k)}} g_{\sigma(k)}\mathbf{u}(k) \\
&\quad + \frac{1}{2}\mathbf{u}^T(k)g_{\sigma(k)}^T \frac{\partial^2 V(z)}{\partial z^2} \Big|_{z=f_{\sigma(k)}} g_{\sigma(k)}\mathbf{u}(k). \quad (19)
\end{aligned}$$

Rearranging equation (18), we obtain

$$\begin{aligned}
& V(f_{\sigma(k)} + g_{\sigma(k)}\mathbf{u}(k)) - V(\mathbf{x}(k)) \\
&= l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)} + \mathbf{y}^T(k)\mathbf{u}(k) \\
&\quad + 2l_{\sigma(k)}^T e_{\sigma(k)}\mathbf{u}(k) + \mathbf{u}^T(k)e_{\sigma(k)}^T e_{\sigma(k)}\mathbf{u}(k) \\
&= l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)} + h_{\sigma(k)}^T \mathbf{u}(k) \\
&\quad + 2l_{\sigma(k)}^T e_{\sigma(k)}\mathbf{u}(k) + \mathbf{u}^T(k)(e_{\sigma(k)}^T e_{\sigma(k)} \\
&\quad + J_{\sigma(k)})\mathbf{u}(k). \quad (20)
\end{aligned}$$

Comparing equations (19) and (20), we obtain equations (14a)-(14c).

ii) A similar argument with equations (17) and (19) yields equations (15a)-(15c).

iii) Sum equations (17) and (18) from $k = 0$ to T . Because the switched system (9) is passive, i.e., the inequality (13) holds, the inequality (16) must be satisfied.

(Sufficiency) The sufficiency proof also has three parts.

i) First, substitution of equations (14a)-(14c) into equation (19) yields equation (18).

ii) A similar argument yields equation (17) from equations (15a)-(15c).

iii) Finally, to show that the switched system (9) is passive according to Definition 4.1, we proceed as follows.

Sum up equation (18) for all the time steps when $\sigma(k) \in \mathcal{S}_1$ and equation (17) for all the time steps when $\sigma(k) \in \mathcal{S}_2$ up to T . This yields

$$\begin{aligned}
V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) &= \sum_{k=0}^T \mathbf{u}(k)\mathbf{y}(k) \\
&+ \sum_{\substack{k:\sigma(k) \in \mathcal{S}_1 \\ k \leq T}} \left((l_{\sigma(k)} + e_{\sigma(k)}\mathbf{u}(k))^T (l_{\sigma(k)} + e_{\sigma(k)}\mathbf{u}(k)) \right. \\
&\quad \left. - m_{\sigma(k)}^T m_{\sigma(k)} \right) - \sum_{\substack{k:\sigma(k) \in \mathcal{S}_2 \\ k \leq T}} \left((l_{\sigma(k)} + e_{\sigma(k)}\mathbf{u}(k))^T \right. \\
&\quad \left. (l_{\sigma(k)} + e_{\sigma(k)}\mathbf{u}(k)) + m_{\sigma(k)}^T m_{\sigma(k)} \right). \quad (21)
\end{aligned}$$

If inequality (16) holds, (21) implies that $V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) \leq \sum_{k=0}^T \mathbf{u}(k)\mathbf{y}(k)$, i.e., the switched system (9) is passive under Definition 4.1 with the switching sequence Σ . ■

V. GENERALIZED FEEDBACK PASSIVITY

In this section, we extend the notion of passivity in the sense of Definition 4.1 to generalized feedback passivity. We relate the feedback passivity of system (9) with its zero dynamics (12). Similar to Definition 4.1, we define the passivity for system zero dynamics (12).

Definition 5.1: The zero dynamics (12) of the switched system (9) are said to be *locally passive* for a given switching sequence Σ if there exists a positive definite storage function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+$, such that for the state evolution under the switching sequence Σ , the following passivity inequality holds,

$$\begin{aligned}
V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) &\leq 0, \\
\forall T \in \{0\} \cup \mathbb{Z}_+, \mathbf{x} &\in \mathbf{X} \quad (22)
\end{aligned}$$

with \mathbf{X} being a neighborhood of the equilibrium point $\mathbf{x}^* = \mathbf{0}$.

Consider a new control input \mathbf{w} for the transformed dynamics (11) of the switched system (9)

$$\mathbf{y}(k) = \mathbf{v}(k) = \bar{h}_{\sigma(k)} + \bar{J}_{\sigma(k)}\mathbf{w}(k).$$

We make the following assumption for the new control input

Assumption 8: $\bar{J}_{\sigma(k)}$ is symmetric.

Furthermore, we set

$$\begin{aligned}
\bar{J}_{\sigma(k)} &= \left(\frac{1}{2}g_{\sigma(k)}^*{}^T \frac{\partial^2 V}{\partial z^2} \Big|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^* \right)^{-1} \\
\bar{h}_{\sigma(k)} &= -\bar{J}_{\sigma(k)} \left(\frac{\partial V}{\partial z} \Big|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^* \right)^T. \quad (23)
\end{aligned}$$

The new system dynamics are given by

$$\begin{cases} \mathbf{x}(k+1) \\ = f_{\sigma(k)}^* + g_{\sigma(k)}^* \bar{h}_{\sigma(k)} + g_{\sigma(k)}^* \bar{J}_{\sigma(k)}\mathbf{w}(k) \\ \mathbf{y}(k) = \bar{h}_{\sigma(k)} + \bar{J}_{\sigma(k)}\mathbf{w}(k). \end{cases} \quad (24)$$

Along the lines of the definition of feedback passivity in [26], we can use Definition 4.1 to define feedback passivity of system (9) as follows.

Definition 5.2: The switched system (9) is said to be *locally feedback passive* for a given switching sequence Σ if there exists a positive definite storage function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for the state evolution under the switching sequence Σ , the following passivity inequality holds, $\forall \mathbf{x} \in \mathbf{X}, \mathbf{w}, T \in \{0\} \cup \mathbb{Z}_+$,

$$V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) \leq \sum_{k=0}^T \mathbf{w}^T(k)\mathbf{y}(k). \quad (25)$$

Lemma 5.1: If the switched system (9) is locally feedback passive for a given switching sequence Σ , then its zero dynamics (12) are also locally passive for the same switching sequence Σ .

Proof. Because system (9) is locally feedback passive for the switching sequence Σ , the inequality (25) holds. The zero dynamics enforces $\mathbf{y}(k) = \mathbf{0}$. Hence, the inequality (25) is converted to the inequality (22). That is, the zero dynamics (12) are also locally passive for the same switching sequence Σ . ■

Theorems 2.1 and 2.2 along with this result yields the following observation.

Lemma 5.2: The passive and feedback passive modes of the switched system (9) correspond to the passive modes of the zero dynamics (12). The non-feedback passive modes of the switched system (9) correspond to the nonpassive modes of

the zero dynamics (12).

Next, based on Theorem 4.1, we obtain the necessary and sufficient conditions for the switched system (9) to be feedback passive for a given switching sequence. We relate the passivity of system zero dynamics (12) to the feedback passivity of the switched nonlinear system (9) and provide a method to check the passivity of a given switched nonlinear system. This yields results analogous to those derived for discrete-time nonlinear (non-switched) systems in [12, Theorem 3.11], [13, Theorem 11], and in [26, Theorem 7.2].

Theorem 5.1: Suppose there exists a \mathcal{C}^2 storage function V , which is positive definite and $V(f_{\sigma(k)} + g_{\sigma(k)}\mathbf{u}(k))$ is quadratic in $\mathbf{u}(k)$ for every possible value of $\sigma(k)$. Then, the switched nonlinear system (9) is locally feedback passive for a given switching sequence Σ with the storage function V if and only if its zero dynamics (12) are locally passive for the same switching sequence.

Proof. The necessity follows directly from Lemma 5.1. We next prove the sufficiency based on Theorem 4.1. More specifically, we prove that if the zero dynamics (12) are locally passive for a given switching sequence $\Sigma = \{\sigma(0), \sigma(1), \dots\}$, conditions (14a)-(16) hold for the transformed system dynamics (24) with proper choices of the real functions $l_{\sigma(k)}$, $e_{\sigma(k)}$ and $m_{\sigma(k)}$. Therefore, according to Theorem 4.1, the switched system (9) is locally feedback passive in the sense of Definition 5.2 for the switching sequence Σ .

We begin by relating the transformed system dynamics (24) with the zero dynamics (12) in terms of the storage function. Because $V(f_{\sigma(k)} + g_{\sigma(k)}\mathbf{u}(k))$ is quadratic in \mathbf{u} , the Taylor series expansion for $V(f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)})$ can be expressed as follows:

$$\begin{aligned} & V(f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}) \\ &= V(f_{\sigma(k)}^*) + \left. \frac{\partial V}{\partial z} \right|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^*\bar{h}_{\sigma(k)} \\ & \quad + \frac{1}{2} \bar{h}_{\sigma(k)}^T (g_{\sigma(k)}^*)^T \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^*\bar{h}_{\sigma(k)}. \end{aligned} \quad (26)$$

Using equations (23) and (26), we have

$$\begin{aligned} & V(f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}) = V(f_{\sigma(k)}^*) \\ & \quad - \bar{h}_{\sigma(k)}^T \left(\bar{J}_{\sigma(k)}^{-1} \right)^T \bar{h}_{\sigma(k)} + \bar{h}_{\sigma(k)}^T (\bar{J}_{\sigma(k)})^{-1} \bar{h}_{\sigma(k)} \\ & \quad = V(f_{\sigma(k)}^*). \end{aligned} \quad (27)$$

We now consider two cases separately.

i) First consider the case when $\sigma(k) \in \mathcal{S}_1^* \cup \mathcal{S}_2$. According to Lemma 5.2, in this case, the corresponding zero dynamics are passive. Based on equations (17) and (27), it follows that

$$\begin{aligned} & V(f_{\sigma(k)}^*) - V(\mathbf{x}(k)) \\ &= V(f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}) - V(\mathbf{x}(k)) \\ &= - (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)})^T (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}) \\ & \quad - m_{\sigma(k)}^T m_{\sigma(k)}. \end{aligned} \quad (28)$$

Equation (28) gives the passivity condition (15a) with $l_{\sigma(k)} = l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}$ and $m_{\sigma(k)} = m_{\sigma(k)}$ for the transformed system dynamics (24).

Next, subtract V from equation (26) and substitute equation (28) into the resulting equation to obtain

$$\begin{aligned} & V(f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}) - V(\mathbf{x}(k)) \\ &= - (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)})^T (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}) \\ & \quad - m_{\sigma(k)}^T m_{\sigma(k)} + \left. \frac{\partial V}{\partial z} \right|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^*\bar{h}_{\sigma(k)} \\ & \quad + \frac{1}{2} \bar{h}_{\sigma(k)}^T (g_{\sigma(k)}^*)^T \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f_{\sigma(k)}^*} g_{\sigma(k)}^*\bar{h}_{\sigma(k)}. \end{aligned} \quad (29)$$

Differentiating both sides of equation (29) with respect to $\bar{h}_{\sigma(k)}$, right multiplying the result by $\bar{J}_{\sigma(k)}$, and substituting equation (23), we obtain

$$\begin{aligned} & \left. \frac{\partial V}{\partial z} \right|_{z=f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}} g_{\sigma(k)}^* \bar{J}_{\sigma(k)} \\ &= \bar{h}_{\sigma(k)}^T - 2 (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)})^T e_{\sigma(k)} \bar{J}_{\sigma(k)}. \end{aligned} \quad (30)$$

Therefore, equation (30) yields the passivity condition (15b) with $l_{\sigma(k)} = l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}$ and $e_{\sigma(k)} = e_{\sigma(k)}\bar{J}_{\sigma(k)}$ for the transformed system dynamics (24).

Now take the second-order derivatives of equation (29) with respect to $\bar{h}_{\sigma(k)}$. Left multiply the results by $\bar{J}_{\sigma(k)}^T$ and right multiply by $\bar{J}_{\sigma(k)}$, and use equation (23) to obtain

$$\begin{aligned} & \left[g_{\sigma(k)}^* \bar{J}_{\sigma(k)} \right]^T \left. \frac{\partial V}{\partial z} \right|_{z=f_{\sigma(k)}^* + g_{\sigma(k)}^*\bar{h}_{\sigma(k)}} g_{\sigma(k)}^* \bar{J}_{\sigma(k)} \\ &= \bar{J}_{\sigma(k)}^T + \bar{J}_{\sigma(k)} + 2 \bar{J}_{\sigma(k)}^T e_{\sigma(k)}^T e_{\sigma(k)} \bar{J}_{\sigma(k)} \end{aligned} \quad (31)$$

Therefore, equation (31) provides the passivity condition (15c) with $e_{\sigma(k)} = e_{\sigma(k)}\bar{J}_{\sigma(k)}$ for the transformed system dynamics (24).

ii) Now consider the case when $\sigma(k) \in \mathcal{S}_1 \setminus \mathcal{S}_1^*$. For this case, we have

$$\begin{aligned} & V(f_{\sigma(k)}^*) - V(\mathbf{x}(k)) \\ &= (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)})^T (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}) \\ & \quad - m_{\sigma(k)}^T m_{\sigma(k)}. \end{aligned} \quad (32)$$

In this case, a similar argument as above will yield the passivity conditions (14a)-(14c).

iii) All that remains to prove is that for all $\mathbf{w}(k)$ the feedback passivity condition (16) holds with $l_{\sigma(k)} = l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}$, $e_{\sigma(k)} = e_{\sigma(k)}\bar{J}_{\sigma(k)}$ and $m_{\sigma(k)} = m_{\sigma(k)}$ with equality holding if and only if $l_{\sigma(k)} = e_{\sigma(k)} = m_{\sigma(k)} = \mathbf{0}$.

To see this, again consider the two cases separately. If $\sigma(k) \in \mathcal{S}_1^* \cup \mathcal{S}_2$, the zero dynamics are passive and the equation $V(f_{\sigma(k)}^*) - V(\mathbf{x}(k)) = -l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)}$ holds. According to equation (28), we have $-l_{\sigma(k)}^T l_{\sigma(k)} - m_{\sigma(k)}^T m_{\sigma(k)} = - (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)})^T (l_{\sigma(k)} + e_{\sigma(k)}\bar{h}_{\sigma(k)}) - m_{\sigma(k)}^T m_{\sigma(k)}$. For this to hold for any $\bar{h}_{\sigma(k)}$, it follows that $e_{\sigma(k)} = \mathbf{0}$. A similar derivation also holds when $\sigma(k) \in \mathcal{S}_1 \setminus \mathcal{S}_1^*$.

With $e_{\sigma(k)} = \mathbf{0}$, the condition (16) reduces to

$$\begin{aligned} & \sum_{\substack{k:\sigma(k)\in\mathcal{S}_1\setminus\mathcal{S}_1^* \\ k\leq T}} l_{\sigma(k)}^T l_{\sigma(k)} - \sum_{\substack{k:\sigma(k)\in\mathcal{S}_2\cup\mathcal{S}_1^* \\ k\leq T}} l_{\sigma(k)}^T l_{\sigma(k)} \\ & \leq \sum_{k=0}^T m_{\sigma(k)}^T m_{\sigma(k)}. \end{aligned} \quad (33)$$

Because the zero dynamics of the switched system are passive under the passivity definition 4.1 for the switching sequence Σ , the above inequality naturally holds. Hence, all the conditions in Theorem 4.1 are satisfied and the switched system (9) is locally feedback passive for the switching sequence Σ . ■

VI. SWITCHING SEQUENCE FOR PASSIVE ZERO DYNAMICS

We now formalize the intuition that a switched system is passive for a given switching sequence if the nonpassive modes are active infrequently enough. To this end, we derive a lower bound on the ratio of the total activation time of (feedback) passive versus nonpassive modes such that the system zero dynamics (12) are passive.

According to Lemma 5.2, the passive and feedback passive modes of the switched system (9) correspond to the passive modes of the zero dynamics. When $\sigma \in \mathcal{S}_2 \cup \mathcal{S}_1^*$, we assume that the storage function V evolve such that

$$\begin{aligned} & V(f_{\sigma}^*(\mathbf{x})) \leq L_{\sigma} V(\mathbf{x}), \\ & 0 < L_{\sigma} < 1, \quad \sigma \in \mathcal{S}_2 \cup \mathcal{S}_1^*, \quad \forall \mathbf{x}. \end{aligned} \quad (34)$$

Let $L_1 = \max\{L_{\sigma} | \sigma \in \mathcal{S}_2 \cup \mathcal{S}_1^*\}$ be the maximum bound among all passive modes. Admittedly, $L_{\sigma} < 1$ being constant for all $\mathbf{x} \in \mathbf{X}$ may be a strong assumption.

Similarly, according to Lemma 5.2, the non-feedback passive modes of the switched system correspond to the nonpassive modes of the zero dynamics. Therefore, when $\sigma \in \mathcal{S}_1 \setminus \mathcal{S}_1^*$, the storage function V evolves in a way such that

$$\begin{aligned} & V(f_{\sigma}^*(\mathbf{x})) \leq L_{\sigma} V(\mathbf{x}), \\ & L_{\sigma} > 1, \quad \sigma \in \mathcal{S}_1 \setminus \mathcal{S}_1^*, \quad \forall \mathbf{x}. \end{aligned} \quad (35)$$

Let $L_2 = \max\{L_{\sigma} | \sigma \in \mathcal{S}_1 \setminus \mathcal{S}_1^*\}$ be the maximum bound among all nonpassive modes.

Theorem 6.1: Consider the zero dynamics (12) with multiple passive and nonpassive modes. Assume that conditions (34) and (35) hold when the zero dynamics evolve in passive and nonpassive modes, respectively. Design the switching sequence such that

$$\frac{K^-(0, T)}{K^+(0, T)} \geq \frac{\ln L_2 - \ln L_0}{\ln L_0 - \ln L_1}, \quad (36)$$

where $L_0 \in (L_1, 1)$, $K^-(0, T)$ is the total activation time of the passive and feedback passive modes, and $K^+(0, T)$ is the total activation time of the non-feedback passive modes during time interval $[0, T]$, $\forall T \in \{0\} \cup \mathbb{Z}_+$. The zero dynamics (12) are passive under the switching sequence governed by (36).

Proof. Let $0 = k_0 < k_1 < k_2 \dots$ denote the switching points and $\sigma(k_{i-1}) = p_i$. Assume that k_i is the time step corresponding to the i _{th} switch, the storage function of the

zero dynamics evolves as

$$\begin{aligned} & V(\mathbf{x}(k_i)) \leq L_{p_i}^{k_i - k_{i-1}} V(\mathbf{x}(k_{i-1})) \\ & \leq \begin{cases} L_1^{k_i - k_{i-1}} V(\mathbf{x}(k_{i-1})), & p_i \in \mathcal{S}_2 \cup \mathcal{S}_1^* \\ L_2^{k_i - k_{i-1}} V(\mathbf{x}(k_{i-1})), & p_i \in \mathcal{S}_1 \setminus \mathcal{S}_1^* \end{cases}. \end{aligned}$$

At time $T \in [k_i, k_{i+1})$, we obtain

$$\begin{aligned} & V(\mathbf{x}(T)) \leq L_{p_{i+1}}^{T - k_i} V(\mathbf{x}(k_i)) \\ & \leq L_{p_{i+1}}^{T - k_i} L_{p_i}^{k_i - k_{i-1}} V(\mathbf{x}(k_{i-1})) \leq \dots \\ & \leq L_{p_{i+1}}^{T - k_i} L_{p_i}^{k_i - k_{i-1}} \dots L_{p_1}^{k_1 - k_0} V(\mathbf{x}(0)). \end{aligned}$$

Based on conditions (34) and (35), at time T , we have

$$V(\mathbf{x}(T)) \leq L_1^{K^-(0, T)} L_2^{K^+(0, T)} V(\mathbf{x}(0)).$$

From the switching sequence governed by (36), we have

$$\begin{aligned} & \ln \left(\exp \left(\ln \frac{L_0}{L_1} \right)^{K^-(0, T)} \right) \geq \ln \left(\exp \left(\ln \frac{L_2}{L_0} \right)^{K^+(0, T)} \right), \\ & \text{or } \left(\frac{L_0}{L_1} \right)^{K^-(0, T)} \geq \left(\frac{L_2}{L_0} \right)^{K^+(0, T)}, \\ & \text{or } L_0^{K^-(0, T) + K^+(0, T)} \geq L_1^{K^-(0, T)} L_2^{K^+(0, T)}. \end{aligned}$$

Because $K^-(0, T) + K^+(0, T) = T$, it follows that

$$\begin{aligned} & V(\mathbf{x}(T)) \leq L_1^{K^-(0, T)} L_2^{K^+(0, T)} V(\mathbf{x}(0)) \\ & \leq L_0^T V(\mathbf{x}(0)). \end{aligned} \quad (37)$$

Because $L_0 \in (L_1, 1)$, we have $V(\mathbf{x}(T)) \leq V(\mathbf{x}(0))$, or equivalently, the zero dynamics (12) are passive. ■

Remark 6.1: The switching law (36) indicates that the closer L_0 is to 1 the less conservative the bound is.

VII. STABILITY & COMPOSITIONALITY PROPERTIES

In this section, we prove that feedback passivity definition 5.2 preserves two useful properties of the classical passivity definition. Specifically, we show that a feedback passive switched system of the form (9) can achieve asymptotic stability using output feedback if the system is zero state detectable, and that the parallel and negative feedback interconnections of two such feedback passive switched systems remain feedback passive.

Definition 7.1: A system is said to be locally zero state detectable (ZSD) [7] if there exists a neighbourhood \mathbf{N} of the origin such that $\forall \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{N}$,

$$\mathbf{y}(k)|_{\mathbf{u}(k)=\mathbf{0}} = h(\psi(k; \mathbf{x}_0)) = \mathbf{0}, \quad \forall k \in \mathbb{Z}_+$$

implies

$$\lim_{k \rightarrow +\infty} \psi(k; \mathbf{x}_0) = \mathbf{0},$$

where $\psi(k; \mathbf{x}_0)$ is a trajectory of the uncontrolled system $\mathbf{x}(k+1) = f(\mathbf{x}(k))$ from $\mathbf{x}(0) = \mathbf{x}_0$.

Theorem 7.1: If the zero dynamics (12) of system (9) satisfy conditions (34) and (35), then system (9) is locally feedback passive for any switching sequence governed by (36) with a \mathcal{C}^2 storage function. Let $\phi: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be any smooth function such that $\phi(\mathbf{0}) = \mathbf{0}$ and $\mathbf{y}^T \phi(\mathbf{y}) > 0$ for all $\mathbf{y} \neq \mathbf{0}$. Then

the smooth feedback control law $\mathbf{w}(k) = -\phi(\mathbf{y}(k))$ locally asymptotically stabilizes the equilibrium $\mathbf{x}^* = \mathbf{0}$, provided that (9) is locally ZSD.

Proof. According to Theorem 6.1, under the switching sequence governed by (36), system zero dynamics (12) satisfying the condition (34) and (35) are feedback passive. From Theorem 5.1, the switched system (9) is locally feedback passive given the passive zero dynamics (12) for a given switching sequence governed by (36).

When $\sigma(k) \in \mathcal{S}_2 \cup \mathcal{S}_1^*$, we have

$$V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \leq \mathbf{w}^T(k)\mathbf{y}(k).$$

Therefore, any trajectory of the (feedback) passive system satisfies

$$V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \leq -\mathbf{y}^T\phi(\mathbf{y}(k)) \leq 0.$$

Under the switching sequence governed by (36), when $\sigma(k) \in \mathcal{S}_1 \setminus \mathcal{S}_1^*$, the storage function may increase. However, the increase is always bounded since $V(\mathbf{x}(k)) \leq L_0^k V(\mathbf{x}(0))$, $L_0 \in (L_1, 1)$. Now define the sequence of time steps $\{t_i\}$ such that $t_0 = 0$ and t_i is the least time $> t_{i-1}$ such that $\sigma(t_i-1) \in \mathcal{S}_1 \setminus \mathcal{S}_1^*$ and $\sigma(t_i) \in \mathcal{S}_2 \cup \mathcal{S}_1^*$. Then, by an argument identical to above, we can derive a series of inequalities

$$\begin{aligned} V(\mathbf{x}(t_{i+1})) - V(\mathbf{x}(t_i)) &\leq 0, \\ \forall i = 0, 1, \dots, \forall \mathbf{x} \in \mathbf{X}. \end{aligned} \quad (38)$$

Since the frequency in which $\sigma(t_i) \in \mathcal{S}_2 \cup \mathcal{S}_1^*$ is infinitely often, $\{t_i\}$ is an infinite sequence. Then V is a Lyapunov function for the system \mathcal{S} which by Theorem 2 in [25] implies that the system is Lyapunov stable. The asymptotic stability now follows from ZSD. ■

Next, we show the preservation of generalized passivity for interconnected systems.

Theorem 7.2: If two switched nonlinear systems of the form (9) are both feedback passive, then their parallel and negative feedback interconnections (as defined in Figure 1) are both feedback passive.

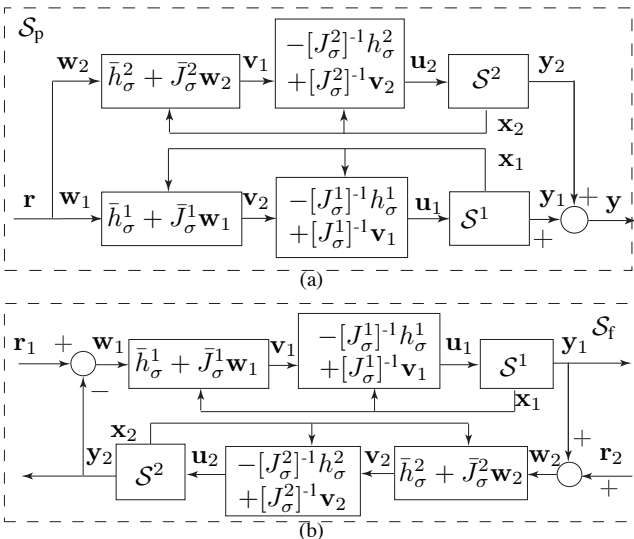


Fig. 1. (a) Parallel, and (b) negative feedback interconnections for two feedback passive switched nonlinear systems \mathcal{S}^1 and \mathcal{S}^2 .

Proof. Let the control inputs for \mathcal{S}^i be $\mathbf{w}_i(k)$, the corresponding output be $\mathbf{y}_i(k)$ and the storage function be $V_i(k)$. For the parallel interconnection, we have for the interconnected system \mathcal{S}_p , the control input $\mathbf{r}(k) = \mathbf{w}_1(k) = \mathbf{w}_2(k)$ and the output $\mathbf{y}(k) = \mathbf{y}_1(k) + \mathbf{y}_2(k)$. Consider the storage function $V(k) = V_1(k) + V_2(k)$. For any time $T \in \{0\} \cup \mathbb{Z}^+$, we have $V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) \leq \sum_{k=0}^T \mathbf{r}^T(k)\mathbf{y}(k)$.

Similarly, for the negative feedback interconnection, we have for the interconnected system \mathcal{S}_f , the control inputs and outputs as $\mathbf{r}_1(k) = \mathbf{w}_1(k) + \mathbf{y}_2(k)$ and $\mathbf{r}_2(k) = \mathbf{w}_2(k) - \mathbf{y}_1(k)$. Consider the storage function $V(k) = V_1(k) + V_2(k)$. For any time $T \in \{0\} \cup \mathbb{Z}^+$, we have $V(\mathbf{x}(T+1)) - V(\mathbf{x}(0)) \leq \sum_{k=0}^T (\mathbf{r}_1^T(k)\mathbf{y}_1(k) + \mathbf{r}_2^T(k)\mathbf{y}_2(k))$. ■

VIII. EXAMPLES

In this section, we provide two examples to present the feedback passivity results proposed in Sections V and VI. We first give an example of a thermal system with linear switched dynamics and then discuss a more general numerical example of nonlinear switched system.

A. Quenching with Variable Bath Properties

Quenching is a process in which a heated object is placed into a liquid bath. The process can improve hardness and other properties of metal by rapid cooling. Consider a metal cube immersed in liquid bath, the model of the cube temperature $T_1(t)$ and liquid bath temperature $T_2(t)$ can be represented by the following linear differential equation:

$$\begin{aligned} C_1 \frac{dT_1(t)}{dt} &= -\frac{1}{R_1}(T_1(t) - T_2(t)) \\ C_2 \frac{dT_2(t)}{dt} &= \frac{1}{R_1}(T_1(t) - T_2(t)) \\ &\quad - \frac{1}{R_2}(T_2(t) - T_o(t)), \end{aligned}$$

where C_1, C_2 are the capacitances of the cube and the liquid bath, respectively, R_1 is the convective resistance between the cube and the bath, R_2 is the combined convective/conductive resistance of the container wall and the liquid surface, and T_o is the temperature outside the bath. Using Euler approximation to discretize the system, we have

$$\begin{aligned} T_1(k+1) &= (1 - \frac{T_s}{R_1 C_1})T_1(k) + \frac{T_s}{R_1 C_1}T_2(k) \\ T_2(k+1) &= \frac{T_s}{R_1 C_2}T_1(k) \\ &\quad + (1 - \frac{T_s}{R_1 C_2} - \frac{T_s}{R_2 C_2})T_2(k) + u(k), \end{aligned}$$

where T_s is the sampling time and the control input is set as $u(k) = \frac{T_s}{R_2 C_2}T_o(k)$. Let the output be

$$y(k) = \frac{T_s}{C_3 R_3}(T_1(k) - T_2(k)) + u(k).$$

In the simulation, we set the capacitance and resistance of the metal cube as $C_1 = 100, R_1 = 0.018$. Now we vary the bath properties C_2, R_2 so that 3 modes representing different bath properties can be simulated. We have Mode

TABLE II
VARIABLE BATH PROPERTIES

| Mode | 1 | 2 | 3 |
|------------|--------|--------|--------|
| C_2 | 50 | 100 | 55 |
| R_2 | 0.2 | 0.019 | 0.03 |
| C_3 | 900 | 900 | 122.22 |
| R_3 | 0.0165 | 0.0092 | 0.0112 |
| L_σ | 1.2812 | 0.6029 | 0.5089 |

1 (nonpassive), Mode 2 (passive), and Mode 3 (feedback passive). Table II lists the bath properties C_2, R_2 , output parameters C_3, R_3 , and L_σ in conditions (34) and (35) for the 3 modes.

The sampling time is set as $T_s = 1$ and we choose the Lyapunov function as $V = \frac{1}{2}(T_1^2(k) + T_2^2(k))$ which is C^2 and $V(f_{\sigma(k)} + g_{\sigma(k)}\mathbf{u})$ quadratic in \mathbf{u} . The storage function is chosen as $V = \frac{1}{2}(x_1^2(k) + x_2^2(k))$ which satisfies A.6 and A.7. The resulting switching law is

$$K^-(0, T) \geq 0.5557K^+(0, T). \quad (39)$$

More insight can be obtained if we consider the system operating over a finite horizon. Consider the system operation from $k = 1$ to $T = 16$. Figure 2(a) shows the system passivity check under the classical definition, i.e., inequality (2). We see from the figure that the inequality does not always hold when the system is in the non-feedback passive Mode 1, i.e., at time steps 2, 6, 9, 13. Hence, the switched system is considered to be nonpassive in the classical passivity theory. Figure 2(b) shows the passivity check under the generalized definition, i.e., inequality (25) which is satisfied at every time step.

B. Numerical Nonlinear Example

Consider a switched nonlinear system consisted of the following three modes

(Mode 1)

$$\begin{aligned} x_1(k+1) &= (2x_2(k) + 0.8x_1^2(k) + u(k)) \cos(x_2(k)) \\ x_2(k+1) &= (1.6x_1(k) + 0.2x_2^2(k) + u(k)) \sin(x_2(k)) \\ y(k) &= x_1(k) + x_2(k) + u(k). \end{aligned}$$

Mode 1 is nonpassive and can not be passivated using feedback and $L_\sigma = 1.5506$.

(Mode 2)

$$\begin{aligned} x_1(k+1) &= x_2^2(k) + u(k) \\ x_2(k+1) &= 0.6x_1^2(k) + u(k) \\ y(k) &= 0.8x_1^2(k) + 0.75x_2^2(k) + u(k). \end{aligned}$$

Mode 2 is locally passive and $L_\sigma = 0.2666$.

(Mode 3)

$$\begin{aligned} x_1(k+1) &= x_1^2(k) + x_2^2(k) + u(k) \\ x_2(k+1) &= x_1^2(k) - 0.3x_2^2(k) + u(k) \\ y(k) &= x_1^2(k) + x_2^2(k) + u(k). \end{aligned}$$

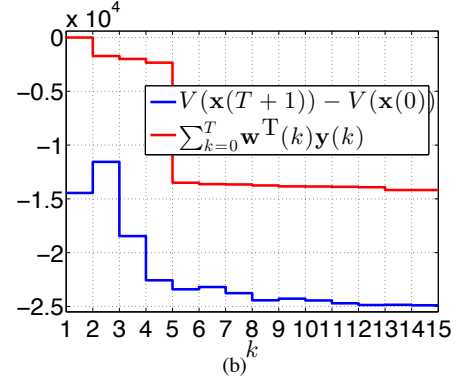
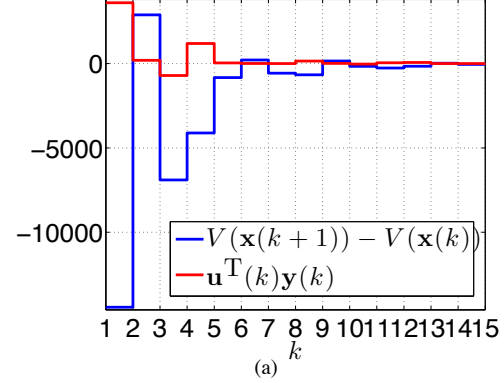


Fig. 2. (a) Passivity check for the switched system according to the classical passivity inequality (2), and (b) Feedback passivity check for the switched system according to the generalized passivity inequality (25).

Mode 3 is a locally feedback passive mode and $L_\sigma = 0.3702$. Therefore, we have $L_1 = 0.3702$ and $L_2 = 1.5506$ and we choose $L_0 = 0.9999$. The resulting switching law is

$$K^-(0, T) \geq 0.4416K^+(0, T). \quad (40)$$

Modes 2 and 3 are ZSD. The design of feedback control is presented as follows. We take Mode 3 as an example, and the derivation of the other two modes follows similarly as in Section V.

$$\begin{aligned} \bar{J}_3(k) &= 2 \\ \bar{h}_3(k) &= -2(2x_1^2(k) + 0.7x_2^2(k) \\ &\quad - 2(x_1^2(k) + x_2^2(k))) \\ w(k) &= -200y(k) \end{aligned}$$

The storage function is chosen as $V = \frac{1}{2}(x_1^2(k) + x_2^2(k))$.

Under the switching law (40), the zero dynamics of the switched system satisfy the generalized passivity inequality as shown in Figure 3(a)(i). Figure 3(a)(ii) shows that the evolution of storage function for zero dynamics satisfy the inequality (37). Figure 3(b) shows the evolution of state dynamics and both states converge to the equilibrium point. Figure 4(a) shows the system passivity check under the classical definition, i.e., inequality (2). Figure 4(b) shows the passivity check under the generalized definition, i.e., inequality (25) which is always satisfied. Table III lists the activation time of each mode during the horizon. The switching condition (40) can be verified at every time step till T .

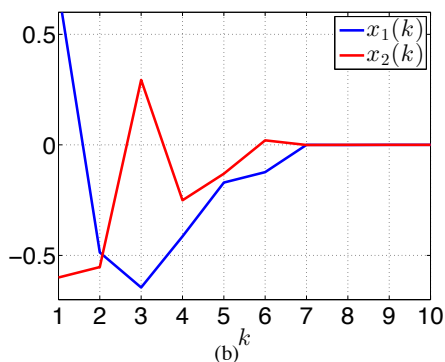
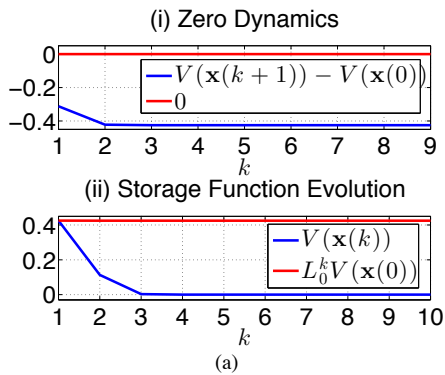


Fig. 3. (a) (i) Passivity check for system zero dynamics according to inequality (22), (a) (ii) Check for inequality (37), and (b) Evolution of state dynamics using control law $w(k)$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|---|----|
| Mode 1 | | ✓ | | | ✓ | | ✓ | | | |
| Mode 2 | ✓ | | ✓ | ✓ | | | | | | |
| Mode 3 | | | | | | ✓ | | ✓ | ✓ | ✓ |

TABLE III
ACTIVATION TIME.

IX. CONCLUSION

In this paper, we generalize the classical passivity definition for a class of discrete-time switched nonlinear systems consisting of both passive and nonpassive modes. We introduce necessary and sufficient conditions for such systems to be locally passive according to the generalized definition. We further apply these results to switched systems with passive modes, feedback passive modes, and modes that cannot be rendered passive using feedback. The switched nonlinear system is proved to be locally feedback passive if and only if its zero dynamics are locally passive according to the generalized passivity definition. A lower bound on the total activation time of (feedback) passive modes versus non-feedback passive modes is derived to guarantee passive zero dynamics. We prove that the system equilibrium point can achieve asymptotic stability using output feedback and the interconnections of two generalized passive systems remain passive.

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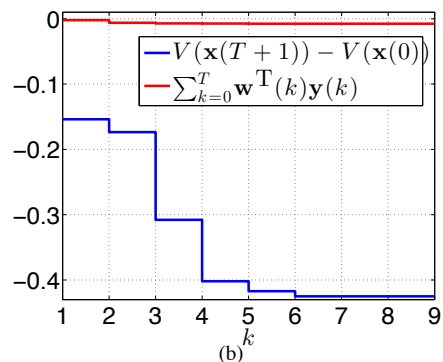
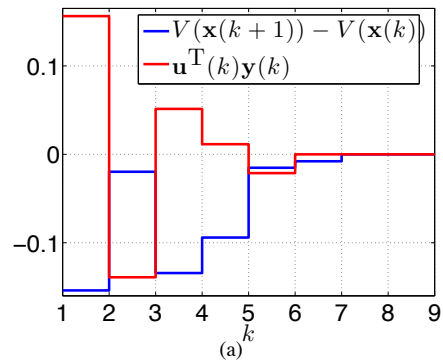
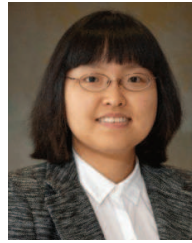


Fig. 4. (a) Passivity check for the switched system according to the classical passivity definition (2) using control law $u(k)$, and (b) Feedback passivity check for the switched system according to the generalized passivity definition (25) using control law $w(k)$.

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Yue Wang is an Assistant Professor in the Department of Mechanical Engineering at Clemson University. She received her B. S. degree from Shanghai University, China, and M.S. and Ph.D. degrees from Worcester Polytechnic Institute. Prior to joining Clemson, she was a postdoctoral research associate in the Electrical Engineering Department at the University of Notre Dame. Her research interests include compositional design and control of cyber-physical systems, decision-making for multi-agent systems, and collaborative human robot teams.



topics at the interaction of communication, computation and control. He received the NSF Career award in 2009 and the Donald P. Eckman award in 2013.

Vijay Gupta is an Associate Professor in the Department of Electrical Engineering at the University of Notre Dame. He received his B. Tech degree from the Indian Institute of Technology, Delhi and the M.S. and Ph.D. degrees from the California Institute of Technology, all in Electrical Engineering. He has served as a research associate in the Institute for Systems Research at the University of Maryland, College Park, and as a consultant to the Systems Group at the United Technology Research Center, Hartford, CT. His research interests include various



and on hybrid and discrete event dynamical systems. He had co-authored two research monographs on discrete event systems, two graduate textbooks on Linear Systems and has co-edited six books on Intelligent Autonomous Control, Hybrid Systems and Networked Embedded Control Systems. He is IEEE, IFAC and AAAS Fellow, the 2006 recipient of the Engineering Alumni Medal of Brown University and a 2012 honorary doctorate recipient from the University of Lorraine, France. He is the Editor-in-Chief of the *IEEE Transactions on Automatic Control*.

Panos J. Antsaklis is the Brosey Professor of Electrical Engineering at the University of Notre Dame. He is a graduate of the National Technical University of Athens, Greece, and holds MS and PhD degrees from Brown University. His research addresses problems of control and automation and examines ways to design control systems that will exhibit high degree of autonomy. His recent research focuses on Cyber-Physical Systems and addresses problems in the interdisciplinary research area of control, computing and communication networks,