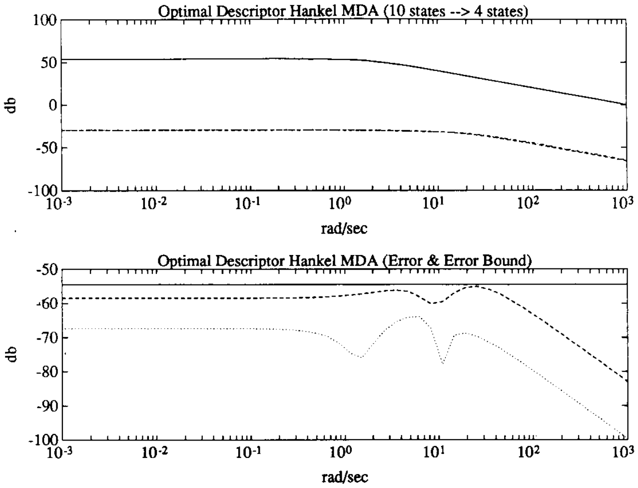


Fig. 1. Optimal Hankel MDA, 10-state MIMO example.

$H(s)$ determined by (10), or equivalently by (46), differs slightly from the $H(s)$ in Glover's Theorem 8.7, including in effect an extra feedback around $K$ which, as shown in (33)–(39), implicitly replaces $K$ by a $\tilde{\tilde{K}}$ which satisfies the Glover constraint $C_2 + K(s)B_2^* = 0$. Consequently, the descriptor representation of Theorem 1 takes the same simple form in both optimal ($\rho = \sigma_{k+1}$) and suboptimal ($\sigma_k > \rho > \sigma_{k+1}$) cases. The price one pays for this increased simplicity is that, in the optimal case $\rho = \sigma_{k+1}$, there is a certain amount of redundancy in the matrix $K(s)$ of Theorem 1, the effective dimension of the matrix $K(s)$ being reduced by the multiplicity of $\sigma_{k+1}$. Of course, in most practical situations, one simply wishes to find the $\hat{G}$ or $\tilde{G}$ corresponding to $K(s) = 0$, and the fact that those $\hat{G}$ and $\tilde{G}$ could also be obtained from other values of $K(s)$ is not an issue.

The numerical superiority of our formulation of the optimal Hankel model reduction results over that of Glover [5] is made transparent by our example. The nonminimal ($\sigma_9, \sigma_{10} = 0$) and nearly nonminimal ($\sigma_8 = 6.7051 \times 10^{-9}$) modes of the example make it numerically infeasible to apply the balancing approach of [5]. Attempting to do so leads inevitably to a computer program crash or very substantial numerical errors. By bypassing the numerically difficult step of balancing, our results make it practical to apply the optimal Hankel model reduction to those systems which stand to benefit the most from model order reduction, namely nonminimal and nearly nonminimal systems.

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# Hidden Modes of Two Degrees of Freedom Systems in Control Design

O. R. GONZÁLEZ AND P. J. ANTSAKLIS

*Abstract*—**A complete treatment of the hidden modes of two degrees of freedom control systems is presented. The uncontrollable and/or unobservable hidden modes are characterized in terms of transfer matrices of the interconnected system and in terms of design parameters. This characterization leads directly to design conditions, which can be used to**

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**adequately control the hidden modes, thus avoiding unnecessarily high order controllers and undesirable behavior. Internal stability is guaranteed via a stability theorem which adds significant insight to the problem.**

## I. Introduction

The hidden modes of a compensated system correspond to the compensated system's eigenvalues which are uncontrollable and/or unobservable from a given input or output, respectively. The hidden modes for single degree of freedom and for particular two degrees of freedom controlled systems have been studied in the literature [1]-[16]. In this note we characterize the hidden modes for the general linear two degrees of freedom controlled system in terms of the frequency domain control design tools: transfer matrices and design parameters. The hidden modes can, of course, be characterized using internal descriptions [4], [6]-[9], and this is useful mainly in the analysis of control systems. In frequency domain control design methods, where transfer matrices and design parameters are used, an internal description characterization of the hidden modes is not as helpful. A characterization of the hidden modes in terms of the design tools, however, leads directly to *design conditions*, which can be used to adequately control the hidden modes in control design, thus avoiding unnecessarily high order controllers and undesirable behavior. It is recognized that if the interconnected system is internally stable, then the hidden modes, if any, will be stable. An example of undesirable behavior is the transient response introduced by a pair of lightly damped eigenvalues that correspond to unobservable modes from the output. From the results in this note, the modes that are unobservable from the output will be observable at the plant input where they could result in saturation problems. In digital systems this is seen as the ringing phenomena reported by Åström in [18]. Another problem with stable hidden modes is reported by Fossard in [26], where an uncontrollable mode with slower dynamics than the controllable and observable modes results in "disappointing behavior in the transient dynamic response."

In this note, a complete treatment of the hidden modes of two degrees of freedom control systems is presented, which extends and unifies several results which have appeared in the literature for particular configurations. The methods used are based on polynomial matrix internal descriptions, however, all results are expressed so that they can be directly used in control design. Internal stability is guaranteed via a stability theorem which adds significant insight to the problem. The method is also used to characterize the hidden modes of a particular two degrees of freedom configuration and could be applied to a class of interconnected systems; for this we use an aggregate system representation [4].

Assuming that the plant and controller are controllable and observable, then the hidden modes of the controlled system are introduced exclusively by the interconnections. Under this assumption, the hidden modes are completely characterized in terms of the transfer matrices and design parameters. The implementation of the controller, $C$, is usually done by interconnecting available subcontrollers, where each subcontroller is designed to handle a particular task such as stability and regulation. Therefore, the resulting controller is not necessarily controllable and observable, and it introduces additional hidden modes; these are also characterized. Proofs of the theorems and lemmas can be found in [19] and [20].

## II. Preliminaries

The study of hidden modes is done starting with internal descriptions of each system. In particular, consider the following polynomial matrix description of the controlled system:

$$\mathcal{P}(s)z(s) = \mathcal{Q}(s)u(s), \qquad y(s) = \mathcal{R}(s)z(s) + \mathcal{W}(s)u(s), \quad (2.1)$$

where $\mathcal{P}(s)$, $\mathcal{Q}(s)$, $\mathcal{R}(s)$, $\mathcal{W}(s)$ are polynomial matrices. For the system described in (2.1), it is well known that the uncontrollable (unobservable) modes from $u(y)$ correspond to the roots of the determinant of a g.c.l.d. of $(\mathcal{P}(s), \mathcal{Q}(s))$ [g.c.r.d. of $(\mathcal{R}(s), \mathcal{P}(s))$] [6], [7], where g.c.l.(r.)d. denotes greatest common left (right) divisor. These uncontrollable and unobservable modes are the hidden modes of the system. In this note we consider the interconnection of systems where each system is completely described by its transfer matrix. The interconnected system is said to be completely characterized by its proper rational transfer matrix if and only if an internal description of the overall system is controllable from the input and observable from the output, that is, if (2.1) is the polynomial matrix description of the overall system, then $(\mathcal{P}(s), \mathcal{Q}(s))$ is left coprime and $(\mathcal{R}(s), \mathcal{P}(s))$ is right coprime. Notice that if the transfer matrix does not completely characterize the interconnected system, then the hidden modes are due exclusively to the interconnections since every interconnected system is assumed to be completely characterized by its transfer matrix.

In classical control design of scalar systems, it is straightforward to characterize the hidden modes in terms of pole-zero cancellations in the products of the transfer functions. In the frequency domain control design of multivariable systems, the hidden modes can also be characterized by considering "pole-zero cancellations" in the products of transfer matrices.[1] In this case, however, the characterization is not as direct mainly due to the fact that "pole-zero cancellations" are not as well defined in the multivariable case, and also because of the difficulty in associating hidden modes with specific cancellations. Results that refer to particular control configurations have been reported in the literature [1]-[3]. In [20], these results have been formalized and extended; they are the basis of the results presented here.

*Cancellations* in products of transfer matrices are not simple extensions of pole-zero cancellations in products of transfer functions. For example, it is possible to have a cancellation where a pole of one transfer matrix does not cancel with a zero of another transfer matrix as in

$$T_1(s)T_2(s) = [1, -1]\begin{bmatrix} \dfrac{2s+3}{(s+1)(s+2)} \\ \dfrac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}. \qquad (2.2)$$

In (2.2), $T_1(s)$ has no zeros and the pole of $T_2$ at $-2$ cancels in $T_1(s)T_2(s)$. Let $\delta T_i(s)$ denote the McMillan degree of $T_i(s)$, $i = 1, 2$. Notice that $\delta T_1(s) = 0$ and $\delta T_2(s) = 2$, while $\delta T_1T_2(s) = 1$. This reduction in the McMillan degree confirms the fact that a pole was cancelled in $T_1(s)T_2(s)$; the pole of $T_2$ at $-2$. This pole corresponds to a hidden mode from the input and/or output, for example, when $T_1(s)T_2(s)$ denotes the transfer matrix of the cascade connection of $T_2(s)$ followed by $T_1(s)$.

It is now clear that cancellations in products of transfer matrices should be taken as *pole cancellations* rather than pole-zero cancellations. It is also important to notice that not every cancelled pole needs to correspond to a hidden mode. A cancelled pole in a product of transfer matrices corresponds to a hidden mode from the input and/or the output if and only if it is a root of the characteristic polynomial of the interconnected system. Additional observations on "multivariable cancellations" can be found in [20], [8]-[12], and [17].

Another way to express the controllability and observability conditions of a polynomial matrix description is given in Lemma 2.1; this lemma is used to characterize the hidden modes in terms of pole cancellations in the products of transfer matrices of the interconnected systems.

*Lemma 2.1:* The system described by (2.1) is controllable from $u$ (observable from $y$) if and only if the McMillan degree of the transfer matrix from $u$ to $z$ ($\mathcal{Q}(s)u(s)$ to $y$) is the same as the degree of $|\mathcal{P}(s)|$.

Lemma 2.1 specifies the products of transfer matrices in which a cancellation may result in a hidden mode; the uncontrollable (unobservable) modes are associated with pole cancellations in $\mathcal{P}(s)^{-1}\mathcal{Q}(s)(\mathcal{R}(s)\mathcal{P}(s)^{-1})$. To characterize the hidden modes in terms of the transfer matrices of the interconnected systems, appropriate transformations are used to map these products into products of transfer matrices. Transformations which yield equivalent polynomial matrix descriptions are used. In particular, we apply transformations that maintain system equivalence in the Rosenbrock sense [6].

## III. Stability Theorem—Parameterizations

The two degrees of freedom linear controller $S_C$ implements the control law $u = C[y', r']' = -C_y y + C_r r$, where $C = [-C_y, C_r]$ as seen in Fig. 1.

[1] For a controllable and observable system, if (2.1) is the polynomial matrix description of the system, then the poles of the system's transfer matrix correspond to the roots of $|\mathcal{P}(s)|$, the characteristic polynomial of the system. The zeros of the transfer matrix are the finite transmission zeros of the system.

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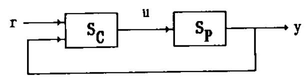


Fig. 1. The controlled system.

$S_P$ is the linear plant described by $y = Pu$ with $P$ its proper transfer matrix. It is assumed that $|I + PC_y| = |I + C_yP| \neq 0$, and that every input-output map is proper. Under these assumptions, the controlled system is said to be internally stable if the inverse of the denominator matrix in a polynomial matrix description is stable. If the controlled system is internally stable, we say that $S_C$ is an internally stabilizing controller for $S_P$.

A significant step toward better understanding the role of $C$ in plant compensation was recently accomplished by parametrically characterizing all internally stabilizing two degrees of freedom controllers $C$, thus extending the results on parametric characterization of all feedback controllers $C_y$ [21]–[25], [1], [2], [20] which have greatly contributed to control design methods. All internally stabilizing controllers $C$ can be parametrically characterized using two independent stable parameters $K$ and $X$ as

$$C = (x_1 - K\tilde{N})^{-1}[-(x_2 + K\tilde{D}), X], \tag{3.1}$$

where $\tilde{N}$, $\tilde{D}$, $x_1$, $x_2$ are polynomial matrices, and they are derived from coprime fractional representations of the plant $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ and the associated Bezout–Diophantine equation $x_1D + x_2N = I$. In (3.1), $K$ must be such that $|x_1 - K\tilde{N}| \neq 0$, and for $C$ proper need $D(x_2 + K\tilde{D})$ proper and $D(x_1 - K\tilde{N})$ biproper ($D(x_1 - K\tilde{N})$ and its inverse proper).

It is evident that if exogenous signals (such as disturbances and noise) are assumed to be injected at various points in Fig. 1, all possible transfer matrices from all inputs can be derived in terms of $K$ and $X$ by direct substitutions of (3.1); in this way, all "admissible" responses, under internal stability, can be characterized.

It is advantageous to study internal stability of the system in Fig. 1 using Theorem 3.1 [10].

*Theorem 3.1:* The compensated system is internally stable if and only if

i) $u = -C_y y$ internally stabilizes the system $y = Pu$, and

ii) $C_r$ is such that $M := (I + C_yP)^{-1}C_r$ satisfies $D^{-1}M = X$, a stable rational, where $C_y$ satisfies i) and $P = ND^{-1}$ is a right coprime polynomial factorization.

Theorem 3.1 separates the role of $C_y$, the feedback part of $C$, and $C_r$ in achieving internal stability. Clearly, if only feedback action is considered, only i) is of interest; and if open loop control is desired, $C_y = 0$, i) implies that $P$ must be stable, and $C_r = M$ must satisfy ii). In ii), the parameter $M (=DX)$ appears rather naturally, and in i) the way is open to use any desired feedback parameterization.

From Theorem 3.1, we can directly characterize the input–output maps attainable from $r$ with internal stability. In particular, consider the command/output and command control response maps described by $y = Tr$ and $u = Mr$, respectively.

*Theorem 3.2:* A pair $(T, M)$ is realizable with internal stability via a two degrees of freedom configuration if and only if $(T, M) = (NX, DX)$ with $X$ stable.

There are many choices in parametrically characterizing all feedback stabilizing controllers $C_y$, and these are extensively discussed by Antsaklis and Sain in [2]. The stabilizing controllers $C$ can therefore be expressed, in addition to (3.1), as, for example,

$$C = (I - QP)^{-1}[-Q, DX] = ((I - LN)D^{-1})^{-1}[-L, X], \tag{3.2}$$

where $Q = DL$, $DX = M$ with $L$, $X$ stable, and $D^{-1}(I - QP) = (I - LN)D^{-1}$ stable ($|I - QP| \neq 0$ or $|I - LN| \neq 0$). Parametric characterizations of all internally stabilizing controllers $C$, proper and nonproper, are given in (3.2). For $C$ proper, $M$ and $Q$ are chosen proper and such that $(I - QP)$ is biproper; note that if $P$ is strictly proper, $Q$ proper always implies that $(I - QP)^{-1}$ is proper. Notice that $L$ or $Q$ in (3.2) must satisfy certain conditions, in addition to being stable, in contrast to $K$ in (3.1); however, alternative to $K$ parameterizations, such as in (3.2), are very useful, since they do have certain additional desirable properties (see [2]).

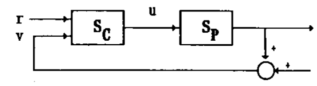


Fig. 2. A two degrees of freedom control system.

The relations between the parameters are

$$L = x_2 + K\tilde{D} = D^{-1}Q$$
$$Q = DL = C_y(I + PC_y)^{-1} = (I + C_yP)^{-1}C_y$$
$$X = (x_1 - K\tilde{N})C_r = D^{-1}M \quad M = DX = (I + C_yP)^{-1}C_r. \tag{3.3}$$

These relations will be useful in Section IV-B where the hidden modes of two degrees of freedom systems are characterized in terms of these parameters.

## IV. Hidden Modes in Two Degrees of Freedom Controlled Systems

In this section, the hidden modes of two degrees of freedom controlled systems, as depicted in Fig. 2, will be studied. In Section IV-A, the hidden modes from given inputs and outputs will be characterized in terms of transfer matrices. This characterization is done when $S_P$ and $S_C$ are completely described by their transfer matrices, and when $S_P$ is completely described by its transfer matrix, but $S_C$ is not. In Section IV-B, the hidden modes are characterized in terms of the design parameters: $K$, $X$, and $L$ when $S_P$ and $S_C$ are completely described by their transfer matrices. Using these characterizations, we then give conditions in terms of the parameters of interest to avoid the introduction of hidden modes. These conditions can be incorporated in the control system design.

### A. Hidden Modes in Terms of I/O Maps

Consider Fig. 2 where the vector of fictitious inputs $\eta_2$ is introduced to help with the interpretation of the uncontrollable hidden modes; the other variables were described in Section III.

First, consider $S_C$ to be completely described by its transfer matrix, that is, $S_C$ is controllable from $[v^t, r^t]^t$ and observable from $u$. A polynomial matrix description for $S_C$ is $\tilde{D}_c z_c = -\tilde{N}_y y + \tilde{N}_r r - \tilde{N}_y \eta_2$, $u = z_c$, where $C = \tilde{D}_c^{-1}[-\tilde{N}_y, \tilde{N}_r]$ is left coprime and $v = \eta_2 + y$. A polynomial matrix description for $S_P$ is $Dz = u$, $y = Nz$, where $(N, D)$ is right coprime. Combining these descriptions gives a polynomial matrix description for the two degrees of freedom controlled system

$$D_o z = \tilde{N}_r r - \tilde{N}_y \eta_2, \quad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} z \tag{4.1.1}$$

where $D_o = \tilde{D}_c D + \tilde{N}_y N$. Since $S_P$ and $S_C$ are assumed to be completely characterized by their transfer matrices, the hidden modes are due exclusively to the interconnection.

A preliminary characterization of the hidden modes follows directly from (4.1.1). The uncontrollable modes from $r\,(\eta_2)$ correspond to the poles of $D_o^{-1}$ that cancel in $D_o^{-1}\tilde{N}_r (D_o^{-1}\tilde{N}_y)$. The unobservable modes from $y(u)$ correspond to the poles of $D_o^{-1}$ that cancel in $ND_o^{-1}$ $(DD_o^{-1})$. This characterization gives insight into the controllability and observability properties of two degrees of freedom systems. For example, notice that the controlled system is observable from $[y^t, u^t]^t$, that is, the unobservable modes from $y$ are observable from $u$ and vice versa.

Notice that even though $S_C$ is completely characterized by $C$, there could be uncontrollable modes from $r$ or from $y$. However, the uncontrollable modes from $r$ of $S_C$ are controllable from $y$ and vice versa. Furthermore, no uncontrollable modes of $S_C$ from $r$ will be uncontrollable from $r$ of the two degrees of freedom controlled system.

Before giving the main result in Theorem 4.1.1, it is useful to characterize the poles of $(I + PC_y)^{-1}$ and of $(I + C_yP)^{-1}$; the characterization is used to determine when a cancellation of poles of $(I + PC_y)^{-1}$ and of $(I + C_yP)^{-1}$ can correspond to a hidden mode from an input and/or output. Notice that only a set containment ($\subset$) condition is given for the poles of $(I + C_yP)^{-1}$.

*Lemma 4.1.1:* The following relations are true.[2]

[2] In Lemmas 4.1.1 and 4.2.1, the definition of a set is extended in an obvious manner to include multiple eigenvalues (e.g., $\{-1, -1\} \neq \{-1\}$).

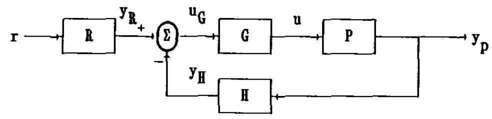


Fig. 3. An $\{R; G, H\}$ controlled system.

i) {poles of $(I + PC_y)^{-1}$} = {{closed-loop eigenvals.} − {uncontrollable eigenvals. from $\eta_2$} − {unobservable eigenvals. from $y$} + {eigenvals. that are uncontrollable from $\eta_2$ and unobservable from $y$}}.

ii) {poles of $(I + C_yP)^{-1}$} $\subset$ {{closed-loop eigenvals.) − {unobservable eigenvals. from $u$} − {roots of $|G_y|$ that do not correspond to unobservable modes from $u$}}.

The hidden modes are determined by considering cancellations in the products of transfer matrices given in Lemma 4.1.2.

*Lemma 4.1.2:* The hidden modes are characterized by considering cancellations in the following products of transfer matrices.

Unobservable modes from $y$: $(I + PC_y)^{-1}P[\tilde{D}_c^{-1}, I]$.

Unobservable modes from $u$: $(I + C_yP)^{-1}[\tilde{D}_c^{-1}, I]$.

Uncontrollable modes from $r$: $\begin{bmatrix} I \\ P \end{bmatrix}(I + C_yP)^{-1}\tilde{D}_c^{-1}\tilde{N}_r$.

The main result when $S_P$ and $S_C$ are completely characterized by their transfer matrices is given next.

*Theorem 4.1.1:* The hidden modes of a two degrees of freedom control system are characterized as follows. The unobservable modes from $y(u)$ correspond to the poles of $C(P)$ that cancel in $PC(\tilde{N}_yP)$. The uncontrollable modes from $r$ correspond to the poles of $(I + C_yP)^{-1}$ that cancel in $(I + C_yP)^{-1}C_r$, and to the poles of $P$ that cancel in both $PM$ and $C_yP$.

The next corollary specializes the conditions in Theorem 4.1.1 for a particular two degrees of freedom configuration.

*Corollary 4.1.1:* In the error feedback configuration ($C_r = C_y$), the unobservable modes from $y(u)$ correspond to poles of $C_y(P)$ that cancel in $PC_y(C_yP)$. The uncontrollable modes from $r$ correspond to the poles of $P$ that cancel in $PC_y$.

*Remark 4.1.1:* The conditions in Theorem 4.1.1 for unobservable modes from $y$ can be written in terms of $C_y$ or $C_r$ in the following cases. If $(\tilde{D}_c, \tilde{N}_r)$ is l.c., then the poles of $C_r$ that cancel in $PC_r$ correspond to unobservable modes from $y$. If $(\tilde{D}_c, \tilde{N}_y)$ is l.c., then the poles of $C_y$ that cancel in $PC_y$ correspond to unobservable modes from $y$.

Theorem 4.1.1 characterizes the hidden modes introduced by the interconnection of systems in Fig. 2. It is also of interest to characterize the hidden modes when the controller is not completely characterized by its transfer matrix. The conditions for the general case of uncontrollability and unobservability, which requires another internal description of $S_C$, are given in Theorem 4.1.2.

*Theorem 4.1.2:* The uncontrollable modes of $S_C$ from $[v', r']'$ will be uncontrollable from $r$, and the unobservable modes of $S_C$ from $u$ will be unobservable from $y$.

Theorem 4.1.2 demonstrates that when $S_C$ is not completely described by $C$, the two degrees of freedom control system considered here maintains the hidden modes of $S_C$ only from appropriate inputs and outputs. Furthermore, additional hidden modes are introduced because of the interconnection. Similar results follow directly when $S_P$ is not completely characterized by $P$ and $S_C$ is completely characterized by $C$. If both $S_P$ and $S_C$ are not completely characterized by their transfer matrices, then the characterization of hidden modes can also be obtained using the method described here.

### B. Hidden Modes in Terms of Design Parameters

The results in the last subsection could be used to give conditions to avoid the introduction of hidden modes, but they would not be simple to implement in control design. In this subsection, the hidden modes will be characterized in terms of the parameters utilized in the design of a control system. This characterization leads directly to design conditions to avoid unnecessary hidden modes. In particular, the hidden modes will be characterized in terms of $K$, $X$, and $L$, which were used in Section III to parameterize the internally stabilizing controllers.

In Lemma 4.2.1, the poles of the parameters of interest are characterized. First, let $G_y$ be a g.c.l.d. of $(\tilde{D}_c, \tilde{N}_y)$, and let $G_P$ be a g.c.l.d. of $(\tilde{D}_k, \tilde{N}_{C_y})$, where $K = \tilde{D}_k^{-1}\tilde{N}_k$ is l.c. It is shown in [20] that the poles of $G_p^{-1}$ are poles of $P$ that cancel in $PC_y$. The poles of $G_p^{-1}$ are also given by the poles of $P$ that do not cancel in $(I - LN)D^{-1}$.

*Lemma 4.2.1:* The following relations are true.

i) {poles of $K$} = {{closed-loop eigenvals.} − {roots of $|G_y|$}}

ii) {poles of $L$} = {{closed-loop eigenvals.} − {uncontrollable eigenvals. from $\eta_2$}}

iii) {poles of $X$} = {{closed-loop eigenvals.} − {uncontrollable eigenvals. from $r$}}.

The characterization of hidden modes in terms of the design parameters is given next.

*Theorem 4.2.1:* The unobservable modes from $y$ correspond to the poles of $[X, L]$ that cancel in $N[X, L]$. The poles of $[X, L]$ that cancel in $D[X, L]$ and the poles of $P$ (in $G_p^{-1}$) that cancel in $D_o^{-1}[\tilde{N}_y, \tilde{N}_r]$ correspond to unobservable modes from $u$. The uncontrollable modes from $r$ correspond to the poles of $P$ in $G_p^{-1}$ and to the poles of $L$ that are not poles of $X$.

The next corollary specializes the conditions in Theorem 4.2.1 to the error feedback configuration ($C_y = C_r$); the results agree with known results in [2].

*Corollary 4.2.1:* When $C_y = C_r$, the unobservable modes from $u$ correspond to the poles of $L(L = X)$ which cancel in $NL$. The uncontrollable modes from $r$ correspond to the poles of $P$ that do not cancel in $(I - LN)D^{-1}$, that is, the poles of $G_p^{-1}$.

The final result of this section gives the design conditions that can be used to avoid unnecessary hidden modes. Notice that these conditions could be used the other way around when it is desirable to introduce a cancellation that does not affect internal stability. In control design, additional hidden modes are usually required when specifications are given on the command/output ($T$) and command/control ($M$) response maps, and on the output disturbance sensitivity matrix ($S = (I + PC_y)^{-1}$). These conditions follow directly from Theorem 4.2.1.

*Design Conditions for No Hidden Modes:* To avoid unobservable modes from $y$ do not choose poles of $[X, L]$ that cancel in $N[X, L]$. To avoid uncontrollable modes from $r$, make all the poles of $L$ and the poles of $P$ in $G_p^{-1}$ poles of $X$. To avoid unobservable modes from $u$, do not choose poles of $[X, L]$ as poles of $P$.

The characterization of hidden modes in terms of transfer matrices of a particular system interconnection can be done starting with the results in Section IV-A. However, it is usually simpler to apply Lemma 2.1 to a polynomial matrix description of the interconnected system. For illustration, the following example characterizes the hidden modes of the $\{R; G, H\}$ controlled system in Fig. 3.

*Example:* Consider Fig. 3 where the interconnected systems are completely described by their transfer matrices $P$, $R$, $H$, and $G$. The $\{R; G, H\}$ controller is an implementation of a two degrees of freedom compensator, where $C_y = GH$ and $C_r = GR$.

The hidden modes are characterized as follows. The unobservable modes from $y$ correspond to the poles of $G$, $H$, and $R$ that cancel in $PG$, $(PG)H$, and $(I + PGH)^{-1}(PG)R$, respectively. The unobservable modes from $u$ correspond to the poles of $H$, $P$, and $R$ that cancel in $GH$, $(GH)P$, and $(I + GHP)^{-1}GR$, respectively. The uncontrollable modes from $r$ correspond to the poles of $G$ that cancel in $G[HP, R]$; the poles of $P$ that cancel in $HPG$ and in $PM$; the poles of $H$ that cancel in $HPG$; and the poles of $(I + GHP)^{-1}$ that cancel in $(I + GHP)^{-1}GR$. These characterizations extend and simplify the results originally presented in [5].

## V. Conclusions

The results presented here on the hidden modes of two degrees of freedom control systems in terms of transfer matrices and design parameters extend and unify the results in the literature. The emphasis here was in control design. The results and the methodology presented are not limited to the applications shown, but they can be applied to a class of interconnected systems where the study of the hidden modes introduced by the interconnections is of interest.

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# On the Analysis of Discrete Linear Time-Invariant Singular Systems

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*Abstract*—**The discrete singular equation over an interval can represent a two-point boundary-value problem or it can be considered as a dynamical relation developing forward in time. Here, we provide a theory that encompasses both interpretations by giving analytic solutions and discussing system properties in both cases. Of fundamental importance in our approach is the *relative fundamental matrix*, which generalizes the notion of the state-transition matrix.**

## I. Introduction

Although some work has been done on the analysis of discrete singular systems [1]–[3], [11]–[20], there seems to be some confusion when trying to interpret the results in an overall framework. This is because, associated with a singular system defined over an interval, there are actually several *distinct* problems of interest. The discrete singular relation is subject to *several different interpretations*.

In [1], [11]–[16], [19], [20] the two-point boundary-value problem was investigated. That is, given the input sequence and the initial and final values of the semistate it is desired to find the intermediate semistate values. We call this the *symmetric solution*. In [3], the *forward* solution was investigated. In this case it is desired to find the semistate sequence given the inputs and the initial semistate value. Complementary to this solution is the *backward* solution, where the inputs and the final semistate value are prescribed.

In this note we attempt to provide some unification between these different interpretations by deriving analytic solutions for the forward, backward, and symmetric case. We also briefly discuss system properties, making the point that reachability and observability are different depending on how the discrete singular equations are interpreted.

For the state-space equation $\dot{x} = Ax$ an analysis may be accomplished in terms of $A$. It is well known that in the singular case $E\dot{x} = Ax$ an analysis in terms of $E$ and $A$ is not possible. Auxiliary quantities that have been used in the analysis of these systems have included the Drazin inverse of a related matrix [3], [5], the transformations to Weierstrass form [1], [6], [24] [equivalently, the eigenvectors of $(zE - A)$], and deflating subspaces [13]. Other approaches, algorithmic in nature, are also important [14]–[16].

Here, we make it clear that in the singular case a complete analysis of solutions and properties in the forward, backward, and symmetric cases is possible in terms of $E$, $A$, $\phi_0$, and $\phi_{-1}$, with $\phi_k$ the *fundamental matrix*.

One of our main goals is to show the fundamental importance in the analysis of discrete singular systems of the fundamental matrix sequence $\phi_k$, which represents the $n \times n$ matrix coefficient sequence of the Laurent expansion about infinity of the resolvent matrix $(zE - A)^{-1}$. In [19], [20] an analysis of the symmetric solution was performed after a preliminary conversion to a system in which $E$ and $A$ commute. In that case the analysis is possible only in terms of $E$ and $A$. However, we choose not to take this approach since it obscures the importance of the fundamental matrix, which does not explicitly appear in [19], [20].

## II. Fundamental Matrix

Subspaces shall be denoted by boldface and superscript "$-1$" shall denote the inverse image of a linear operator, or the usual inverse, if it exists, of its matrix representation.

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