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Cyclicity and Controllability in Linear Time-Invariant Systems

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***Abstract*—A number of results, involving the notions of controllability (observability) and cyclicity in a linear time-invariant control system, are derived using a new basic theorem. New tests of cyclicity and controllability (observability), together with new algorithms to evaluate the controllable (observable) modes and the minimal polynomial are also presented.**

I. Introduction

The controllability (observability) of a linear time-invariant system $\{A,B\}$ and the cyclicity of a square matrix A have been dealt with extensively in the literature in recent years, and many important properties have been shown using a variety of methods. In this paper, a basic theorem (Theorem 1) dealing with the linear independence of the K matrices $B, AB, \cdots, A^{K-1}B$ is presented, and a combined simple test of controllability and cyclicity is given (Corollary 1). A method to evaluate the minimal polynomial of A_c, the controllable part of A, is then introduced (Theorem 2), and its use in evaluating the minimal (or characteristic) polynomial of A as well as the controllable modes of the system is indicated. Corollary 3 presents a new test for the cyclicity of A and in Section IV, a new simple proof to an important property of linear control systems, namely, the ability to reduce a multiinput system to a single input controllable system, is given.

II. Preliminaries

Assume that the linear time-invariant system

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1}$$

is given where $A\in R^{n\times n}$, $B\in R^{n\times m}$, and $x(t)$, $u(t)$ are the state and the input vectors, respectively. It is known [1] that there exists an equivalence transformation matrix Q such that

$$\bar{A}\triangleq QAQ^{-1}=\begin{bmatrix} A_c & A_{c\bar{c}} \\ 0 & A_{\bar{c}}\end{bmatrix},\quad \bar{B}\triangleq QB=\begin{bmatrix} B_c \\ 0\end{bmatrix} \tag{2}$$

with $A_c\in R^{\bar{n}\times\bar{n}}$, $B_c\in R^{\bar{n}\times m}$, and $\{A_c,B_c\}$ completely controllable. Furthermore, $|sI-A|=|sI-\bar{A}|=|sI-A_c|\cdot|sI-A_{\bar{c}}|$ where $|sI-A_c|$ is the polynomial with roots the $\bar{n}\leqslant n$ controllable poles of the system. Clearly, if $\rho M\triangleq$ rank M, then

$$\begin{aligned}\rho[B,AB,\cdots,A^{n-1}B]&=\rho[\bar{B},\bar{A}\bar{B},\cdots,\bar{A}^{n-1}\bar{B}]\\ &=\rho\begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{n-1}B_c \\ 0 & 0 & \cdots & 0\end{bmatrix}=n.\end{aligned} \tag{3}$$ [1]

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[1] See also [2].

Let $q(\leqslant\bar{n})$ be the degree of the minimal polynomial of A_c; note that q is uniquely determined by (2), since any other $\tilde{A}_c$ which results from an equivalent to (2) representation $\{\tilde{A},\tilde{B}\}$ with the same structure will obey a relation of the form $\tilde{A}_c=\tilde{Q}A_c\tilde{Q}^{-1}$, which preserves [3] the minimal polynomial. Also let b_i, $i=1,2,\cdots,m$ denote the ith column of B and define

$$p_i\triangleq\begin{bmatrix} A^ib_1 \\ A^ib_2 \\ \vdots \\ A^ib_m\end{bmatrix};\quad P_{AB}^K\triangleq[p_0,p_1,\cdots,p_{K-1}]. \tag{4}$$

The main theorem of this paper can now be stated and proved.

III. Main Results

Theorem 1: The K matrices $B,AB,\cdots,A^{K-1}B$ are linearly independent ($\rho P_{AB}^K=K$) if and only if $K\leqslant q$.

Proof Necessity: If $K>q$, then, from the definition of the minimal polynomial, there exist reals a_i, $i=0,1,\cdots,K-1$ such that $\sum_{i=0}^{K-1}a_iA_c^i=0$; this in turn implies that

$$\sum_{i=0}^{K-1}a_iA_c^iB_c=0,\quad \sum_{i=0}^{K-1}a_i\begin{bmatrix}A_c^iB_c\\0\end{bmatrix}=0,\quad \sum_{i=0}^{K-1}a_i\bar{A}^i\bar{B}=0,\quad \sum_{i=0}^{K-1}a_iA^iB=0,$$

i.e., $B,AB,\cdots,A^{K-1}B$ are not linearly independent. Thus, $K\leqslant q$ is necessary.

Sufficiency: Let $K\leqslant q$, but $B,AB,\cdots,A^{K-1}B$ linearly dependent, i.e., there exist reals a_i, $i=0,\cdots,K-1$ such that $\sum_{i=0}^{K-1}a_iA^iB=0$ or $\sum_{i=0}^{K-1}a_iA_c^iB_c=0$. If the last relation is premultiplied in turn by $A_c,A_c^2,\cdots,A_c^{\bar{n}-1}$, the relation

$$[a_0I_{\bar{n}}+a_1A_c+\cdots+a_{K-1}A_c^{K-1}]\cdot[B_c,A_cB_c,\cdots,A_c^{\bar{n}-1}B_c]=0$$

is obtained from which $a_0I_{\bar{n}}+a_1A_c+\cdots+a_{K-1}A_c^{K-1}=0$, since $\{A_c,B_c\}$ is controllable. This clearly implies that $K-1\geqslant q$ or $K>q$, i.e., $K\leqslant q$ is also sufficient. Note that in view of (4), $\rho P_{AB}^K=K$ if and only if the matrices $B,AB,\cdots,A^{K-1}B$ are linearly independent, which establishes the part in parentheses.

Corollary 1: The n matrices $B,AB,\cdots,A^{n-1}B$ are linearly independent ($\rho P_{AB}^n=n$) if and only if $\{A,B\}$ is completely controllable and A is cyclic.

Proof: From Theorem 1, $\rho P_{AB}^n=n$ iff $n\leqslant q$. Note, however, that $q\leqslant\bar{n}\leqslant n$ always. Consequently, $\rho P_{AB}^n=n$ iff $q=\bar{n}=n$.

Remark: Corollary 1 clearly suggests a new rank test (rank (P_{AB}^n)) to determine whether or not the given system (1) has two important properties, namely, if it is completely controllable, *and* if the state matrix A is cyclic. Note that this test can be easily carried out since, in view of the above, P_{AB}^n can be directly constructed from the controllability matrix of $\{A,B\}$ and the full rank of P_{AB}^n can also be tested using the determinant of the $(n\times n)$ Grammian matrix [4] $[g_{ij}]$ where $g_{ij}\triangleq p_{i-1}^T p_{j-1}$, $i,j=1,2,\cdots,n$.

The following corollary is important in establishing Theorem 2.

Corollary 2: $A^KB+\sum_{i=0}^{K-1}a_iA^iB=0$ ($A_c^KB_c+\sum_{i=0}^{K-1}a_iA_c^iB_c=0$) implies $A_c^K+\sum_{i=0}^{K-1}a_iA_c^i=0$ if and only if $K\geqslant q$.

Proof: Note that the relation inside the parentheses is equivalent to the first relation, since it is derived from the equivalent to (1) representation (2). Clearly, in view of Theorem 1, $K\geqslant q$ is necessary for the linear dependence of $B,AB,\cdots,A^KB$; it is also necessary for the existence of an annihilating polynomial of A_c of degree K. If the relation inside the parentheses is now premultiplied in turn by $A_c,A_c^2,\cdots,A_c^{\bar{n}-1}$, the relation $[A_c^K+\sum_{i=0}^{K-1}a_iA_c^i][B_c,A_cB_c,\cdots,A_c^{\bar{n}-1}B_c]=0$ is obtained, or the desired $A_c^K+\sum_{i=0}^{K-1}a_iA_c^i=0$ since $\{A_c,B_c\}$ is controllable. Thus, $K\geqslant q$ is also a sufficient condition.

Remark: If $A^KB+\sum_{i=0}^{K-1}a_iA^iB=0$ is written as

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$$P_{AB}^{K}\begin{bmatrix} a_0 \\ \vdots \\ a_{K-1} \end{bmatrix} = -p_K \tag{5}$$

then it is clear that the set of K reals a_i, $i=0,\cdots,K-1$ which satisfies (5) is unique iff $\rho P_{AB}^{K}=K$, or in view of Theorem 1, iff $K \leqslant q$. Consequently, in view of Corollary 2, a unique annihilating polynomial of A_c of degree K is implied by the linear dependence of $B, AB, \cdots, A^K B$ iff $K=q$. Note that this agrees with the well-known result of the uniqueness of the minimal polynomial (of A_c) and the nonuniqueness of any annihilating polynomial (of A_c) of higher degree.

An important theorem is now presented, which relates the columns of the controllability matrix $[B, AB, \cdots, A^{n-1}B]$ to the coefficients of the minimal polynomial of the controllable part of (1). Namely:

Theorem 2: The minimal polynomial of A_c is $s^q+\sum_{i=0}^{q-1}a_i s^i=0$ where $q \triangleq \rho P_{AB}^{n}$ and a_i, $i=0,1,\cdots,q-1$ is the unique set of reals which satisfy

$$P_{AB}^{q}\begin{bmatrix} a_0 \\ \vdots \\ a_{q-1} \end{bmatrix} = -p_q \qquad \left(A^q B + \sum_{i=0}^{q-1} a_i A^i B = 0\right).$$

Proof: In view of Theorem 1, it is clear that only the first q columns of P_{AB}^{n} (only $B,\cdots,A^{q-1}B$) are linearly independent, i.e., $\rho P_{AB}^{n}=\rho P_{AB}^{q}=q$. This implies that there exists a unique set of reals a_i, $i=0,\cdots,q-1$ such that $P_{AB}^{q}[a_0,\cdots,a_{q-1}]^T=-p_q$ $(A^qB+\sum_{i=0}^{q-1}a_iA^iB=0)$. Corollary 2 together with its Remark directly now imply that $A_c^q+\sum_{i=0}^{q-1}a_iA_c^i=0$, i.e., $s^q+\sum_{i=0}^{q-1}a_is^i$ is the unique minimal polynomial of A_c.

Remark: Theorems 1 and 2 are quite general and perhaps their generality obscures their usefulness and applicability, which is best shown through some special cases. Observe that if $\{A,B\}$ is controllable, then Corollary 1 provides a new rank test for the cyclicity of A, and Theorem 2 suggests a direct method of calculating the minimal polynomial of A (or the characteristic polynomial if A is cyclic). Furthermore, if A_c is cyclic (or A is cyclic in which case, as it can be easily shown, A_c will be cyclic as well), Theorem 2 gives the part of the characteristic polynomial of the state matrix A which contains the controllable modes of the system, i.e., $|sI_{\bar{n}}-A_c|$.

Corollary 3: $\rho P_{AI}^{n}=n$ if and only if A is cyclic. Furthermore, the minimal polynomial of A is $s^q+\sum_{i=0}^{q-1}a_is^i$, with $q \triangleq \rho P_{AI}^{n}$ and a_i, $i=0,\cdots,q-1$ the unique set of reals which satisfies

$$P_{AI}^{q}\begin{bmatrix} a_0 \\ \vdots \\ a_{q-1} \end{bmatrix} = -\begin{bmatrix} A^q e_1 \\ \vdots \\ A^q e_n \end{bmatrix}$$

where e_i is the zero column vector with unit at the ith position.

Proof: Corollary 3 is Theorem 2 for the case $B=I_n$ and $A_c=A$. Note that $\{A,I\}$ is completely controllable for any A.

Remark: Corollary 3 gives a simple new rank test for the cyclicity of A, and at the same time, provides an algorithm for the evaluation of the minimal (or the characteristic in case A is cyclic) polynomial of A. Note that this algorithm is similar to Krylov's algorithm[2] for the evaluation of the minimal polynomial, although it does not depend on the choice of a vector x such that $x, Ax, \cdots, A^{q-1}x$ are linearly independent (a drawback of the method). Similarly, contrary to the existing methods, the above test for cyclicity depends strictly on the matrix A.

IV. A New Proof to an Important Property

The above results can also be used to provide a simple new proof to a known important property of linear control theory, namely:

Corollary 4: Given $\{A,B\}$ there exists a column vector g such that $\{A,Bg\}$ is controllable if and only if $\{A,B\}$ is completely controllable and A cyclic. If this is the case, then almost any g will suffice. (It should be noted at this point that this useful property was first shown by Wonham [5, Lemma 3] in a complicated manner. The following proof is completely different and much simpler.)

Proof: Let $g=\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}$ and premultiply P_{AB}^{n} by the $(nm \times nm)$ matrix

$$S \triangleq \begin{bmatrix} g_1 I_n & g_2 I_n & \cdots & g_m I_n \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & I_n \end{bmatrix}.$$

If there exists a vector g such that $\{A,Bg\}$ is controllable, then the $(mn \times n)$ matrix product SP_{AB}^{n} will have full rank n. This follows from the fact that the first n rows of the SP_{AB}^{n} will be linearly independent since, as a simple calculation shows, they are the n rows of the $(n \times n)$ controllability matrix $[Bg, ABg, \cdots, A^{n-1}Bg]$. But the rank of a matrix product is always less than or equal to the rank of the factors, which implies that $\rho P_{AB}^{n}=n$ and in view of Corollary 1, that $\{A,B\}$ is completely controllable and A is cyclic. Assume now that $\{A,B\}$ is completely controllable and A is cyclic, i.e., $\rho P_{AB}^{n}=n$. Then, the first n rows of SP_{AB}^{n}, i.e., $[Bg, ABg, \cdots, A^{n-1}Bg]$, will be linearly independent for almost any g, since $|Bg, \cdots, A^{n-1}Bg|$, which is the sum of the products of all the nth order minors of $[g_1I_n, g_2I_n, \cdots, g_mI_n]$ multiplied by the corresponding nth order minors of P_{AB}^{n} (at least one of which is nonzero), is actually a multivariable polynomial in $g_1, g_2, \cdots, g_m$ and becomes zero only when $g_1, g_2, \cdots, g_m$ take on values equal to the roots of this polynomial.

Remark: Using duality, similar results involving observability instead of controllability can be directly derived.

V. Conclusions

In this paper, it has been shown that a number of important results involving the notions of controllability (observability) and cyclicity can be derived from a basic, simple theorem (Theorem 1). A new proof has been given to a useful property (the reduction of a multiinput system to a single-input controllable system), tests for cyclicity and controllability have been presented, and methods to evaluate the controllable (observable) modes of the system and the minimal polynomial have been shown.

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Some Properties of the Value Matrix in Infinite-Time Linear–Quadratic Differential Games

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Abstract—**The null space (and range space) of the value matrix in an infinite-time linear quadratic differential game is characterized.**

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