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INTERNAL MODELS IN REGULATION, STABILIZATION, AND TRACKING

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ABSTRACT

The internal model concept is fundamental in control problems. In linear control systems, internal models have been described in terms of poles in the unstable (bad) region of the complex plane which contain the needed information for the control system to attain the desired objective. It has been erroneously believed for some time that such internal models did not exist in the nonrobust regulation problem when a generalized plant was considered. It is shown here that such internal models always exist not only in regulation, but also in stabilization, and tracking if the appropriate physically meaningful maps are considered, thus completely resolving the existing discrepancy between abstract results and intuition on one hand, and linear regulation literature results on the other. Both robust and nonrobust control problems are considered and a complete treatment of the internal models in all basic problems is presented.

I. INTRODUCTION

The concept of internal models is very general and it is applicable in diverse fields such as epistemology, neurology, psychology, and artificial intelligence [26]. Intuitively, in control theory, the internal model concept can be explained as follows: for acceptable performance, a system needs to have "enough" information about the conditions under which it has to perform. For example, if a system needs to perform under the effects of undesirable exogenous signals, then acceptable performance is possible if a copy of a model of the dynamics of the exosystem generating the exogenous signals is present in the compensated system. This copy is called an internal model, which provides the necessary information to counteract the effects of the undesirable behavior. Currently, the role of internal models in the regulation of linear, lumped, time-invariant, continuous, and discrete time systems is well understood. Furthermore, recent work has shown the existence of internal models for a class of systems that are described over rings (linear, lumped, and distributed) as well as for a class of nonlinear systems in the solution of the robust regulation and robust asymptotic tracking problems [1,2,7,27,28]. Also, internal models are present in the solution of the nonrobust asymptotic tracking problem in linear time varying discrete time systems [31]. Both robust and nonrobust control problems have been considered in the literature of internal models with emphasis on the robust case. In the former case, the internal model is a property of the controller, while in the latter case, it is a property of the cascade connection of the controller and plant.

In the literature, it is generally believed that in the regulation of a fixed generalized (2–input, 2–output) plant, where the measured and controlled variables are not necessarily the same, internal models do not exist. In this paper we show that internal models do exist but they manifest themselves in the compensated system in ways that have not been considered up to now. In particular, *we show, using a description of systems over rings, that there is always a map containing the internal model, not only in the solution of regulation, but also in stabilization, and asymptotic tracking problems, even when robustness is not required.* This map may appear explicitly in the compensated system, but this is not necessary. In particular, the map between the regulated variables and the plant's output contains the internal model. In this sense, an internal model is always present in the solution of most control problems, contrary to current beliefs. These results are very appealing in control theory because they formalize the intuitive concept of internal models and explicitly verify their existence in all basic control problems.

The detailed original contributions of this paper are to show the existence of internal models in the following control problems when robustness is not required: regulation; asymptotic tracking using one and two degrees of freedom controllers; and internal stability. In the first two cases, the measured and regulated variables do not coincide. For completeness, a brief outline of known results when robustness is required is also included.

In Section II, we briefly discuss internal models; present definitions of internal models over a desirable ring, $\mathbb{R}_g(s)$; and introduce a set of maps that contain the internal model of the transfer matrix of the exosystem in a generic formulation of the regulation problem. In Section III, the existence of internal models in the regulation of plants, where the measured and controlled variables are not necessarily the same, is established. To this effect, two sets of solvability conditions are derived, a parameterization of controllers is presented, and the structure of the controller that solves RPIS over $\mathbb{R}_g(s)$ is characterized. This section is an extension of the work presented in [3,4,19], giving simpler conditions for the solvability of RPIS over $\mathbb{R}_g(s)$ than other recent ones in [5] and [6]. Furthermore, our approach, which characterizes the controller's structure, provides a direct treatment of internal models. In Section IV, the asymptotic tracking problem over $\mathbb{R}_g(s)$ is considered. We show that there exists a map that always contains the internal model of the transfer matrix of the exosystem. In Section V, the presence of internal models in internally stable systems is established. The full version of this paper in [32] contains two appendices; Appendix A analyzes $\mathbb{R}_g$–stability and Appendix B presents some of the proofs (the rest of the proofs can be found in [9]).

II. INTERNAL MODELS

The study of internal models in multivariable systems started in the early 1970's (for example, see [12–15]). In these papers, the researchers investigated the necessary controller structure required to achieve robust regulation with internal stability. The main result is known as the Internal Model Principle (IMP). The IMP states that the robust regulation problem with internal stability is solvable if and only if feedback of the controlled variables is used and the controller includes a replication of the exogenous system dynamics in its denominator. The IMP is implicit in the results presented in [16].

The role of internal models in the regulation problem with internal stability (RPIS) is to make the undesirable modes of the exosystem unobservable from the variables to be regulated. In this way, the exogenous signals will not affect the output variables of interest. In terms of multivariable zeros and poles, this is equivalent to say that appropriate transmission zeros, corresponding to the poles of the exosystem's transfer matrix, are introduced in the map between the variables to be regulated and the point of injection of the exogenous signal. When the controlled (y_c) and measured (y_m) variables coincide, then it is known that the synthesis of such a controller should be such that it introduces in the feedback loop an appropriate model of the dynamic structure of the exosystem which must make itself present at the injection point of the exogenous signals [21,23,7,9].

In 1977, a characterization of internal models in the frequency domain appeared in [17,18]. In particular, Bengtsson, in [17], gave a definition of internal models without the robustness requirement. In this case the internal model is a property of the loop gain, that is, the transfer function matrix of the cascade connection of the plant and controller. In this way, the regulation problem is solved utilizing any available structure in the plant.

Before introducing the internal model definitions, the algebraic structure and notation are defined. Factorizations of transfer function matrices over a desirable ring, $\mathbb{R}_g(s)$, are used to represent the systems, that is, a given transfer function matrix is modeled as the *ratio* of two rational matrices with entries in $\mathbb{R}_g(s)$. Let $\mathbb{R}_g(s)$ be a nonempty subset of $\mathbb{R}_p(s)$, the ring of proper rational functions with real coefficients, consisting of the proper rational functions that have all their poles in S_g. S_g corresponds to the *good* region of the complex plane; S_g is symmetric with respect to the real axis and contains at least one real point. For a description of the properties of $\mathbb{R}_g(s)$ see [8,9].

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Let $\mathbf{M}(\mathbb{R}_g(s))$ denote the set of all matrices with entries in $\mathbb{R}_g(s)$, regardless of dimensions. The background to develop the theory can be found in [1,2,10,11]. We will develop the theory in the context of linear, time–invariant, continuous and discrete systems, but it can be easily extended to consider other systems (for the appropriate algebraic tools see [1,2,10,11]).

The internal model part of our work builds on the results presented in [17]. The following definition can be considered to be an extension of the internal model definition given in [17].

Definition 2.1 Let R(s), V(s) be arbitrary proper rational matrices with the same number of rows. Let $R=\tilde{Q}'_r{}^{-1}\tilde{P}'_r$ and $V=\tilde{Q}'_v{}^{-1}\tilde{P}'_v$ be left coprime (l.c.) over $\mathbb{R}_g(s)$ where $\tilde{Q}'_r$, $\tilde{P}'_r$, $\tilde{Q}'_v$, $\tilde{P}'_v \in \mathbf{M}(\mathbb{R}_g(s))$, and $\tilde{Q}'_r$ and $\tilde{Q}'_v$ are square, nonsingular and biproper. Then R(s) contains an internal model of V(s) if $\tilde{Q}'_r\tilde{Q}'_v{}^{-1} \in \mathbf{M}(\mathbb{R}_g(s))$.

From the definition we see that R(s) contains an internal model of V(s) if and only if $\tilde{Q}'_r = \tilde{D}'_r\tilde{Q}'_v$ where $\tilde{D}'_r \in \mathbf{M}(\mathbb{R}_g(s))$, that is, R(s) contains a copy of the *bad* poles (in $\Omega = \mathbb{C}\backslash S_g$) of V(s) with appropriate structure, in the form of a right divisor of its denominator matrix. A particular form of an internal model that is useful in the solution of robust problems is described in Section 3.2.

A second definition of internal models when the transfer function matrices have the same number of columns is given below.

Definition 2.2 Let R(s) and V(s) be arbitrary proper rational matrices with the same number of columns. Let $R=P'_rQ'_r{}^{-1}$ and $V=P'_vQ'_v{}^{-1}$ be right coprime (r.c.) over $\mathbb{R}_g(s)$. Then R(s) contains an internal model of V(s) if and only if $Q'_v{}^{-1}Q'_r \in \mathbf{M}(\mathbb{R}_g(s))$, that is, Q'_v is a left divisor of Q'_r.

It is useful to consider now a particular generic formulation of the regulation problem. Its solution which is utilized in the following sections provides clear insight into the mechanism of regulation and the role of internal models. Consider the analysis of a compensated linear, lumped, time–invariant control system, where the controller is such that the system is $\mathbb{R}_g$–stable (the $\mathbb{R}_g$–stability problem is to place all the system's eigenvalues in S_g, a desired region of the complex plane; see Appendix A [32]). Acting upon this system there are some undesirable exogenous signals. Let w be the vector containing the exogenous variables. It is assumed that w can be modeled as the output of a causal, linear, lumped, time–invariant, system described by $w=T_{wd}d$, where d is a bounded vector, and T_{wd} is antistable, that is, all the poles of T_{wd} are in Ω. One interpretation for d is as the vector of initial conditions of the exosystem. No assumptions are made on the structure of the controller at this time, except that open-loop controllers are allowed only with stable plants with no uncertainty.

Now, let A be the characterization of maps attainable with $\mathbb{R}_g$–stability from w to y_c, that is,

$$y_c = Aw \tag{2.1}$$

where $A \in M(\mathbb{R}_g(s))$, y_c is the vector of variables to be regulated. The regulation problem over $\mathbb{R}_g(s)$ is to make $AT_{wd} \in M(\mathbb{R}_g(s))$. It is well known, that if this problem is solved that appropriate transmission zeros will be introduced in A [23]. One interpretation for the transmission zeros in terms of internal models is given in Theorem 2.1.

Theorem 2.1. The regulation problem is solved if and only if $A = \hat{A}A_d$, where $FA_d{}^{-1}$ contains an internal model of T_{wd}, $F,\hat{A},A_d \in M(\mathbb{R}_g(s))$, and (F,A_d) a right coprime (r.c.) $\mathbb{R}_g$–factorization.

Theorem 2.1 characterizes a set of maps, $FA_d{}^{-1}$, that contain an internal model of the exosystem. In the solution of the problems studied in this paper, there is a map containing the internal model explicitly in the loop (open or closed, depending upon the controller structure). But, there are problems in which no map containing the internal model appears explicitly in the loop; nevertheless, the problem is solvable as shown in the following sections.

The conditions in Theorem 2.1 are valid for both the robust and nonrobust versions of the regulation problem. In robust regulation, parameter perturbations in the plant and in part of the controller are such that the controlled system remains $\mathbb{R}_g$–stable. In this case, the internal model usually appears in the controller, and it is suitably reduplicated (see Section 3.2).

Theorem 2.1 generates a large class of maps which contains an internal model of the exosystem. Some of these maps are of particular significance in the control system. One way to select such maps that contain the internal model and at the same time have physical significance is as follows. Assume that A is square and nonsingular, $B \in M(\mathbb{R}_g(s))$, and (B,A) r.c. $\mathbb{R}_g$–factorizations. Since (B–A,A) is also r.c. then

$$(B - A)A^{-1} \tag{2.6}$$

contains an internal model of T_{wd}. Appropriate choices for A and B in (2.6) will be used in the following sections to form the map between the regulated variables and the output of the plant. It will be shown that this map contains an internal model of the transfer matrix of the exosystem.

III. RPIS AND INTERNAL MODELS

In this section the regulation problem with internal stability (RPIS) is considered. The problem is formulated in the next subsection. Then the robust and nonrobust problems are analyzed. In Section 3.3.2 it is shown that there is always a map that contains the internal model even when the controlled and measured variables do not coincide and robustness is not required

3.1. Problem Formulation

Consider the system $\Sigma(S_p,S_c)$ in Figure 1,

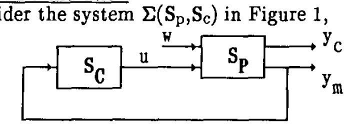

Figure 1. The compensated system $\Sigma(S_p,S_c)$.

where S_p and S_c denote the plant and controller, respectively. For a recent overview of the regulation problem with internal stability for $\Sigma(S_p,S_c)$ see [19]. Assume that S_p and S_c are controllable and observable. Let an input–output description of the plant be

$$\begin{bmatrix} y_m \\ y_c \end{bmatrix} = P \begin{bmatrix} u \\ w \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{3.1.1}$$

where $P_{ij} \in \mathbf{M}(\mathbb{R}_p(s))$; the vectors y_c, y_m contain the variables of the plant to be controlled and the ones that are measured, respectively; the vector w contains all the variables that affect the plant, but are not manipulated by the controller (for example, nonmeasurable disturbances and initial conditions); and u is the vector of control inputs. This general plant model is used because it unifies the study of plants where the controlled and measured variables are not necessarily the same ($y_c \neq y_m$), and where an exogenous signal w is present. Let the control u be given by $u=-C_yy_m$. Also, assume that the controlled system $\Sigma(S_p,S_c)$ is well–defined, that is, $|I+P_{11}C_y|=|I+C_yP_{11}|\neq 0$, and that all input output maps are proper. As in Section II it is assumed that w is the output of a linear system with input–output relation given by $w=T_{wd}d$, where d is a bounded vector and T_{wd} is antistable.

We call RPIS over $\mathbb{R}_g(s)$ the problem of finding a causal, linear controller S_c that makes $T_{cd} \in \mathbf{M}(\mathbb{R}_g(s))$, and $\Sigma(S_p,S_c)$ $\mathbb{R}_g$–stable. A controller that solves this problem is usually called a regulator.

The generalized plant representation can be simplified when a particular relation between y_c and y_m is known or needs to be established. A simplified plant representation that includes most common relations between y_c and y_m was presented in [25] and is considered in the next example. This plant representation is used to gain additional insight in Sections 3.2 and 3.3.

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Example 1. Consider the system represented in Figure 2.

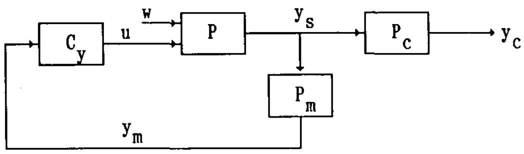

Figure 2. A representation of plants when $y_c \neq y_m$.

An input-output description of this system is given by

$$y_s = P\begin{bmatrix} u \\ w \end{bmatrix}, \quad P = [P_u, \quad P_w]$$

$$\begin{bmatrix} y_m \\ y_c \end{bmatrix} = \begin{bmatrix} P_m P_u & P_m P_w \\ P_c P_u & P_c P_w \end{bmatrix} \tag{3.1.2}$$

and
$$u = -C_y y_m. \tag{3.1.3}$$

Some typical special cases are obtained as follows. If $P_c=P_m$, then $y_m=y_c$. In particular, the classical unity feedback system with disturbances at the output of the plant corresponds to $P_c=P_m=P_w=I$; and the classical plant-sensor configuration with disturbances at the output of the plant corresponds to $P_c=P_w=I$, $P_m=H_s$. Two other special cases are: if $P_m=H_sP_c$, then $y_m=H_sy_c$, and if $P_c=\hat{Q}_mP_m$, then $y_c=\hat{Q}_my_m$. In the latter case, it is said that the controlled variables are *readable* from the measured variables.

We make the following assumptions: $Q_m=P_cP_m^{-1}$, $Q_m^{+}=P_mP_c^{+} \in M(\mathbb{R}_g(s))$, where P_m^{-1} exists and $P_cP_c^{+}=I$. ◻

3.2. Robust RPIS

By robust RPIS over $\mathbb{R}_g(s)$, we understand that parameter variations in the plant and in part of the controller are such that the compensated system remains $\mathbb{R}_g$–stable and the controller should be designed so that regulation over $\mathbb{R}_g(s)$ is maintained. Robust RPIS of the system in Figure 1 is well understood and the necessary conditions have been given in terms of the Internal Model Principle (IMP) [15]. There are two parts to the IMP. The first part characterizes the kind of systems that admit robust RPIS over $\mathbb{R}_g(s)$. The systems must be such that y_c is readable from y_m [13], in the sense that there exists $\hat{Q}'_m \in M(\mathbb{R}_g(s))$ so that

$$y_c = \hat{Q}'_m y_m. \tag{3.2.1}$$

If this condition is not satisfied, then robust RPIS over $\mathbb{R}_g(s)$ is not possible. Notice that the condition in (3.2.1) is satisfied by the system in Figure 2, where $\hat{Q}'_m=Q_m=P_cP_m^{-1}$. In particular, the special cases considered in Example 1 admit a robust regulator.

The second part of the IMP gives a way to synthesize the controllers that attain robust RPIS over $\mathbb{R}_g(s)$. Let $\Phi(s)$ be the largest invariant factor, that is, the one divisible by all the other invariant factors, of $\tilde{D}'_w$ then the controller that solves the robust RPIS over $\mathbb{R}_g(s)$ *must be of the form*

$$C_y = \hat{C}_y(\Phi I)^{-1} \tag{3.2.2}$$

where $\hat{C}_y$ must internally stabilize $P_{11}(\Phi I)^{-1}$. Because of $\Phi(s)I$, C_y contains an internal model of T_{wd}. This internal model can commute with matrices, so that it can be present at the injection point where the exogenous signals come in

3.3 Nonrobust RPIS

3.3.1 Solvability Conditions

First, consider the special case when $y_c=y_m$, then $y_c=[P_{11}, \quad P_{12}][u^t, \ w^t]^t$. In this case, $T_{cw} = (I + P_{11}C_y)^{-1}P_{12}$, and the regulation condition is satisfied when

$$T_{cd} = (I + P_{11}C_y)^{-1}P_{12}T_{wd} \in M(\mathbb{R}_g(s)). \tag{3.3.1}$$

The following definitions are needed to derive the solvability conditions. Let

$$P_{11} = N'_1D'^{-1}_1 = \tilde{D}'^{-1}_1\tilde{N}'_1, \text{ and } C_y = \tilde{D}'^{-1}_c\tilde{N}'_c \tag{3.3.2}$$

be coprime over $\mathbb{R}_g(s)$, and let a doubly coprime factorization of P_{11} be

$$U'\,U'^{-1} = \begin{bmatrix} x'_1 & x'_2 \\ -\tilde{N}'_1 & \tilde{D}'_1 \end{bmatrix}\begin{bmatrix} D'_1 & -\tilde{x}'_2 \\ N'_1 & \tilde{x}'_1 \end{bmatrix}, \tag{3.3.3}$$

where x'_1, x'_2, $\tilde{x}'_1$, $\tilde{x}'_2 \in M(\mathbb{R}_g(s))$ [1,22]. In order to combine both the regulation and $\mathbb{R}_g$–stability requirements, substitute the characterization of all $\mathbb{R}_g$–stabilizing controllers C_y in (3.3.1). The characterization is given in (A4) (in Appendix A). The substitution gives

$$T_{cd} = (\tilde{x}'_1 - N'_1K')\tilde{D}'_1P_{12}T_{wd} \tag{3.3.4}$$

So RPIS over $\mathbb{R}_g(s)$ *is solvable if and only if there exists* $K' \in M(\mathbb{R}_g(s))$ such that (3.3.2) is satisfied. The solvability condition is given in Theorem 3.3.1.

Theorem 3.3.1. RPIS over $\mathbb{R}_g(s)$ for the system with $y_c=y_m$ is solvable if and only if there exist $e'_1, e'_2 \in M(\mathbb{R}_g(s))$ satisfying

$$\tilde{N}'_1e'_1 + e'_2\tilde{D}'_{2w} = I. \tag{3.3.5}$$

This is a direct extension of the results in [17], the proof is given in [9].

When $y_c \neq y_m$, the characterization of stabilizing controllers in (A4) [32] is also used to combine the regulation and $\mathbb{R}_g$–stability conditions into one. First, note that

$$T_{cw} = P_{22} - P_{21}C_y(I + P_{11}C_y)^{-1}P_{12} \tag{3.3.6}$$

and substituting

$$P'_1 := P_{21}D'_1, \; P'_2 := \tilde{D}'_1P_{12}, \; P'_3 := P_{22} - P_{21}D'_1x'_2P_{12} \in M(\mathbb{R}_g(s)).$$

and (A4) [32] into (3.3.6) gives

$$T_{cw} = P'_3 - P'_1K'P'_2. \tag{3.3.7}$$

The regulation over $\mathbb{R}_g(s)$ condition is satisfied if and only if there exists $K' \in M(\mathbb{R}_g(s))$ so that $T_{cd}=T_{cw}T_{wd} \in M(\mathbb{R}_g(s))$, which can be written using (3.3.7) as

$$T_{cd} = (P'_3 - P'_1K'P'_2)\tilde{D}'^{-1}_w\tilde{N}'_w \in M(\mathbb{R}_g(s)), \tag{3.3.8}$$

where $T_{wd}=\tilde{D}'^{-1}_w\tilde{N}'_w$ is l.c. over $\mathbb{R}_g(s)$. So **RPIS over $\mathbb{R}_g(s)$** is solvable if and only if there exists $K' \in M(\mathbb{R}_g(s))$ such that (3.3.8) is satisfied. Let

$$P'_2\tilde{D}'^{-1}_w = N'_{2w}D'^{-1}_{2w} = \tilde{D}'^{-1}_{2w}\tilde{N}'_{2w}$$

and
$$P'_3\tilde{D}'^{-1}_w = N'_{3w}D'^{-1}_{3w}, \tag{3.3.9}$$

where (N'_{2w}, D'_{2w}), $(\tilde{D}'_{2w}, \tilde{N}'_{2w})$, (N'_{3w}, D'_{3w}) are coprime over $\mathbb{R}_g(s)$. Then there exist $x'_{1w}, x'_{2w} \in M(\mathbb{R}_g(s))$, satisfying

$$x'_{1w}D'_{2w} + x'_{2w}N'_{2w} = I. \tag{3.3.10}$$

The solvability conditions are given in Theorem 3.3.2.

Theorem 3.3.2. RPIS over $\mathbb{R}_g(s)$ for the system $\Sigma(S_p,S_c)$ when $y_c \neq y_m$ *is solvable if and only if either (i) or (ii) is satisfied.*

(i) There exist $K', V' \in M(\mathbb{R}_g(s))$ so that

$$V'\tilde{D}'_w + P'_1K'P'_2 = P'_3 \tag{3.3.11}$$

(ii) (a) $D'_{1w} \in M(\mathbb{R}_g(s))$, where $D'_{1w}=D'^{-1}_{3w}D'_{2w}$ (3.3.12)

and (b) there exist $K', R' \in M(\mathbb{R}_g(s))$ so that

$$P'_1K' - R'\tilde{D}'_{2w} = N'_{3w}D'_{1w}x'_{2w}. \tag{3.3.13}$$

The theorem is proven in [9]. A similar condition to the one in (3.3.11) has appeared in [1], for the case when $y_c=y_m$. The conditions in (ii) demonstrate once again that the solvability condition of the regulation problem with internal stability depends on the solution of a *skew–prime* equation.

The solvability conditions in Theorem 3.3.2 can be simplified for the system in Figure 2. First, let

$$P_cP_u = N'_{cu}D'^{-1}_{cu} = \tilde{D}'^{-1}_{cu}\tilde{N}'_{cu} \tag{3.3.14}$$

be coprime factorizations,

$$U'_{cu}\,U'^{-1}_{cu} = \begin{bmatrix} x'_{1cu} & x'_{2cu} \\ -\tilde{N}'_{cu} & \tilde{D}'_{cu} \end{bmatrix}\begin{bmatrix} D'_{cu} & -\tilde{x}'_{2cu} \\ N'_{cu} & \tilde{x}'_{1cu} \end{bmatrix}, \tag{3.3.15}$$

be a doubly coprime factorization, and

$$\begin{aligned}\hat{C}_y = C_yQ_m^{+} &= (x'_{1_{cu}} - K'_{cu}\tilde{N}'_{cu})^{-1}(x'_{2_{cu}} + K'_{cu}\tilde{D}'_{cu}) \\ &= (\tilde{x}'_{2_{cu}} + D'_{cu}K'_{cu})(\tilde{x}'_{1_{cu}} - N'_{cu}K'_{cu})^{-1}\end{aligned} \tag{3.3.16}$$

be a characterization of all $\hat{C}_y$ that internally stabilize P_cP_u.

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Corollary 3.3.2.1. RPIS over $R_g(s)$ for the system in Figure 2 is solvable if and only if there exist $c'_1, c'_2 \in M(\mathbb{R}_g(s))$ satisfying

$$\tilde{N}'_{cu}c'_1 + c'_2\tilde{D}'_{3w} = I, \qquad (3.3.17)$$

where $\tilde{D}'_{cu}P_cP_w\tilde{D}'_w{}^{-1} = \tilde{D}'_{3w}{}^{-1}\tilde{N}'_{3w}$, a coprime factorization.

Under the assumed relationship between y_c and y_m, this corollary shows that the solvability of RPIS over $\mathbb{R}_g(s)$ depends on the solution of a skew–prime equation with an identity on the right hand side as in Theorem 3.3.1. This is shown to be a sufficient condition for the explicit presence of internal models in the next section.

3.3.2. Internal Models and Controller Structure

First, consider the case when $y_m=y_c$. If in addition, $P_{12}=I$, the structure of the controller and its relation to internal models is well understood (see [17]). A direct extension of this result is to consider the case when $P_{12}\neq I$. The study of internal models in this case is simplified by letting P_{12} be part of the exosystem. Then the exosystem's transfer matrix is $P_{12}T_{cw}$ and $\hat{T}_{cw}=(I+P_{11}C_y)^{-1}$. The internal model result is given in Theorem 3.3.3.

Theorem 3.3.3. Let $y_c=y_m$, if RPIS over $\mathbb{R}_g(s)$ is solvable, then $P_{11}C_y$ contains an internal model of $P_{12}T_{wd}$.

If $y_{11}=P_{11}u$, then $y_{11}=-P_{11}C_yy_c$, that is, $P_{11}C_y$ is the map from the regulated signal to y_{11}, which can be considered to be the "plant's output." This is a general property shown in this paper: the map from the regulated signal to the "plant's output" always contains an internal model of the exosystem's transfer matrix. Since $y_c=y_{11}+P_{12}w$, it can be seen that the role of the internal model in $P_{11}C_y$ is to introduce appropriate modes into y_{11} that will counterbalance the effect of the exogenous signal $P_{12}w$. Since $P_{11}C_y$ contains the internal model, the internal model is present explicitly in the feedback system.

The structure of the controller that solves the RPIS over $\mathbb{R}_g(s)$ in this case is of the form

$$C_y = G'_d{}^{-1}\underline{C}, \qquad (3.3.18)$$

where the controller $\underline{C}$ $\mathbb{R}_g$–stabilizes $P_{11}G'_d{}^{-1}$ and $G'_d \in M(\mathbb{R}_g(s))$. Note that G'_d contains the poles of T_{wd} that are not poles of the plant with similar structure, and by Remark A1 (in Appendix A [32]), P_{11} contains an internal model of P_{12}.

Similar results follow for the system described in Figure 2 because the skew-prime solvability condition has an identity on the right hand side. In this case, $P_cP_u\hat{C}_y$ contains an internal model of $P_cP_wT_{wd}$, this internal model appears explicitly in the feedback system, and the structure of the controller $\hat{C}_y$ is of the same form as in (3.3.18).

When $y_c\neq y_m$ it is known that internal models appearing explicitly in the loop are not necessary [29], but under some conditions internal models will be present [4]. In Theorem 3.3.4, we give a map that always contains an internal model of the exosystem, under the assumption that T_{cw} is square and nonsingular, and $(P'_3, P'_1K'P'_2)$ a r.c. $\mathbb{R}_g$–factorization.

Theorem 3.3.4. Let $y_c\neq y_m$, if RPIS over $\mathbb{R}_g(s)$ is solvable, then $P'_1K'P'_2(P'_3-P'_1K'P'_2)^{-1}$ contains an internal model of T_{wd}.

Let $y_{21}=P_{21}u$ and consider y_{21} to be the "plant's output." Note that $y_{21}=P'_1K'P'_2(P'_3-P'_1K'P'_2)^{-1}y_c$, so once again the map between the regulated signal and the plant's output contains the internal model, which does not have to appear explicitly in the loop. This is illustrated in the following example. It was considered in [29] to illustrate the absence of an internal model and in [4] the conditions where given for an internal model to be present.

Example 2. Consider the following system description: $P_{11}=1/(s+2)$, $P_{12}=P_{21}=1$, and $P_{22}=-2/(s+1)$. Note that $C_y=-1$ would regulate with internal stability a step disturbance $1/s$; this fact was used in [29] to show that an internal model need not be present in the loop. Nevertheless, observe that $P_{11}=(1/(s+\alpha)) \,/\, ((s+2)/(s+\alpha))$, where $\alpha>0$, $\alpha\neq 1$, $x'_1=x'_2=1$, $K'=-(\alpha-1)(s+\alpha)/(s+1)$, $P'_1=(s+2)/(s+\alpha)$, $P'_2=(s+2)/(s+\alpha)$, and $P'_3=-[2/(s+1) + (\alpha-2)(s+2)/(s+\alpha)]$. With these definitions, the map given in Theorem 3.3.4 is $P'_1K'P'_2(P'_3-P'_1K'P'_2)^{-1}=-(\alpha-1)(s+2)^2/(s(s+\alpha))$, demonstrating the presence of the internal model in this map, even though it does not appear explicitly in the control system in this case. □

The characterization of the structure of the controller when $y_c\neq y_m$ is given in Lemma 3.3.1.

Lemma 3.3.1. If RPIS over $\mathbb{R}_g(s)$ is solvable, then the structural conditions that must be satisfied by C_y are

$$\tilde{D}'_cD'_1 + \tilde{N}'_cN'_1 = I \qquad (3.3.19)$$

$$\tilde{D}'_c(-(\bar{x}'_2+D'_1K'_p)) + \tilde{N}'_c(\bar{x}'_1-N'_1K'_p) = W'\tilde{D}'_{2w}. \qquad (3.3.20)$$

Notice that (3.3.19) is the $\mathbb{R}_g$–stability condition and that (3.3.20) corresponds to the RPIS over $\mathbb{R}_g(s)$ requirement. A characterization of controllers that solve RPIS over $\mathbb{R}_g(s)$ can be obtained from (3.3.19) and (3.3.20) and is written below

$$\tilde{D}'_c = \bar{x}'_1 - W'\tilde{D}'_{2w}\tilde{N}'_1 \text{ and } \tilde{N}'_c = \bar{x}'_2 + W'\tilde{D}'_{2w}\tilde{D}'_1. \qquad (3.3.21)$$

Notice that the set characterized in (3.3.21) is nonempty if RPIS over $\mathbb{R}_g(s)$ is solvable. Suppose that $W'=0$ or $\tilde{D}'_{2w} \in \mathrm{Ker}(W')$, then a compensator that solves RPIS over $\mathbb{R}_g(s)$ is described by

$$\tilde{D}'_c = \bar{x}'_1 \triangleq x'_1 - K'_p\tilde{N}'_1 \text{ and } \tilde{N}'_c = \bar{x}'_2 \triangleq x'_2 + K'_p\tilde{D}'_1. \qquad (3.3.22)$$

The characterization of controllers given in (3.3.21) depends on the particular choice of W'. One way to characterize the part of the controller that is independent of the choice of W' is by defining

$$G'_d = \text{a g.c.r.d. } (x'_1, \tilde{D}'_{2w}\tilde{N}'_1)$$

and

$$G'_n = \text{a g.c.r.d. } (x'_2, \tilde{D}'_{2w}\tilde{D}'_1), \qquad (3.3.23)$$

then

$$\tilde{D}'_c = D'_cG'_d\,,\ \tilde{N}'_c = N'_cG'_n$$

$$\Rightarrow\ C_y = G'_d{}^{-1}\underline{C}G'_n, \qquad (3.3.24)$$

where $\underline{C}=D'_c{}^{-1}N'_c$ and g.c.r.d. denotes greatest common right divisor. The significance of G'_d and G'_n is that they are introduced solely because of the regulation over $\mathbb{R}_g(s)$ requirement, while $\underline{C}$ must satisfy both structural conditions: (3.3.19) and (3.3.20). Note that $|G'_d||G'_n|$ divides $|\tilde{D}'_{2w}|$ and that $\underline{C}$ satisfies the $\mathbb{R}_g$–stability condition (3.3.19) if and only if

$$|G'_d|\ |G'_n| \text{ and } |\tilde{D}'_{2w}| \text{ are associates[1].} \qquad (3.3.25)$$

If (3.3.25) is satisfied and if RPIS is solvable then

$$\tilde{D}'_c = D'_cG'_d\,,\ \tilde{N}'_c = N'_cG'_n \qquad (3.3.26)$$

is a necessary and sufficient condition for regulation over $\mathbb{R}_g(s)$. In this case, the controller must contain part of the exosystem dynamics in G'_d and/or G'_n. This is one way for the map in Theorem 3.3.4 to have the appropriate internal model.

When $y_c\neq y_m$, we have seen that a map from the regulated variables, y_c, to the output of the plant, y_{21}, always contains the internal model, and that the controller that solves RPIS over $\mathbb{R}_g(s)$ is given by (3.3.24). When (3.3.25) is satisfied, then the internal model is introduced in the map in Theorem 3.3.4 via poles of C_y (in $G'_d{}^{-1}$) and zeros of C_y (in G'_n), in addition to some appropriate structure on the plant. But, this is not necessary in general, that is, the internal model does not need to appear explicitly in the feedback system. However, if $y_c=y_m$ or if y_c and y_m are related as shown in Figure 2, then the internal model will be present explicitly in the feedback system.

[1]Two elements a, b $\in \mathbb{R}_g(s)$ are associates if they differ by u, a unit in $\mathbb{R}_g(s)$, that is, a = ub.

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IV. ASYMPTOTIC TRACKING AND INTERNAL MODELS

4.1 Problem Formulation

Consider the two degrees of freedom system

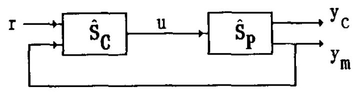

Figure 3. Two degrees of freedom control system.

where w=0, the plant, $\hat{S}_p$, has transfer matrix $[P^t_{11} \;\; P^t_{21}]^t$, the two degrees of freedom controller, $\hat{S}_c$, has transfer matrix $C=[-C_y, \;\; C_r]$, and r is the vector of request inputs. It is assumed that r can be modeled as the output of a causal, linear, time-invariant, finite dimensional system described by $r=T_{rv}v$, where v is a bounded vector and T_{rv} is antistable.

The asymptotic tracking problem with internal stability over $\mathbb{R}_g(s)$ (TPIS over $\mathbb{R}_g(s)$) consists on finding a linear two degrees of freedom controller so that

$$(I - T_{cr})T_{rv} \in \mathbf{M}(\mathbb{R}_g(s)) \tag{4.1}$$

and the compensated system in Figure 3 is $\mathbb{R}_g$–stable. This is because the vector of error variables is given by $(I - T_{cr})T_{rv}v$ and the objective of TPIS is to drive the error variables asymptotically to zero while maintaining $\mathbb{R}_g$–stability.

4.2 Nonrobust TPIS

4.2.1 Solvability conditions

First, note that in Appendix A [32] it is shown that a characterization of all attainable transfer function matrices with $\mathbb{R}_g$–stability from r to y_c is

$$T_{cr}=P_{21}D'_1X'=P'_1X' \tag{4.2.1}$$

where X' is the response parameter used in the characterization of $\mathbb{R}_g$–stabilizing controllers C_r. Let

$$T_{rv} = \tilde{D}'_r{}^{-1}\tilde{N}'_r \tag{4.2.2}$$

be an $\mathbb{R}_g$–coprime factorization. The main result of this section is Theorem 4.2.1.

Theorem 4.2.1. TPIS over $\mathbb{R}_g(s)$ for the system in Figure 3 is solvable if and only if

$$P'_1X' + V'_r\tilde{D}'_r = I. \tag{4.2.3}$$

A special case of the above result is when $y_c=y_m$. In this case, $P_{11}=P_{21}$, then $P'_1=P_{21}D'_1=N'_1$, and the solvability condition is

$$N'_1X' + V'_r\tilde{D}'_r = I \tag{4.2.4}$$

which is well known. For scalar systems, the condition in (4.2.3) ((4.2.4)) reduces to the zeros of P'_1 (P_{11}) must be disjoint from the poles of T_{rv}. This last comment has also been generalized to the multivariable case. One way to present this disjointedness is as a coprimeness condition. For example, once an X' and a V'_r have been found in (4.2.3), then (4.2.3) can be rewritten as

$$I(P'_1X') + V'_r\tilde{D}'_r = I, \tag{4.2.5}$$

showing that $(P'_1X')\tilde{D}'_r{}^{-1}$ is coprime.

4.2.2 Internal Models

The solvability condition in Theorem 4.2.1 is of the same form as the solvability condition of RPIS over $\mathbb{R}_g(s)$ when $y_c=y_m$ (see Theorem 3.3.1). This indicates that TPIS over $\mathbb{R}_g(s)$ is solvable if and only if an appropriately defined regulation problem is solvable. This is true even though we are using a two degrees of freedom controller to solve TPIS over $\mathbb{R}_g(s)$. The difference is that the solution of TPIS over $\mathbb{R}_g(s)$ as seen from (4.2.3) imposes constraints on C_r via X', while the solution of RPIS over $\mathbb{R}_g(s)$ imposes constraints on C_y and C_r via K' (see Theorem 3.3.2, and (A3) and (A4) in Appendix A [32]).

We present now a new result that shows that the solution of TPIS over $\mathbb{R}_g(s)$ implies that there exists a map that contains an internal model of T_{rv}.

Theorem 4.2.2. Let $y_c \neq y_m$, if TPIS over $\mathbb{R}_g(s)$ is solvable, then $P_{21}\hat{C}$, where $\hat{C}=M(I-P_{21}M)^{-1}$, contains an internal model of T_{rv}.

In general, the internal model characterized in Theorem 4.2.2 does not appear explicitly in the loop. Note that $\hat{C}(I-P_{21}M)r=Mr$ implies that $u=\hat{C}e$ or $P_{21}\hat{C}$ is the transfer matrix between e, the signal to be regulated, and y_{21}, the output of the "plant", where $y_{21}=P_{21}u$, that is, the map from the regulated variables, e, to the output of the plant, y_{21}, contains the internal model.

A relation between $\hat{C}$ and C_y and C_r is given by

$$\hat{C} = (I + C_yP_{11})^{-1}C_r(I - P_{21}M)^{-1} \tag{4.2.6}$$

$$= (I + C_yP_{11})^{-1}C_r(I - P_{21}(I + C_yP_{11})^{-1}C_r)^{-1}. \tag{4.2.7}$$

The expression for $\hat{C}$ can be simplified for some typical cases. For example, if $C_y=C_r$, then

$$\hat{C}=C_y(I+(P_{11}-P_{21})C_y)^{-1}, \tag{4.2.8}$$

and if in addition, $y_c=y_m$ then $\hat{C}=C_y=C_r$. Similarly, if $y_c=y_m$ and TPIS over $\mathbb{R}_g(s)$ is solvable, then $P_{11}M(I-P_{11}M)^{-1}= N'_1X'(I-N'_1X')^{-1}$ contains the internal model. In this case,

$$\hat{C}= (I + C_yP_{11})^{-1}C_r[I + P_{11}(C_y - C_r)]^{-1}(I + P_{11}C_y), \tag{4.2.9}$$

and if in addition $C_y=C_r$, then $\hat{C}=C_y=C_r$, as expected. Clearly, when $y_c=y_m$ and $C_y=C_r$, the internal model is contained in the product $P_{11}C_y$ (Theorem 3.3.3).

As a trivial case, it is interesting to note that the above results on internal models also hold for open-loop systems when no uncertainty is present. This is verified by example.

Example 3. Consider an open-loop plant with transfer function $1/(s+1)$, and a feedforward proportional controller with unity gain. This simple system can track a step input if no uncertainty is present. It follows that $P\hat{C}=1/s$, but the internal model is not present in the open-loop system.

4.3 Robust TPIS

The analysis and synthesis of robust TPIS has appeared in the literature [1,2,24,25]. In these papers the necessity of an internal model in C_y is shown. The same results characterizing the structure of the controller and the necessity of an internal model can be derived starting with Theorem 4.2.2. Note that robust TPIS is solvable if $\hat{C}$ contains an internal model of T_{rv} and the compensated system remains $\mathbb{R}_g$–stable as the plant varies. When $y_c=y_m$ and a single degree of freedom controller is used, it is well known that TPIS and RPIS are mathematically equivalent. In this case, $\hat{C}=C_y$ contains an internal model of T_{rv} with a structure similar to the one in (3.2.2). Solvability and structural conditions are given in, for example, [1,2]. When $y_c=y_m$ and a two degrees of freedom controller is used, robust TPIS has been analyzed in [1,24]. Their results could also be derived from $\hat{C}$ in (4.2.9). If $y_c \neq y_m$ and the plant parameters are allowed to vary, then it is known that robust TPIS is not solvable. This is easy to see when $C_y=C_r$ from (4.2.8). Notice that even if C_y is of the form in (3.2.2) that $\hat{C}$ will not contain the desired internal model. Recently, the analysis of TPIS for the system in Figure 2 with a two degrees of freedom controller has appeared in [25]; structural conditions for the two degrees of freedom controller are given, which include an internal model

V. STABILIZATION AND INTERNAL MODELS

In this section we consider the controlled system in Figure 1 with $y_c=y_m$ and w=0. Since it has been assumed that the controlled system is well-defined and that all input-output maps are proper, then $\mathbb{R}_g$–stability means that all the eigenvalues of an internal description of the controlled system are in S_g, the good region of the complex plane. The analysis of $\mathbb{R}_g$–stability is presented in Appendix A [32]. One way to interpret the role of the control action in stabilization is that the unstable modes (those modes that do not correspond to eigenvalues in S_g) of the open-loop plant, S_p, and controller, S_c, must be regulated. In this way, in the absence of external inputs and for all initial conditions, every signal in the feedback system will go asymptotically to zero and can be considered to be the regulated signal. So we can think of the plant and the

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controller as the exosystems whose unstable modes we need to regulate. Assume that the internal descriptions of S_p and S_c are completely characterized by their transfer matrices $P=P_{11}$ and C_y, respectively. Then we are interested in determining maps that contain an internal model of P and C_y. These maps are given in Theorem 5.1.

Theorem 5.1. If the system in Figure 1 with $y_c=y_m$ is $\mathbb{R}_g$–stable then PC_y and C_yP contain an internal model of P and C_y.

Proof of Theorem 5.1 Under the stated assumptions, the controlled system is $\mathbb{R}_g$-stable if and only if

$$D'^{-1}_1[(I+C_yP)^{-1}, \quad (I+C_yP)^{-1}C_y] \in M(\mathbb{R}_g(s)), \tag{5.1}$$

if and only if

$$\begin{bmatrix} (I+PC_y)^{-1} \\ C_y(I+PC_y)^{-1} \end{bmatrix} \tilde{D}'^{-1}_1 \in M(\mathbb{R}_g(s)). \tag{5.2}$$

This is a direct extension of results in [30]. In particular, to show that C_yP contains an internal model of P, note that (5.1) implies

$$(I + C_yP)^{-1} = D'_1V', \tag{5.3}$$

where $V' \in M(\mathbb{R}_g(s))$. By (2.6) with B=I and $A=(I + C_yP)^{-1}$, C_yP contains an internal model of P. Similarly, it can be shown that PC_y contains an internal model of P. To show that PC_y and C_yP contain an internal model of C_y, consider C_y to be the "plant" and P the "controller" and follow the same approach. □

Note that once again a map between a signal that is regulated and the output of the plant contains an internal model, in this case PC_y. In addition, other maps must contain the internal model, such as C_yP. Clearly, the presence of the internal model in PC_y and C_yP indicates that there are no unstable cancellations in PC_y and in C_yP. These are necessary conditions for $\mathbb{R}_g$–stability that guarantee that no unstable uncontrollable and/or unobservable modes are introduced [30,9].

In Appendix A [32] it is made clear that if $y_c \neq y_m$ or if a two degrees of freedom controller is used, then the internal representations of S_p and S_c must be admissible. Admissibility requires that additional internal model relations be satisfied

VI. CONCLUSION

A treatment of internal models over rings has been presented. For a generic formulation of the regulation problem, a set of maps containing the internal model of the transfer matrix of the exosystem was characterized. This result was used in the study of the problems of RPIS and TPIS over $\mathbb{R}_g(s)$ as well as $\mathbb{R}_g$–stability without a robustness requirement to show that in each case at least the map from the "regulated" variables to the "output" of the plant contained the internal model. In RPIS over $\mathbb{R}_g(s)$ when $y_c \neq y_m$ the map containing the internal model does not need to appear explicitly in the feedback system. However, if $y_c=y_m \circ y_c$ and y_m are related as in Figure 2, then the skew–prime solvability equation has an identity on the right hand side and the internal model appears explicitly in the feedback system. In TPIS over $\mathbb{R}_g(s)$ when $y_c \neq y_m$ and a two degrees of freedom controller is used, then there is a map containing the internal model that does not need to appear explicitly in the feedback system This map containing the internal model appears explicitly if $y_c=y_m$ and $C_y=C_r$. For stabilization we considered the plant and controller to be the exosystems and characterized two maps that contain the internal model. The existence of the internal model in these maps was interpreted as a necessary condition to avoid the introduction of unstable uncontrollable and/or unobservable modes. These results reemphasize the importance of internal models in most control problems.

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