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Stability of the pseudo-inverse method for reconfigurable control systems

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One of the key reconfigurable control methods, the pseudo-inverse method (PIM), is analysed and new insight is obtained which provides the theoretical basis for this practical approach. The main shortcoming of this method, the lack of stability guarantees, is pointed out and a new approach is proposed in which recent results on the stability robustness of linear systems are used to provide stability constraints for the solutions of the PIM. When the original PIM solution results in an unstable closed-loop system, the control redesign problem is treated as a constraint minimization problem. For single-input systems, a closed-form solution is presented; for multi-input systems, a near-optimal solution is found which maintains the stability of the closed-loop system.

1. Introduction

Reconfigurable/restructurable control systems (RCS) are control systems that possess the ability to accommodate system failures automatically based upon *a priori* assumed conditions. The research in this area is largely motivated by control problems encountered in aircraft control system design, where the ideal goal is to achieve the so called 'fault-tolerant' or 'self-repairing' capability in flight control systems. In this way unanticipated failures in the system can be accommodated and the aeroplane can at least be landed safely whenever possible. This problem has drawn the attention of many researchers and many results have been published; recent publications include Caglayan *et al.* (1988), Chandler (1984), Gavito and Collins (1987), Eslinger and Chandler (1988), Howell *et al.* (1983), Huber and McCulloch (1984), Looze *et al.* (1985), Ostroff (1985), Ostroff and Hueschen (1987); Potts and D'Azzo (1981), Rattan (1985) and Raza and Silverthorn (1985).

Research on RCS has drawn extensive attention ever since two accidents in the late 1970s involving commercial aircraft in which control element failures occurred (NTSB 1979, McMahan 1978). In one case the pilot successfully reconfigured the remaining control elements and landed the plane safely (McMahan 1978), while in the other case the plane crashed, although it could have been saved (NTSB 1979). In the second case, the pilot only had 15 s to react. This shows that there is a need to develop a flight control system which has the ability to reconfigure and/or restructure the control law automatically in case of control and/or sensor element failures.

The distinction between reconfigurable control and restructurable control is based upon the degree of pre-planned configuration according to Ostroff (1985). In particular, in reconfigurable control the control law is designed *a priori* to accommodate certain anticipated failures; these control laws are then stored and used

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according to need. Restructurable control implies an automatic design to accommodate unanticipated failures; depending on the size of failures, it may represent a long-term, possibly idealistic objective. However, these definitions have not been fully accepted in the literature and the words reconfigurable and restructurable have been used interchangeably. In this paper we concentrate on the more challenging case where *the failures are unanticipated and automated on-line failure accommodation is required.* We refer to this problem as the 'reconfigurable control problem'.

The pseudo-inverse method (PIM) has been accepted as a key approach to reconfigurable control and has been used quite successfully in flight simulations as reported by Caglayan *et al.* (1988), Huber and McCulloch (1984), Ostroff (1985) and Rattan (1985). The main idea is to modify the feedback gain so that the reconfigured system approximates the nominal system in some sense. It is attractive because of its simplicity in computation and implementation. The pseudo-inverse of a constant matrix has been defined by Stewart (1973) and Golub and Van Loan (1983) as follows.

Let the singular value decomposition of $A \in \mathbb{R}^{n \times m}$ be

$$A = V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^{\mathrm{H}} \tag{1.1}$$

where V and U are unitary matrices. For any unitary matrix W, we have $W^{\mathrm{H}} W = I$ and $\|W\|_2 = 1$; W^{H} is the complex conjugate of W.

Definition (Stewart 1973, pp. 325)

The *pseudo-inverse* of A is defined as

$$A^{+} = U \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} V^{\mathrm{H}} \tag{1.2}$$

Note that $x = A^{+} b$ is the solution of smallest norm for the linear least-squares problem of minimizing $\|Ax - b\|_2^2$.

The main drawback of the PIM is that the stability of the reconfigured system is not guaranteed. As a result, the PIM, if applied with no appropriate safeguard, can lead to instability. It is perhaps equally as worrying that the results in literature do not provide adequate insight as to why the PIM actually works when it does, or under what circumstances it will fail. These are very important questions that need to be answered, because in control system redesign, the first thing to recover has to be stability; once stability is guaranteed, performance, such as command following, can then be taken into consideration. It is therefore necessary in any approach to reconfigurable control, no matter what method is used, to guarantee stability. In § 2, appropriate methods are introduced, the PIM is outlined and its properties are discussed. In § 3, results on stability bounds for linear systems with structured perturbations are first introduced and then, based upon these results and the PIM, the control reconfiguration problem is mathematically formulated and a new approach is presented. Finally, some concluding remarks are presented in § 4.

2. Pseudo-inverse method

The PIM has become a key approach to reconfigurable control. Its main objective is to maintain as much similarity as possible to the original design and

thus to provide graceful degradation in performance. This is achieved by reassigning the feedback gains and the approach is illustrated as follows.

Let the open-loop plant be given by

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{2.1}$$

for $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{q \times n}$. Assume that the nominal closed-loop system is designed by using the state feedback $u = Kx$, $K \in \mathbb{R}^{m \times n}$, and the closed-loop system is

$$\left.\begin{aligned} \dot{x} &= (A + BK)x \\ y &= Cx \end{aligned}\right\} \tag{2.2}$$

where K is the state feedback gain. Suppose that the model of the system, in which failures have occurred, is given by

$$\left.\begin{aligned} \dot{x}_f &= A_f x_f + B_f u_f \\ y_f &= C_f x_f \end{aligned}\right\} \tag{2.3}$$

where $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times m}$ and $C_f \in \mathbb{R}^{q \times n}$, and the new closed-loop system is

$$\left.\begin{aligned} \dot{x}_f &= (A_f + B_f K_f)x_f \\ y_f &= C_f x_f \end{aligned}\right\} \tag{2.4}$$

where K_f is the new feedback gain to be determined. In the PIM due to Ostroff (1985), the objective is to find a K_f such that the closed-loop transition matrix in (2.4) approximates in some sense to the one in (2.2). For this, $A + BK$ is equated to $A_f + B_f K_f$ and an approximate solution for K_f is given by

$$K_f = B_f^+ (A - A_f + BK) \tag{2.5}$$

where B_f^+ denotes the pseudo-inverse of B_f.

Note that K_f can be calculated from (2.5) for many anticipated failures and be stored in the flight-control computer. Once the failure has occurred and is identified, i.e. the model of the system with failure (2.3) is obtained, the feedback gain can then be modified. This is considered as a relatively fast solution to stabilize the impaired aeroplane (Ostroff 1985). This PIM method has also been used for on-line accommodation for unanticipated failures (Caglayan *et al.* 1988, Huber and McCulloch 1984, Ostroff 1985, Rattan 1985) although it appeared in different forms. However, there is one problem which might render the method useless, namely that the solution from (2.5) does not necessarily make the closed-loop system in (2.4) stable (see Example 1 and Example 3). This had led us to introduce the modified PIM in § 3.

2.1. *Properties of the PIM*

Although the PIM has been used widely in the study of the RCS, it is still used in an *ad hoc* manner. Theoretical aspects of this method have not been fully investigated. For example, it is not clear in what sense the closed-loop system in (2.4) approximates to the one in (2.2) when K_f is obtained from (2.5). In the PIM, it is desirable to have

$$A + BK = A_f + B_f K_f \tag{2.6}$$

This equation may or may not have an exact solution depending on the row rank of the matrix B_f. If B_f has full row rank, then (2.5) always satisfies (2.6), otherwise, there is no exact solution to (2.6) and K_f from (2.5) is only an approximate solution. It is interesting to see what the PIM implies in terms of the eigenvalues, which are often used in the specifications of the performance criteria. First we shall look at the following lemma.

Lemma

Let

$$J = \|(A + BK) - (A_f + B_f K_f)\|_F \tag{2.7}$$

where $\|\cdot\|_F$ stands for the Frobenius norm. Then the K_f obtained from (2.5) minimizes J.

Proof

Let $\bar{A} = A + BK - A_f = [\bar{a}_1, \bar{a}_2, ..., \bar{a}_n]$, $K_f = [k_{f1}, k_{f2}, ..., k_{fn}]$, where $\bar{a}_i$ and k_{fi}, $i = 1, 2, ..., n$, are column vectors. Then J can be written (Stewart 1973) as

$$\begin{aligned} J^2 &= \|[\bar{a}_1 - B_f k_{f1}, \bar{a}_2 - B_f k_{f2}, ..., \bar{a}_n - B_f k_{fn}]\|_F^2 \\ &= \|\bar{a}_1 - B_f k_{f1}\|_2^2 + \|\bar{a}_2 - B_f k_{f2}\|_2^2 + ... + \|\bar{a}_n - B_f k_{fn}\|_2^2 \end{aligned} \tag{2.8}$$

Since $k_{fi} = B_f^+ \bar{a}_i$ is the least-squares solution of the equation $\bar{a}_i = B_f k_{fi}$, it minimizes the vector norm $\|\bar{a}_i - B_f k_{fi}\|_2^2$ for $i = 1, 2, ..., n$. Note that the terms in (2.8) are independent of each other, thus K_f from (2.5) minimize (2.8) and (2.7). □

The Lemma shows that solution (2.5), used in the PIM, makes the closed-loop system (2.4) approximate the nominal one (2.2) in the sense that the Frobenius norm of the difference of the A matrices is minimized. The underlying idea in the PIM is that if the norm J is minimized, it is hoped that the behaviour of the reconfigured system will be close to that of the nominal system. Clearly we should like to know just how close it is going to be and in exactly what sense. The relation between variations of closed-loop eigenvalues and the Frobenius norm in (2.7) is given by the following theorem.

Theorem 1

Let $(A + BK)$ be non-defective, that is, it can be reduced to diagonal form by a similarity transformation, and let X be the eigenvector matrix of $A + BK$ in (2.2) and $X^{-1}(A + BK)X = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. The eigenvalues of $A_f + B_f K_f$ in (2.4) lie in the union of the discs

$$D_i = \{\lambda : |\lambda - \lambda_i| \leqslant \|X\|_2 \|X^{-1}\|_2 J\} \tag{2.9}$$

where J is defined in (2.7) and $\|\cdot\|_p$ denotes the matrix p-norms.

Proof

Let $E = (A_f + B_f K_f) - (A + BK)$, then $A_f + B_f K_f = (A + BK) + E$, and $J = \|E\|_F$. By the Bauer–Fike theorem (Stewart 1973, p. 304; Golub and Van Loan 1983, p. 200), an eigenvalue λ of $(A_f + B_f K_f)$ satisfies

$$\min |\lambda - \lambda_i| \leqslant \|X\|_p \|X^{-1}\|_p \|E\|_p$$

and since $\|E\|_2 \leqslant \|E\|_F = J$,

$$\min |\lambda - \lambda_i| \leqslant \|X\|_2 \|X^{-1}\|_2 J \qquad \square$$

The significance of this theorem is that it provides a bound to the variation of the eigenvalues in terms of the eigenvectors of the nominal system and the Frobenius norm J. If the nominal system is robust in the sense that all the eigenvalues are relatively far left of the $j\omega$ axis, then the PIM has a good chance to work provided J is small enough; J can be made small when the failure is not 'too severe'. This fact must be taken into consideration in the design of the nominal system. Theorem 1 also shows the limitations of the PIM. It is clear that by just minimizing J, stability is not guaranteed; the minimum J can be large enough to allow the eigenvalues of the closed-loop system to shift to the right half-plane. Thus the use of the PIM in the automatic reconfiguring control system may result in an unstable closed-loop system unless it is restricted to only certain classes of failure where the value of J is small enough, in which case all the discs in (2.9) lie in the left half-plane.

Example 1

Let the nominal plant in (2.1) be

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad C = I$$

and the feedback gain be given by $K = [-1 \;\; 0]$ which assigns the closed-loop poles at $(-1, -2)$. Let the model of the failed plant be

$$A_f = A, \quad B_f = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C_f = C$$

By using the PIM, (2.5) gives

$$K_f = B_f^+ (A - A_f + BK) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -5 & 0 \end{bmatrix} = [-2 \quad 0]$$

However, the eigenvalues of $(A_f + B_f K_f)$ are $\{1, -1\}$. This shows that the PIM led to an unstable system.

In the next section, a new method for reconfigurable control with guaranteed stability is proposed. It is based on the PIM and some recent results on the stability robustness of systems with structured uncertainties.

3. Modified pseudo-inverse method

In the implementation of the RCS stability is probably the most important property of the system. Although the PIM has many good aspects, lack of stability guarantees puts constraints on its application. In this section, an alternative approach is proposed. This method utilizes recent research results on the stability robustness of linear systems with structured uncertainty (Barlett *et al.* 1988, Barmish 1984, 1988, Yedavalli 1988, Zhou and Khargonekar 1987). The objective of this approach is to maintain closed-loop stability while recovering the performance as much as possible. Equivalently, the objective is to keep the system in (2.4) stable,

while minimizing the norm in (2.7). Clearly that is a constraint minimization problem (CMP). In order to use standard easy-to-implement numerical algorithms on the CMP, the stability constraints will be expressed explicitly in terms of the individual elements of K_f as simple inequalities.

3.1. *Stability bounds for linear systems with structured uncertainties*

In the analysis of system stability with parameter uncertainty in the state-space model, many approaches have been proposed to obtain bounds on the uncertainty that guarantee the stability of the perturbed system (Barlett *et al.* 1988, Barmish 1984, 1988, Yedavalli 1988, Zhou and Khargonekar 1987). Barmish extends Kharitonov's stability theorem to establish a stability bound on the coefficients of the characteristic polynomial. Barlett *et al.* made a breakthrough in dealing with dependent parameter uncertainty. If the uncertainty can be characterized by a family of characteristic polynomials (a polytope), then the stability of the whole family is completely determined by examining only the exposed edges of the polytope. Yedavalli and Zhou *et al.* each gave an algorithm to compute the stability bounds on all the uncertain parameters in the state-space model and these are outlined below. It seems that the method due to Zhou *et al.* is more computationally attractive, while Yedavalli's method seems less conservative. It should be noted, however, that both of these methods can give very conservative results at times.

Consider the state-space model with perturbation E.

$$\dot{x} = (A + E)x \tag{3.1}$$

where A is an $n \times n$ real Hurwitz matrix. Assume that the perturbation matrix E takes the form

$$E = \sum_{i=1}^{m} k_i E_i \tag{3.2}$$

where E_i are constant matrices and k_i are uncertain parameter that are assumed to vary in the intervals around zero.

3.1.1. *Zhou's method.* Let P be the solution of the Lyapunov equation

$$PA + A^{\mathrm{T}}P + 2I = 0 \tag{3.3}$$

and define P_i and P_e as

$$\left.\begin{aligned} P_i &:= (E_i^{\mathrm{T}} P + PE_i)/2, \quad i = 1, 2, \ldots, m \\ P_e &:= [P_1 P_2 \ldots P_m] \end{aligned}\right\} \tag{3.4}$$

Then system (3.1) is stable if

$$(a) \qquad \sum_{i=1}^{m} k_i^2 < 1/\sigma_{\max}^2(P_e) \tag{3.5 a}$$

or

$$(b) \qquad \sum_{i=1}^{m} |k_i| \sigma_{\max}(P_i) < 1 \tag{3.5 b}$$

or

$$(c) \qquad |k_i| < 1/\sigma_{\max}\left(\sum_{j=1}^{m} |P_j|\right), \quad i = 1, 2, ..., m \tag{3.5 c}$$

where $\sigma_{\max}(\cdot)$ denotes the largest singular value and $|\cdot|$ denotes a matrix formed by taking the absolute value of each element.

3.1.2. *Yedavalli's method.* System (3.1) is stable if

$$|k_i| < 1/\sup_{\omega \geqslant 0} \rho\left(\sum_{j=1}^{m} |(j\omega I - A)^{-1}E_i|\right) \quad \text{for } m > 1 \tag{3.6 a}$$

and

$$|k_i| < 1/\sup_{\omega \geqslant 0} \rho((j\omega I - A)^{-1}E_1) \quad \text{for } m = 1 \tag{3.6 b}$$

where $\rho(\cdot)$ denotes the spectral radius, i.e. the size of the largest eigenvalue of the matrix. Note that these inequalities are sufficient conditions for the system in (3.1) to remain stable. These results will be used in constructing the modified PIM in the following section.

3.2. *New algorithm for reconfigurable control with guaranteed stability*

In order to use the PIM as a reliable procedure in the control reconfiguration, the stability must be guaranteed. In this section, a new approach is presented. First, the control reconfiguration problem is reformulated as a constraint minimization problem. The results described in § 3.1 are used to provide a safeguard for stability.

Let us assume that the pair (A_f, B_f) given in (2.3) is stabilizable. If it is not stable, an inner loop can be used to stabilize it, since the method in § 3.1 assumes that A is stable (see also Example 3). From § 3.1, a stability bound δ can be found such that if

$$|K_f(i, j)| < \delta \quad \text{for } i = 1, 2, ..., m \quad \text{and} \quad j = 1, 2, ..., n \tag{3.7}$$

the system in (2.4) will be stable. Such a δ could be, for example, the righ-hand side of (3.5 *c*), (3.6 *a*) or (3.6 *b*). For convenience

$$|K_f(i, j)| \leqslant \delta' \quad \text{for } i = 1, 2, ..., m \quad \text{and} \quad j = 1, 2, ..., n \tag{3.7 a}$$

will be used instead of (3.7), where $\delta' = \delta - \varepsilon$ for some small ε. Now the reconfigurable control problem can be described as follows.

3.2.1. *Problem formulation*

$$\text{Determine } K_f \text{ to minimize } J \text{ in (2.7) subject to (3.7 } a\text{).} \tag{3.8}$$

Without loss of generality, we assume that A_f is stable. Since J^2 can be decomposed as a sum of the square 2-norm of column vectors (see (2.8)), problem (3.8) can be solved as a set of constrained least-squares problems (CLSP). Algorithms to solve constrained minimization problems can be found in the work of Lawson and Hanson (1974) and Gill *et al.* (1981).

When the plant is a single-input system or can be decomposed into SIMO subsystems, then the following theorem gives an explicit solution to the reconfigurable control problem (3.8).

Assume that the PIM solution (2.5) makes the closed-loop system in (2.4) unstable.

Theorem 2

If $B_f \in \mathbb{R}^{n \times 1}$, then the K_f below solves (3.8):

$$K_f(i,j) = \begin{cases} \bar{K}_f(i,j), & \text{if } |\bar{K}_f(i,j)| \leqslant \delta' \\ \operatorname{sgn}(|\bar{K}_f(i,j)|)\delta', & \text{otherwise} \end{cases} \tag{3.9}$$

where $\bar{K}_f$ is obtained from (2.5).

Proof

Let the singular value decomposition of B_f be

$$B_f = V \begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{pmatrix} U^{\mathrm{H}}$$

where $V \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}$ are unitary matrices. Without loss of generality, let $U = 1$ and

$$V^{\mathrm{H}}(A - A_f + BK) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $C_1 \in \mathbb{R}^{1 \times n}$, $C_2 \in \mathbb{R}^{(n-1) \times n}$. Then we have

$$\begin{aligned} J^2 &= \|(A + BK) - (A_f + B_f K_f)\|_{\mathrm{F}}^2 = \|V^{\mathrm{H}}(A - A_f + BK) - V^{\mathrm{H}} B_f K_f\|_{\mathrm{F}}^2 \\ &= \left\| \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - \begin{pmatrix} \sigma \\ 0 \end{pmatrix} K_f \right\|_{\mathrm{F}}^2 = \left\| \begin{matrix} C_1 - \sigma K_f \\ C_2 \end{matrix} \right\|_{\mathrm{F}}^2 \end{aligned}$$

This implies that the constraint minimization problem is equivalent to minimizing

$$J' = \|C_1 - \sigma K_f\|_{\mathrm{F}}^2 = (C_1 - \sigma K_f)(C_1 - \sigma K_f)^{\mathrm{T}} \tag{3.10}$$

subject to (3.7 *a*). From (2.5), the solution of the PIM is

$$\bar{K}_f = C_1/\sigma \tag{3.11}$$

which makes $J' = 0$. If this solution makes the closed-loop system in (2.4) stable, it is the desired solution.

Now consider the case where (2.5) makes (2.4) unstable. Let $C_1 = [c_{11}\ c_{12}\ \dots\ c_{1n}]$, $K_f = [k_{f1}\ k_{f2}\ \dots\ k_{fn}]$ and $\bar{K}_f = [\bar{k}_{f1}\ \bar{k}_{f2}\ \dots\ \bar{k}_{fn}]$, then J' in (3.10) can be written as

$$J' = \sum_{i=1}^{n} (c_{1i} - \sigma k_{fi})^2 \tag{3.10'}$$

Again, in (3.10′), the n terms are independent of each other and can therefore be minimized individually. Thus we can just look at the ith term, $J_i = (c_{1i} - \sigma k_{fi})$, as follows.

(*a*) If $|\bar{k}_{fi}| \leqslant \delta'$, clearly $k_{fi} = \bar{k}_{fi}$ minimize J_i.

(*b*) If $|\bar{k}_{fi}| > \delta'$, or equivalently $|c_{1i}/\sigma| > \delta'$ without loss of generality, assume that $\bar{k}_{fi} = c_{1i}/\sigma > 0$.

$$J_i' = \frac{dJ_i}{dk_{fi}} = -2\sigma(c_{1i} - \sigma k_{fi}) = -2\sigma^2(c_{1i}/\sigma - k_{fi}) \tag{3.12}$$

Note $J_i' < 0$ for $k_{fi} \in [-\delta' \ \delta']$, therefore for $k_{fi} \in [-\delta' \ \delta']$, J_i obtains its minimum at the boundary $k_{fi} = \delta'$ and $(J_i)_{\min} = (c_{1i} - \sigma\delta')^2$. Similarly, it can be shown that when $\bar{k}_{fi} = c_{1i}/\sigma < 0$, $k_{fi} = -\delta'$ minimizes J_i for $k_{fi} \in [-\delta' \ \delta']$. Therefore (3.9) minimizes each term in (3.10 *a*) subject to (3.7 *a*); hence it minimizes J subject to (3.7 *a*). □

For single input systems we have the following algorithm.

Algorithm 1

Step 1. Calculate K_f from (2.5).

Step 2. Check the stability of (2.4) for the K_f obtained in Step 1.

Step 3. If (2.4) is stable, stop; otherwise calculate K_f using (3.9).

We call this approach the modified pseudo-inverse method (MPIM).

Example 2

For the system in Example 1, E can be constructed as:

$$E = B_f K_f = k_{f1} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + k_{f2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

By (3.6), the bound on K_f is found to be $\delta = 0{\cdot}5$; this is non-zero for k_{f1} and k_{f2} in general. If we consider

$$k_{f2} = 0 \quad \text{and} \quad E = k_{f1} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

then $\delta = 0{\cdot}828$. Let $\delta' = 0{\cdot}8$, and use Theorem 2; then $K_f = [-0{\cdot}8 \quad 0]$ is a solution to the reconfigurable control problem (3.8).

Remark 1

The stability constraints obtained in (3.5) and (3.6) are sufficient conditions only. They tend to be more conservative as m in (3.2) becomes larger. Thus, for high-dimensional systems the stability bounds could be too restrictive to be useful. In such cases it is suggested that the E matrix be constructed as $E = \beta(B_f \bar{K}_f)$, where $\bar{K}_f$ is obtained in (2.5), and the bound on β is found using (3.5) or (3.6). In this way, there are bounds to be established on only one parameter β. These bounds appear to be less conservative than before. This is illustrated in Example 3.

Remark 2

In implementing the MPIM in practice, cases when the impaired systems are unstable must be considered. In such cases, the system has to be stabilized first by modifying the feedback gain. Many methods can be used here, but the key issue is that this process has to be computationally efficient. Only after the system is stabilized can the MPIM be applied to improve the performance of the system.

Remark 3

From Remarks 1 and 2 above, it can be seen, in general, that one way to adjust the reconfigured feedback gain is to use $k_{\mathrm{MPIM}} = k_1 + \bar{\beta}\,\Delta k$, where k_1 is the stabilizing feedback gain, $\bar{\beta}$ is the stability bound found for β in Remark 1 above and Δk is obtained by $\Delta k = B_f^+((A + BK) - (A_f + B_f k_1))$.

In a general MIMO system, a closed-form solution to problem (3.8) does not appear to exist. Note that (3.8) can be viewed as a minimization problem subject to simple bounds, and this is a well-studied problem in optimization theory. The choice for the particular optimization algorithm used should be guided by the requirement for rapid feasible solutions. For a quick solution with a reasonable sacrifice in optimality, an algorithm is proposed as follows.

Let $\bar{A} = A + BK - A_f = [\bar{a}_1\ \bar{a}_2\ \dots\ \bar{a}_n]$, $K_f = [k_{f1}\ k_{f2}\ \dots\ k_{fn}]$, where $\bar{a}_i$ and k_{fi}, $i = 1, 2, \dots, n$, are column vectors. Then J can be written as

$$\begin{aligned} J^2 &= \|[\bar{a}_1 - B_f k_{f1}\quad \bar{a}_2 - B_f k_{f2}\quad \dots\quad \bar{a}_n - B_f k_{fn}]\|_F^2 \\ &= \|\bar{a}_1 - B_f k_{f1}\|_2^2 + \|\bar{a}_2 - B_f k_{f2}\|_2^2 + \dots + \|\bar{a}_n - B_f k_{fn}\|_2^2 \end{aligned} \tag{2.8}$$

For the K_f obtained from (2.5), its ith column has the form $k_{fi} = B_f^+ \bar{a}_i$, which is the least-squares solution of the equation $\bar{a}_i = B_f k_{fi}$. Such k_{fi} minimizes the vector norm $\|\bar{a}_i - B_f k_{fi}\|_2^2$ and $\|k_{fi}\|_2^2$ for $i = 1, 2, \dots, n$. Let

$$\bar{k}_{fi} = k_{fi} + \psi_i \tag{3.13}$$

where ψ_i is a column vector with its elements chosen such that all elements in $\bar{k}_{fi}$ satisfy the stability constraints (3.7 *a*) and $\|\psi_i\|_2$ is the smallest possible.

Algorithm 2

Step 1. Calculate K_f from (2.5).

Step 2. Check the stability of (2.4). If it is stable, stop. Otherwise

Step 3. Set $\psi_i(j) = 0$ if $k_{fi}(j)$ satisfies (3.7 *a*), otherwise set $\psi_i(j) = \delta' - \mathrm{sgn}\,(k_{fi}(j)) \cdot |k_{fi}(j)|$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Step 4. Calculate the solutions

$$\bar{K}_f = K_f + \Psi \tag{3.14}$$

where Ψ has ψ_i, $i = 1, 2, \dots, n$, as its columns.

Note that the solution given in (3.14) satisfies the stability constraints (3.7 *a*) by the nature of the algorithm. Although it is no longer the optimal solution, the distance from this solution to the optimal one is bounded by

$$\|\bar{K}_f - K_f\|_F \leqslant \|\Psi\|_F \tag{3.15}$$

A bound on the increase of the performance index can also be found. Let $\bar{J} = \|(A + BK) - (A_f + B_f \bar{K}_f)\|_F$ with the J bounded by

$$\bar{J} - J \leqslant \|B_f\|_F \|\Psi\|_F \tag{3.16}$$

This can be seen from

$$\begin{aligned}\bar{J} - J &= \|(A + BK) - (A_f + B_f\bar{K}_f)\|_F - \|(A + BK) - (A_f + B_f K_f)\|_F \\ &\leqslant \|B_f K_f - B_f \bar{K}_f\|_F \\ &\leqslant \|B_f\|_F \|\Psi\|_F\end{aligned}$$

Example 3: *Flight control design* (Friedland 1986, pp. 321–322)

Let the nominal system (A, B, C) be given by

$$A = \begin{bmatrix} -0.0507 & -3.861 & 0. & -32.17 \\ -0.0012 & -0.5164 & 1.0 & 0. \\ -0.0001 & 1.4168 & -0.4932 & 0. \\ 0. & 0. & 1.0 & 0. \end{bmatrix},$$

$$B = \begin{bmatrix} 0. \\ -0.717 \\ -1.645 \\ 0. \end{bmatrix}, \quad C = \begin{bmatrix} 0. \\ 0. \\ 1.0 \\ 0. \end{bmatrix}^{T}$$

A full state feedback is used with gain $k = [-0{\cdot}0043 \;\; -3{\cdot}872 \;\; -0{\cdot}7186 \;\; -0{\cdot}0988]$, which assigns the closed-loop eigenvalues to the desired locations: $\{-0{\cdot}0095 \pm 0{\cdot}0941i, -1{\cdot}25 \pm 2{\cdot}1655]$. Suppose a failure occurred and the model of the impaired system were given by

$$A_f = A, \quad B_f = \begin{bmatrix} 0{\cdot} \\ -0{\cdot}0717 \\ -0{\cdot}1645 \\ 0{\cdot} \end{bmatrix} \quad C_f = C$$

The PIM method (2.5) gives an unacceptable solution, $k_{PIM} = [-0{\cdot}0367 \;\; -33{\cdot}17 \;\; -6{\cdot}154 \;\; -0{\cdot}8456]$, where the closed-loop 'A' matrix $(A_f + B_f k_{PIM})$ is unstable.

Using the MPIM method, we must first stabilize the impaired system, since neither A_f nor $(A_f + B_f k)$ is stable. There are many ways to accomplish this. Here, it is done by solving the matrix algebraic Riccati equation:

$$0 = SA_f + A_f^T S - SB_f R^{-1} B_f^T S + Q$$

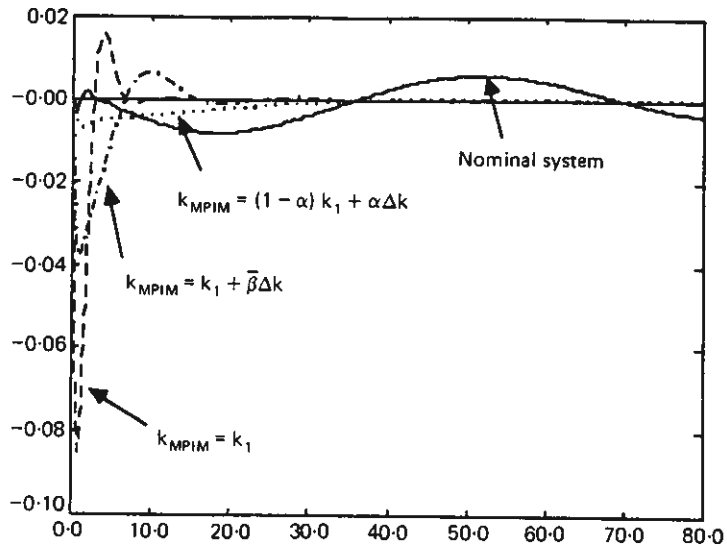
where R and Q are standard weighting matrices used in LQR. The stabilizing feedback gain has the form

$$k_1 = -R^{-1} B_f^T S$$

Note that this solution will guarantee the stability if the impaired system is stabilizable and detectable. To make k_1 reasonably small we choose $Q = 1$ and $R = 10$, and the resulting k_1 is $k_1 = [0{\cdot}2925 \;\; -8{\cdot}83 \;\; -13{\cdot}86 \;\; -16{\cdot}74]$. Although this gain stabilizes the system it does not provide a desirable performance (see the Figure). To recover the performance, the PIM is used in the form:

$$\Delta k = B_f^+((A + BK) - (A_f + B_f k_1))$$

The idea is to adjust k by the amount Δk so that $(A_f + B_f(k_1 + \Delta k))$ is close to the nominal closed-loop 'A' matrix $(A + BK)$. Unfortunately the Δk so obtained does



Comparison of outputs of the reconfigured system with different feedback gains.

not preserve stability; therefore, the stability bound should be found and enforced. As mentioned, the gain obtained by MPIM can be used to establish a stability bound on each element of Δk. However, less-conservative results were obtained using the form $k_{\mathrm{MPIM}} = k_1 + \bar{\beta}\,\Delta k$, where $\bar{\beta} = 0{\cdot}6$ is the stability bound found by using the method due to Yedavalli (1988). The performance is much improved by using this new gain (see the Figure).

It is interesting to mention that in some cases the performance can be further improved by tuning both k_1 and Δk, i.e.

$$K_{\mathrm{MPIM}} = (1-\alpha)k_1 + \alpha\,\Delta k$$

where α is increased from zero to the largest number (smaller than one) before the system becomes unstable. It turns out the gain obtained in this way ($\alpha = 0{\cdot}48$) gives the best transient response (see the Figure). In fact, this gain gives smaller changes in eigenvalues and eigenvectors than the gains obtained above. This approach worked very well for this example. Further investigation is necessary to establish the validity of the approach in general.

4. Conclusions

In this paper, the stability properties of the pseudo-inverse method in reconfigurable control have been analysed. To guarantee stability, the reconfigurable control problem was formulated as a constrained minimization problem and a modified pseudo-inverse method was proposed which guarantees the stability of the reconfigured system. A closed-form solution was derived for single-input systems. For general multi-input/multi-output systems, the problem was formulated as a minimization problem subject to simple bounds. A simple solution is found by sacrificing the optimality of the solution to the stability of the reconfigured system. This new method is illustrated by an aeroplane control example.

References

BARLETT, A. C., HOLLOT, C. V., and LIN, H., 1988, Root locations of an entire polytope of polynomials: it suffices to check the edges. *Math Control Signal Systems*, **1**, 61–71.

BARMISH, B. R., 1984, Invariance of the strict hurwitz property for polynomial with perturbed coefficients. *I.E.E.E. Transactions on Automatic Control*, **29**, 935–936; 1988, New tools for robustness analysis. *Proceedings of the 27th I.E.E.E. Conference on Decision and Control*, 1–6.

CAGLAYAN, A. K., ALLEN, S. M., and WEHMULLER, K., 1988, Evaluation of a second generation reconfiguration strategy for aircraft flight control systems subjected to actuator failure surface damage. *Proceedings of the* 1988 *I.E.E.E. National Aerospace and Electronics Conference*, 520–529.

CHANDLER, P. R., 1984, Self-repairing flight control system reliability and maintainability program executive overview. *Proceedings of the* 1984 *I.E.E.E. National Aerospace and Electronics Conference*, 586–590.

ESLINGER, R. A., and Chandler, P. R., 1988, Self-repairing flight control system program overview. *Proceedings of the* 1988 *I.E.E.E. National Aerospace and Electronics Conference*, 504–511.

FRIEDLAND, B., 1986, *Control System Design* (New York: McGraw-Hill) 322.

GAVITO, V. F., and COLLINS, D. J., 1987, Application of eigenstructure assignment to self reconfiguring aircraft MIMO controllers. *Proceedings of A.I.A.A. Guidance and Control Conference*, 1–12.

GILL, P. E., MURRAY, W., and WRIGHT, M. H., 1981, *Practical Optimization* (London: Academic Press).

GOLUB, G. H., and VAN LOAN, C. F., 1983, *Matrix Computations* (Baltimore, MD: Johns Hopkins University Press).

HOWELL, W. E., BUNDICK, W. T., HUESCHEN, R. M., and OSTROFF, A. J., 1983, Restructurable controls for aircraft. *Proceedings of the A.I.A.A. Guidance and Control Conference*, 646–651.

HUBER, R. R., and MCCULLOCH, B., 1984, Self-repairing flight control system. Society of Automotive Engineers Technical Paper Series, 841552, *Aerospace Congress and Exposition*, Long Beach, California.

LAWSON, C. L., and HANSON, R. J., 1974, *Solving Least Squares Problems*, (Englewood Cliffs, NJ: Prentice Hall).

LOOZE, D. P., WEISS, J. L., ETERNO, J. S., and BARRETT, N. M., 1985, An automatic redesign approach for restructurable control systems. *I.E.E.E. Control Systems Magazine*, 16–22.

MCMAHAN, J., 1978, Flight 1080, *Air Line Pilot*.

NTSB, 1979, National transportation safety board accident report of the American airlines DC10 crash at Chicago–O'Hare International Airport, May 25, 1979, NTAB-AAR-79-17.

OSTROFF, A. J., 1985, Techniques for accommodating control effector failures on a mildly statically unstable airplane. *Proceedings of the American Control Conference*, 906–903.

OSTROFF, A. J. and HUESCHEN, R. M., 1984, Investigation of control law reconfigurations to accommodate a control element failure on a commercial airplane. *Proceedings of the American Control Conference*, 1746–1754.

POTTS, D. W., and D'AZZO, J., 1981, Direct digital design method for reconfigurable multivariable control laws for the A-7D digital II aircraft. *Proceedings of the* 1981 *I.E.E.E. National Aerospace and Electronics Conference*, **3**, 1284–91.

RATTAN, K. S., 1985, Evaluation of control mixer concept for reconfiguration of flight control system. *Proceedings of the* 1985 *I.E.E.E. National Aerospace and Electronics Conference*, 560–569.

RAZA, S. J., and SILVERTHORN, J. T., 1985, Use of pseudo-inverse for design of a reconfigurable flight control system. *Proceedings of the A.I.A.A. Guidance and Control Conference*, 349–356.

STEWART, G. W., 1973, *Introduction to Matrix Computation* (New York: Academic Press).

YEDAVALLI, R. K., 1988, Stability robustness measures under dependent uncertainty. *Proceedings of the American Control Conference*, 820–823.

ZHOU, K., and KHARGONEKAR, P. P., 1987, Stability robustness bounds for linear state-space models with structured uncertainty. *I.E.E.E. Transactions on Automatic Control*, **32**, 621–623.