

Determining Passivity Using Linearization for Systems with Feedthrough Terms

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Abstract—In this paper, we consider the following problem: what passivity properties for a nonlinear system can be inferred when its linearization around an equilibrium point is known to be passive. We consider both continuous-time and discrete-time systems with feedthrough terms. Our main results show that when the linearized model is simultaneously strictly passive and strictly input passive, the nonlinear system is passive as well within a neighborhood of the equilibrium point around which the linearization is done.

I. INTRODUCTION

Passivity and dissipativity characterize the energy consumption of a dynamical system and form a powerful tool in many applications. Passivity is closely related to stability and exhibits a compositional property for parallel and feedback interconnections [1], [2], [3]. Thus, it is especially useful in the analysis of large-scale systems.

In this paper, we are interested in the passivity of a nonlinear system as inferred from studying its linearized model. Using linearized models for nonlinear systems has a long history both in analysis and synthesis [4], [5]. These methods rely on results that can guarantee that certain properties such as stability ‘carry over’ to the nonlinear system from its linearized version. For the properties of passivity and dissipativity that we are interested in, the picture is not as clear. The closest works to ours in the literature are [1], [6]. In [6], nonlinear systems that are affine in control and without any feedthrough term were studied. It was shown that for such systems, strict passivity of the linearized model is sufficient to guarantee local passivity of the nonlinear system. Since a passive discrete time system must have a non-zero feedthrough term, consideration was restricted to continuous time systems only. In [1], once again, the feedthrough term of the linearized model was assumed to be zero. Conditions under which dissipativity of the linearized system can guarantee a similar property of the nonlinear system are then established. However, since these conditions assume the supply rate to be of a particular form that does not include passivity, the results cannot be used to study passivity of the nonlinear system (even when the feedthrough term is assumed to be absent).

The main contribution of this paper is the establishment of one set of sufficient conditions for the linearized model to ensure local passivity of the corresponding nonlinear system, for both continuous-time and discrete-time systems. In particular, we show that if the linearization of a nonlinear

system is simultaneously strictly passive and strictly input passive, then the nonlinear system is guaranteed to be locally passive. We also establish a similar result for the case when the linearization is dissipative. Analogous to the results in [1], [6], for a given nonlinear system, our results hold true within a neighborhood of the equilibrium point around which the linearization is done. We do not constrain the system to have no feedthrough term as was the case in [1], [6] or to be affine in the control input as in [6]. This requires us to use a different analysis technique based on Taylor’s theorem and the positive real lemma. In particular, by allowing the presence of feedthrough terms, we are able to present passivity results for discrete time systems as well. Our results are compatible with those in [6] in the sense that when the system model has the form in [6], our results reduce to theirs.

The rest of the paper is organized as follows. Section II provides background material on passivity and dissipativity. The main results are given in Section III when the linearized model is assumed to be simultaneously strictly passive and strictly input passive. When the linearization is merely strictly passive, the corresponding results are presented in IV. Section V provides some concluding remarks.

Notation: \mathbb{R}^m denotes the Euclidean space of dimension m . The n -dimensional identity matrix is denoted by $I_{n \times n}$ or simply I by omitting the dimensions if clear from the context. For a matrix $P \in \mathbb{R}^{m \times n}$, its transpose is denoted by P^T . For a symmetric matrix $P = P^T$, $P > 0$ denotes that P is positive-definite and $P \geq 0$ denotes that it is positive semi-definite. The maximum eigenvalue of P is denoted by $\bar{\lambda}(P)$ and its minimum eigenvalue is denoted by $\underline{\lambda}(P)$. The 2-norm of a vector $x \in \mathbb{R}^m$ is denoted by $\|x\|$, and analogously, the 2-norm of a matrix $P \in \mathbb{R}^{m \times n}$ is denoted by $\|P\|$. The absolute value of $x \in \mathbb{R}$ is denoted by $|x|$.

II. BACKGROUND MATERIAL

Consider a continuous-time nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x, u), \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^m$ is the system output. The functions f and h are real analytic about $(x = 0, u = 0)$. Without loss of generality, we assume that the pair $(x = 0, u = 0)$ is an equilibrium point for system (1). Thus, $f(0, 0) = 0$ and $h(0, 0) = 0$.

Linearization of system (1) around the equilibrium point

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$(x = 0, u = 0)$ is given by

$$\begin{aligned} \dot{z} &= Az + Bv, \\ w &= Cz + Dv, \end{aligned} \quad (2)$$

where $z \in \mathbb{R}^n$ denotes the system state, $v \in \mathbb{R}^m$ denotes the control input and $w \in \mathbb{R}^m$ denotes the system output. The system matrices $\{A, B, C, D\}$ are given by

$$\begin{aligned} A &= \frac{\partial f}{\partial x} \Big|_{x=0, u=0}, & B &= \frac{\partial f}{\partial u} \Big|_{x=0, u=0}, \\ C &= \frac{\partial h}{\partial x} \Big|_{x=0, u=0}, & D &= \frac{\partial h}{\partial u} \Big|_{x=0, u=0}. \end{aligned} \quad (3)$$

Throughout the paper, we assume that $\{A, B\}$ is controllable and $\{A, C\}$ is observable.¹ The positive real lemma (see e.g. Lemma 6.2 in [4]) states that system (2) is positive real if and only if there exist matrices $P = P^T > 0$, L and W , such that

$$\begin{aligned} PA + A^T P &= -L^T L, \\ PB &= C^T - L^T W, \\ W^T W &= D + D^T. \end{aligned}$$

The Taylor series expansions for f and h about $(0, 0)$ exist and are given by

$$\begin{aligned} f(x, u) &= Ax + Bu + F(x, u), \\ h(x, u) &= Cx + Du + H(x, u), \end{aligned} \quad (4)$$

where $F(x, u)$ and $H(x, u)$ contain higher-order terms corresponding to $f(x, u)$ and $h(x, u)$, respectively (see e.g. [9, p. 388-393]).

Analogously, for a discrete-time nonlinear system given by

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k), u(k)), \end{aligned} \quad (5)$$

we can obtain a linearized model given by

$$\begin{aligned} z(k+1) &= Az(k) + Bv(k), \\ w(k) &= Cz(k) + Dv(k), \end{aligned} \quad (6)$$

with $\{A, B, C, D\}$ defined in (3). Note that as opposed to system (1), system (5) cannot be passive when $y(k) = h(x(k))$ (i.e. if there is no feedthrough term), see e.g. [10].

Definition 1: ([1], [11]) The state-space system (1) is said to be *dissipative* with respect to supply rate $w(u(t), y(t))$, if there exists a nonnegative function $V(x)$, called the *storage function*, satisfying $V(0) = 0$ such that for all $x_0 \in \mathcal{X}$, all $t_1 > t_0$, and all $u \in \mathbb{R}^m$,

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} w(u(t), y(t)) dt, \quad (7)$$

where $x(t_0) = x_0$ and $x(t_1)$ is the state at t_1 resulting from initial condition x_0 and input function $u(\cdot)$. In particular, if (7) holds with strict inequality, (1) is called *strictly dissipative*.

Definition 2: ([10]) The state-space system (5) is said to be *dissipative* with respect to supply rate $W(u(k), y(k))$, if

¹Note that we make no assumption on the global controllability or observability of the nonlinear system (1). The problem considered in this paper assumes that the linearized system (2) satisfies the positive real lemma. Under this framework, controllability and observability is a common assumption (see, e.g., [4, pp. 237-241], [7, pp. 361-372], [8, pp. 218-269]).

there exists a nonnegative function $V(x)$, called the *storage function*, satisfying $V(0) = 0$ such that for all $x_0 \in \mathcal{X}$, all $k > k_0$, and all $u \in \mathbb{R}^m$,

$$V(x(k)) - V(x(k_0)) \leq \sum_{i=k_0}^{k-1} W(y(i), u(i)), \quad (8)$$

where $x(k_0) = x_0$ and $x(k)$ is the state at k resulting from initial condition x_0 and input function $u(\cdot)$. In particular, if (8) holds with strict inequality, (5) is called *strictly dissipative*.

Remark 1: If $V(x)$ is differentiable, (7) is equivalent to

$$\dot{V}(x) \triangleq \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq w(u(t), y(t)). \quad (9)$$

Further, it has been shown in [10] that in discrete-time domain, (8) is equivalent to

$$V(x(k+1)) - V(x(k)) \leq W(u(k), y(k)).$$

Definition 3: ([2], [11]) Suppose system (1) is dissipative. It is called:

- 1) *passive* if (7) holds for $w(u, y) = u^T y$;
- 2) *strictly passive* (SP) if (7) holds with strict inequality for $w(u, y) = u^T y$;
- 3) *strictly input passive* (SIP) if (7) holds for $w(u, y) = u^T y - \nu u^T u$, where $\nu > 0$.
- 4) *strictly passive and strictly input passive* (SSIP) if it is simultaneously SP and SIP.
- 5) (Q, S, R) -*dissipative*, if (7) holds for $w(u, y) = u^T R u + 2y^T S u + y^T Q y$, where $Q = Q^T$, S and $R = R^T$ are matrices of appropriate dimensions. In particular, if $Q = -\rho I$, $S = \frac{1}{2} I$, $R = 0$, then ρ is called the *output feedback passivity index* (OFP); if $Q = 0$, $S = \frac{1}{2} I$, $R = -\nu I$, then ν is called the *input feedforward passivity index* (IFP).

Analogously, we can define passivity and other properties mentioned above for the discrete-time system (5) as well. \square

Remark 2: As shown in [4], a system is SP if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ . SSIP has been defined as input-state strictly passivity in [12] where it is also shown that a system is SSIP if $u^T y \geq \dot{V} + \psi(x) + \epsilon u^T u$ for some positive definite function $\psi(x)$ and some positive constant $\epsilon > 0$.

Definition 4: ([13]) If any of the properties for system (1) or (5) as defined above in Definitions 1, 2, 3 hold in a neighborhood of $(x = 0, u = 0) \in \mathcal{X} \times \mathcal{U}$, it is called a *local property* of system (1) or (5).

Remark 3: In the literature, alternate definitions of local passivity have been proposed. For instance, in [6], local passivity is defined in a ball around $x = 0$ and all control inputs u that do ‘not drive the state too far from the equilibrium point’. In [14], Sobolev spaces have been used to define local passivity by constraining the magnitudes of u and its derivatives. In [15], local dissipativity is defined both in terms of small-gain inputs and local internal stability regions. In this paper, we define local passivity or dissipativity in a neighborhood of $(x = 0, u = 0) \in \mathcal{X} \times \mathcal{U}$ as in [13].

III. MAIN RESULTS

In this section, we establish conditions under which a linear system is SSIP and show that a SSIP linearized model guarantees local SSIP of the nonlinear system.²

A. SSIP: from linearity to nonlinearity

We now present our main result. The following theorem shows that if the linearized model is SSIP, then the nonlinear system is guaranteed to be locally SSIP. Thus, in particular, the nonlinear system is locally passive.

Theorem 1: Consider the continuous-time nonlinear system given by (1) and its linearized model given by (2) and (3). If the linearized model (2) is SSIP, then system (1) is locally SSIP. Analogously, for the discrete-time nonlinear system given by (5) and its linearized model given by (6) and (3). If the linearized model (6) is SSIP, then system (5) is locally SSIP.

Proof: See the Appendix. ■

Remark 4: For the nonlinear systems (1) and (5), the above results hold true only in a neighborhood of the equilibrium point $(x = 0, u = 0)$.

Remark 5: Note that the converse of this result does not hold in general. In other words, if the nonlinear system (1) is SSIP, then the linearized model (2) need not to be SSIP [6], [7].

Remark 6: Our result that SSIP of the linearized model implies local passivity of the corresponding nonlinear system is analogous to the result that asymptotic stability of the linearized model implies local Lyapunov stability of the corresponding nonlinear system.

Example 1: To better understand Theorem 1, we consider a particular form of system (1) which is affine in control and given by

$$\begin{aligned} \dot{x} &= \alpha(x) + \beta(x)u, \\ y &= \gamma(x) + \zeta(x)u, \end{aligned} \quad (10)$$

where α , β , γ and ζ are real analytic at $x = 0$. We assume that $\alpha(0) = 0$ and $\gamma(0) = 0$. Further, let the Taylor series expansions for α , β , γ and ζ about $x = 0$ be given by

$$\begin{aligned} \alpha(x) &= Ax + \tilde{F}(x), & \gamma(x) &= Cx + \tilde{H}(x), \\ \beta(x) &= B + \tilde{G}(x), & \zeta(x) &= D + \tilde{M}(x), \end{aligned} \quad (11)$$

If the linearized model (2) is SSIP, then there exist a storage function $V = \frac{1}{2}z^T Pz$ and positive constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $v^T w - \dot{V} \geq \varepsilon_1 z^T z + \varepsilon_2 v^T v$. Apply $V(x) = \frac{1}{2}x^T P x$ as a storage function for (10) and define $\Upsilon(x, u) \triangleq u^T y - \dot{V}(x)$. From (11), we obtain

$$\begin{aligned} \Upsilon(x, u) &= u^T (\gamma(x) + \zeta(x)u) - \frac{\partial V}{\partial x} (\alpha(x) + \beta(x)u) \\ &= u^T (Cx + Du) + u^T (\tilde{H}(x) + \tilde{M}(x)u) \\ &\quad - x^T P (Ax + Bu) - x^T P (\tilde{F}(x) + \tilde{G}(x)u) \\ &\geq \frac{1}{2} (\varepsilon_1 x^T x + \varepsilon_2 u^T u) + \psi(x, u), \end{aligned} \quad (12)$$

²In this paper, we will present proofs for the case when the system is in continuous-time. The proofs for the case when the system is in the discrete-time domain are largely analogous and are provided in [16].

where $\psi(x, u) \triangleq \frac{1}{2} (\varepsilon_1 x^T x + \varepsilon_2 u^T u) + u^T (\tilde{H}(x) + \tilde{M}(x)u) - x^T P (\tilde{F}(x) + \tilde{G}(x)u)$. Next, we show that $(0, 0)$ is a local minimum of the function $\psi(x, u)$. It is obvious that $\psi(0, 0) = 0$. Through simple calculations, we can obtain that the first derivatives of $\psi(x, u)$ are given by $\frac{\partial \psi}{\partial x}|_{x=0, u=0} = 0$, $\frac{\partial \psi}{\partial u}|_{x=0, u=0} = 0$ and the Hessian matrix is given by

$$\begin{aligned} \mathcal{H} &\triangleq \begin{bmatrix} \frac{\partial^2 \psi}{\partial x^2}|_{x=0, u=0} & \frac{\partial^2 \psi}{\partial x \partial u}|_{x=0, u=0} \\ \frac{\partial^2 \psi}{\partial u \partial x}|_{x=0, u=0} & \frac{\partial^2 \psi}{\partial u^2}|_{x=0, u=0} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \varepsilon_1 I & 0 \\ 0 & \frac{1}{2} \varepsilon_2 I \end{bmatrix} > 0. \end{aligned}$$

Therefore, the point $(0, 0)$ is a local minimum of the function $\psi(x, u)$. In other words, in a sufficiently small neighborhood of $(0, 0)$, we have $\psi(x, u) \geq 0$. Then, from (12), we obtain

$$\Upsilon(x, u) \geq \frac{1}{2} \varepsilon_1 x^T x + \frac{1}{2} \varepsilon_2 u^T u.$$

Therefore, the system (10) is locally SSIP. □

We show that it is relatively straight-forward to check if a linear system is SSIP. Thus, Theorem 1, which shows the SSIP of the linearized system guarantees passivity of the nonlinear system, can be used in a computationally efficient manner.

- Proposition 1:*
- 1) If the linear system (2) is strictly passive and further satisfies $D + D^T > 0$, then it is SSIP.
 - 2) If the linear system (6) is strictly passive with a storage function $V(z) = \frac{1}{2}z^T Pz$ and further satisfies $D + D^T - B^T P B > 0$, then it is SSIP.

Proof: See the Appendix. ■

Remark 7: We note here that if a continuous time linear system of the form (2) is strictly passive and satisfies $D + D^T > 0$, then the system has been called strongly positive real or extended strictly positive real in the literature [3], [7], [17]. Also a discrete-time system of the form (5) cannot be strongly positive real [17].

Remark 8: As shown in [17], [18], to test whether the system (2) is SSIP, we can test if the following LMI has a solution $P > 0$:

$$\begin{bmatrix} A^T P + P A & P B - C^T \\ B^T P - C & -(D^T + D) \end{bmatrix} < 0. \quad (13)$$

Similarly, to test if the LTI system (5) is SSIP, we can test if the following LMI has a solution $P > 0$:

$$\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - (D^T + D) \end{bmatrix} < 0.$$

An example is now given to illustrate the application of Theorem 1.

Example 2: Consider the continuous-time nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_2, \\ \dot{x}_2 &= -x_1 - x_2 + (ax_1 + 1)u, \\ y &= x_1 + 2x_2 + (bx_2 + 1)u, \end{aligned}$$

where $a \neq 0, b \neq 0$. The linearized model of the system around the origin is given by (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 2], \quad D = 1.$$

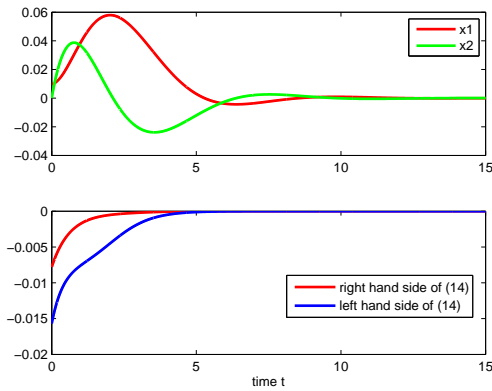


Fig. 1. In the top plot are the state trajectories in Example 2 and in the bottom plot are the trajectories of functions $V - u^T y$ (blue line) and $-\frac{1}{16}(x_1^2 + x_2^2) - \frac{1}{2}u^2$ (red line) in Example 2.

By solving the LMI (13), we can obtain that $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} > 0$. Applying $V(x) = x_1^2 + x_1 x_2 + x_2^2$ as a locally defined storage function for the nonlinear system, we obtain

$$\begin{aligned} \dot{V} - u^T y &\leq -\frac{1}{2}(x_1^2 + x_2^2) - u^2(1 - |bx_2|) \\ &\quad - x_1^2(2x_1 + x_2 - au - |au|) + |au|x_2^2 \\ &\leq -x_1^2\left(\frac{1}{2} - |2x_1| - |x_2| - 2|au|\right) - u^2(1 - |bx_2|) \\ &\quad - x_2^2\left(\frac{1}{2} - |au|\right). \end{aligned}$$

Consider a neighborhood of $(x = 0, u = 0)$ for which $|u| \leq \frac{1}{8|a|}$, $|x_1| \leq \frac{1}{16}$, $|x_2| \leq \min\{\frac{1}{16}, \frac{1}{2|b|}\}$. We obtain $\frac{1}{2} - |2x_1| - |x_2| - 2|au| \geq \frac{1}{16}$ and thus

$$\dot{V} - u^T y \leq -\frac{1}{16}(x_1^2 + x_2^2) - \frac{1}{2}u^2. \quad (14)$$

Therefore, the nonlinear system is locally SSIP as guaranteed by Theorem 1. Now, let $a = 1$, $b = 2$ and $x(0) = [0.01, 0]^T$. Let the control input u be given by $u = \frac{1}{8} \exp(-0.8t)$. It is shown in the top plot of Fig. 1 that $|x_1(t)| \leq \frac{1}{16}$ and $|x_2(t)| \leq \frac{1}{16}$. Consider a neighborhood of $(x = 0, u = 0)$ which is given by $\{(x, u) \mid |x_1| \leq \frac{1}{16}, |x_2| \leq \frac{1}{16}, |u| \leq \frac{1}{8}\}$. It is shown in the bottom plot of Fig. 1 that the function $\dot{V} - u^T y$ is upper bounded by the function $-\frac{1}{16}(x_1^2 + x_2^2) - \frac{1}{2}u^2$ for any $t \geq 0$, i.e. (14) is satisfied. Thus, the system is locally SSIP. \square

B. QSR-dissipative systems

The arguments so far generalize to the case when the linearized model is strictly (Q, S, R) -dissipative. By setting (Q, S, R) to be of particular forms, we can consider the case when the linearized model may not be passive or SSIP as in Section III-A. We present without proof conditions under which a strictly (Q, S, R) -dissipative linearized system implies local (Q, S, R) -dissipativity of the nonlinear system. The proof can be found in [16].

Corollary 1: Consider the continuous-time system (1) and its linearized model given by (2) and (3). Assume that the

linearized model (2) is strictly (Q, S, R) -dissipative. If

$$R + S^T D + D^T S + D^T Q D > 0, \quad (15)$$

then system (1) is locally strictly (Q, S, R) -dissipative. Analogously, for the discrete-time system (5) and its linearized model given by (6) and (3). Assume that the linearized model (6) is strictly (Q, S, R) -dissipative with with a storage function $V(z) = \frac{1}{2}z^T P z$. If

$$R + S^T D + D^T S + D^T Q D - B^T P B > 0. \quad (16)$$

then system (5) is locally strictly (Q, S, R) -dissipative. \square

Remark 9: For the continuous-time case, if $D = 0$, then (15) is reduced to $R > 0$. Particularly, if $R = \gamma^2 I > 0$, $S = \frac{1}{2}I$ and $Q = -I$, then the linearized model has finite gain $\gamma > 0$. Then from Corollary 1, the nonlinear system has local finite-gain γ (see also [1]).

To test if the LTI system (2) satisfies the conditions in Corollary 1, one can test if the following LMI has a positive definite solution P :

$$\Pi \triangleq \begin{bmatrix} A^T P + P A - C^T Q C & P B - \hat{S} \\ B^T P - \hat{S}^T & -\hat{R} \end{bmatrix} < 0, \quad (17)$$

where $\hat{S} \triangleq C^T S + C^T Q D$ and $\hat{R} \triangleq D^T Q D + (D^T S + S^T D) + R$. Similarly, to test if the LTI system (6) satisfies the conditions in Corollary 1, one can test if the following LMI has a positive definite solution P :

$$\begin{bmatrix} A^T P A - P - C^T Q C & A^T P B - \hat{S} \\ B^T P A - \hat{S}^T & B^T P B - \hat{R} \end{bmatrix} < 0,$$

where \hat{S} , and \hat{R} are the same as in (17).

Remark 10: If $\Pi \leq 0$, then the LTI system (2) is (Q, S, R) -dissipative, see e.g. [17]. Further, if $\Pi < 0$, then local (Q, S, R) -dissipativity of the nonlinear system (1) can be guaranteed from (Q, S, R) -dissipativity of its linearization (3).

Corollary 1 can be used to find (local) passivity indices of a nonlinear system from those of its linearized model. The two passivity indices (OFP ρ and IFP ν) informally characterize how passive a dynamical system is. In particular, if $\rho > 0$ or $\nu > 0$, then the system has an excess of passivity. Similarly, if $\rho < 0$ or $\nu < 0$, then the system has a shortage of passivity [3]. The passivity indices can be used in control designs and stability analysis [19].

Corollary 2: Consider the continuous-time system (1) and its linearized model given by (2) and (3). Assume that the linearized model (2) is strictly $(0, \frac{1}{2}I, -\nu I)$ -dissipative. If $\frac{1}{2}(D + D^T) - \nu I > 0$, then system (1) has local IFP(ν). Analogously, for the discrete-time system (5) and its linearized model given by (6) and (3). Assume that the linearized model (6) is strictly $(0, \frac{1}{2}I, -\nu I)$ -dissipative with a storage function $V(z) = \frac{1}{2}z^T P z$. If $\frac{1}{2}(D + D^T) - \nu I - B^T P B > 0$, then system (5) has local IFP(ν). \square

Corollary 3: Consider the continuous-time system (1) and its linearized model given by (2) and (3). Assume that the linearized model (2) is strictly $(-\rho I, \frac{1}{2}I, 0)$ -dissipative. If $\frac{1}{2}(D + D^T) - \rho D^T D > 0$, then system (1) has local OFP(ρ). Analogously, for the discrete-time system (5) and its linearized model given by (6) and (3). Assume that the linearized model

(6) is strictly $(-\rho I, \frac{1}{2}I, 0)$ -dissipative with a storage function $V(z) = \frac{1}{2}z^T Pz$. If $\frac{1}{2}(D + D^T) - \rho D^T D - B^T P B > 0$, then system (5) has local OFP(ρ). \square

IV. DISCUSSION

In the previous section, we assumed that the linearized model is SSIP. SSIP is a more restrictive condition than SP as assumed e.g. in [6]. In this section, we show that the property of SP alone for the linearized model may not be sufficient to guarantee local passivity of a nonlinear system with feedthrough terms. This is particularly relevant for discrete-time systems since a discrete-time system with no feedthrough term cannot be passive. For simplicity, in this section we focus on system models that are affine in control.

Theorem 2: 1) Consider the system (10) and its linearized model given by (2) and (3). If the linearized model (2) is SP and there exists a constant $l \geq 0$ such that

$$\lim_{\|x\|^2 \rightarrow 0} \frac{\|\zeta(x) - D\|}{\|x\|^2} \leq l, \quad (18)$$

then the nonlinear system (10) is locally SP (LSP).

2) Consider the discrete-time nonlinear system given by

$$\begin{aligned} x(k+1) &= \alpha(x(k)) + \beta(x(k))u(k), \\ y(k) &= \gamma(x(k)) + \zeta(x(k))u(k). \end{aligned} \quad (19)$$

where $\zeta(x(k)) \neq 0$ and its linearized model given by (6) and (3). If the linearized model (6) is SP and there exist constants $l_1 \geq 0$ and $l_2 \geq 0$ such that

$$\lim_{\|x\|^2 \rightarrow 0} \frac{\|\zeta(x) - D\|}{\|x\|^2} \leq l_1, \quad \lim_{\|x\|^2 \rightarrow 0} \frac{\|\beta(x) - B\|}{\|x\|^2} \leq l_2, \quad (20)$$

then the nonlinear system (19) is LSP.

Proof: See the Appendix. \blacksquare

Remark 11: For the system (10), if $\zeta(x) \equiv 0$, we obtain the following system with no feedthrough,

$$\begin{aligned} \dot{x} &= \alpha(x) + \beta(x)u, \\ y &= \gamma(x). \end{aligned} \quad (21)$$

The linearization of system (21) is given by (3) with $D \equiv 0$. Then, $\zeta(x) - D \equiv 0$. Therefore, (18) is necessarily satisfied with $l \equiv 0$. The following result is immediate from Theorem 2. Note that Corollary 4 has been established in [6].

Corollary 4: Consider a nonlinear system (21) and its linearized model (3) where $D \equiv 0$. If its linearization is SP, then system (21) is LSP.

V. CONCLUSIONS AND FUTURE DIRECTION

In this paper, we established conditions under which local passivity of a nonlinear system can be obtained by analyzing its linearization. The general result states that if the linearized model is simultaneously strictly passive and strictly input passive (SSIP), then the corresponding nonlinear system is SSIP as well within a neighborhood of the equilibrium point around which the linearization is done. Possible directions for future work may include investigation of local linearizations around trajectories.

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VI. APPENDIX

Proof of Theorem 1: Linear system (2) is SSIP, thus there exist $P = P^T > 0$ and positive constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that $V(z) = \frac{1}{2}z^T Pz$ is a storage function for system (2) and

$$\begin{aligned} v^T w - \dot{V} &= v^T (Cz + Dv) - z^T P (Az + Bv) \\ &\geq \varepsilon_1 z^T z + \varepsilon_2 v^T v. \end{aligned}$$

Apply $V(x) = \frac{1}{2}x^T Px$ as a locally defined storage function for the nonlinear system (1). Define $\Upsilon(x, u) \triangleq u^T y - \dot{V}(x)$. From (4), we obtain

$$\begin{aligned}\Upsilon(x, u) &= u^T h(x, u) - \frac{\partial V}{\partial x} f(x, u) \\ &= u^T (Cx + Du) + u^T H(x, u) \\ &\quad - x^T P(Ax + Bu) - x^T PF(x, u) \\ &\geq \varepsilon_1 x^T x + \varepsilon_2 u^T u + u^T H(x, u) - x^T PF(x, u).\end{aligned}$$

Denote σ_u (resp. σ_x) as the smallest order for u (resp. x) contained in a polynomial of x and u . From Taylor's Theorem, we have either $\sigma_x \geq 2$ or $\sigma_u \geq 2$ for terms contained in polynomials $x^T F(x, u)$ and $u^T H(x, u)$. Thus, the terms contained in polynomial $u^T H(x, u) - x^T PF(x, u)$ can be classified into two categories:

- 1) the terms with $\sigma_x \geq 2$ and $\sigma_u \leq 1$, denoted by $\Gamma_1(x, u)$,
- 2) the terms with $\sigma_u \geq 2$, denoted by $\Gamma_2(x, u)$.

Further, in a neighborhood of $(x = 0, u = 0)$, there exist constants $c_i \geq 0$ where $i = 1, 2, 3, 4$, and at least one $c_i > 0$ (since system (1) is nonlinear), such that

$$\begin{aligned}\|\Gamma_1(x, u)\| &\leq \|x\|^2(c_1\|u\| + c_2\|x\|), \\ \|\Gamma_2(x, u)\| &\leq \|u\|^2(c_3\|u\| + c_4\|x\|).\end{aligned}$$

Consider a neighborhood of $(x = 0, u = 0)$ for which $c_1\|u\| + c_2\|x\| \leq \frac{1}{2}\varepsilon_1$ and $c_3\|u\| + c_4\|x\| \leq \frac{1}{2}\varepsilon_2$. Then, in this neighborhood, we can obtain

$$\begin{aligned}\Upsilon(x, u) &\geq \|x\|^2(\varepsilon_1 - c_1\|u\| - c_2\|x\|) + \|u\|^2(\varepsilon_2 - c_3\|u\| - c_4\|x\|) \\ &\geq \frac{1}{2}\varepsilon_1\|x\|^2 + \frac{1}{2}\varepsilon_2\|u\|^2.\end{aligned}$$

Therefore, the nonlinear system (1) is locally SSIP. ■

Proof of Proposition 1: We prove the result for the continuous-time system (2). The proof for discrete-time system (6) is along similar lines and is omitted here. The proof can be found in [16].

Since system (2) is SP, there exist a constant $\varepsilon > 0$, matrices $P > 0$, L and W , for which $V(z) = \frac{1}{2}z^T Pz$ is a storage function for the system and (see e.g. Lemma 6.4 [4])

$$\dot{V} - v^T w = -\frac{1}{2}(Lz + Wv)^T(Lz + Wv) - \frac{1}{2}\varepsilon z^T Pz. \quad (22)$$

Further, $W^T W > 0$ because we assume that $D + D^T > 0$. For $b \in \mathbb{R}$ such that $0 < b^2 < 1$, we can obtain

$$\begin{aligned}& -\frac{1}{2}(Lz + Wv)^T(Lz + Wv) - \frac{1}{2}\varepsilon z^T Pz \\ &= -\frac{1}{2}\left(\frac{1}{b}Lz + bWv\right)^T\left(\frac{1}{b}Lz + bWv\right) \\ &\quad - \frac{1}{2}z^T \left(\varepsilon P - \left(\frac{1}{b^2} - 1\right)L^T L\right)z - \frac{1}{2}(1 - b^2)v^T W^T Wv \\ &\leq -\frac{1}{2}v^T Q_1 v - \frac{1}{2}z^T Q_2 z\end{aligned} \quad (23)$$

where $Q_1 \triangleq (1 - b^2)W^T W$ and $Q_2 \triangleq \varepsilon P - \left(\frac{1}{b^2} - 1\right)L^T L$. We have $Q_1 > 0$ and $\lambda(Q_1) > 0$. Next, we prove that there exists $b \in \mathbb{R}$ such that $0 < b^2 < 1$ and

$$\lambda(Q_2) = \lambda(P)\varepsilon - \left(\frac{1}{b^2} - 1\right)\bar{\lambda}(L^T L) > 0.$$

Two cases are possible:

- 1) if $\bar{\lambda}(L^T L) = 0$, we obtain $\lambda(Q_2) = \lambda(P)\varepsilon > 0$;
- 2) if $\bar{\lambda}(L^T L) > 0$, we obtain $\lambda(Q_2) > 0$ when b satisfies $0 < \frac{\bar{\lambda}(L^T L)}{\varepsilon\lambda(P) + \bar{\lambda}(L^T L)} < b^2 < 1$.

Thus, we obtain $\lambda(Q_2) > 0$ and $Q_2 > 0$ for appropriate choice of b . Together with (22) and (23), we have shown that there exist $\lambda(Q_1) > 0$ and $\lambda(Q_2) > 0$ for which

$$\dot{V} - v^T w \leq -\frac{1}{2}\lambda(Q_1)v^T v - \frac{1}{2}\lambda(Q_2)z^T z.$$

Therefore, the linear system (2) is SSIP. ■

Proof of Theorem 2: Since the linearized model (2) is SP, there exist a constant $\varepsilon > 0$ and $P = P^T > 0$ such that (see e.g. Lemma 6.4 [4])

$$v^T (Cz + Dv) - z^T P(Az + Bv) \geq \frac{1}{2}\varepsilon z^T Pz.$$

Apply $V(x) = \frac{1}{2}x^T Px$ as a locally defined storage function for system (10). Define $\Upsilon(x, u) \triangleq u^T y - \dot{V}(x)$. From (11), we obtain

$$\begin{aligned}\Upsilon(x, u) &= u^T (\gamma(x) + \zeta(x)u) - \frac{\partial V}{\partial x} (\alpha(x) + \beta(x)u) \\ &= u^T (Cx + Du) - x^T P(Ax + Bu) \\ &\quad + u^T (\tilde{H}(x) + \tilde{M}(x)u) - x^T P(\tilde{F}(x) + \tilde{G}(x)u) \\ &\geq \frac{1}{2}\varepsilon x^T Px + u^T (\tilde{H}(x) + \tilde{M}(x)u) - x^T P(\tilde{F}(x) + \tilde{G}(x)u) \\ &\geq \frac{1}{2}\varepsilon\lambda(P)\|x\|^2 + \Lambda(x, u),\end{aligned}$$

where $\Lambda(x, u) \triangleq u^T (\tilde{H}(x) + \tilde{M}(x)u) - x^T P(\tilde{F}(x) + \tilde{G}(x)u)$. We have the following relation

$$\begin{aligned}|\Lambda(x, u)| &\leq \|x\|\|P\|\|\tilde{F}(x)\| + \|u\|^2\|\tilde{M}(x)\| \\ &\quad + \|u\|\|\tilde{H}(x)\| + \|u\|\|\tilde{G}(x)\|\|P\|\|x\|.\end{aligned}$$

From Taylor's theorem, there exist a constant $c \geq 0$ and a ball around $x = 0$ for which

$$\begin{aligned}\|x\|\|P\|\|\tilde{F}(x)\| + \|u\|\|\tilde{H}(x)\| + \|\tilde{G}(x)\|\|P\|\|x\| \\ \leq c\|x\|^2(\|x\| + \|u\|).\end{aligned} \quad (24)$$

Further, from assumption (18), in a neighborhood of $x = 0$, we obtain

$$\|u\|^2\|\tilde{M}(x)\| \leq \ell\|u\|^2\|x\|^2. \quad (25)$$

Thus, consider a ball around $(x = 0, u = 0)$ such that (24), (25) hold and for some $\theta \in (0, 1)$,

$$c(\|x\| + \|u\|) + \ell\|u\|^2 \leq \frac{1}{2}\theta\varepsilon\lambda(P). \quad (26)$$

In this ball, we obtain $|\Lambda(x, u)| \leq \frac{1}{2}\theta\varepsilon\lambda(P)\|x\|^2$. Thus, we have the following inequality

$$\begin{aligned}\Upsilon(x, u) &\geq \frac{1}{2}\varepsilon\lambda(P)\|x\|^2 - |\Lambda(x, u)| \\ &\geq \frac{1}{2}(1 - \theta)\varepsilon\lambda(P)\|x\|^2.\end{aligned}$$

Therefore, system (10) is SP in a neighborhood of $(x = 0, u = 0)$, i.e. LSP. ■