III. An Example

To illustrate the preceding, let us consider the following example which is given in [2, Sec. 7.3]: In particular, suppose

$$T_S(s)=\frac{\begin{bmatrix} -s^2+1 & 0 \\ -s & s^2 \end{bmatrix}}{-s^3+s^2}=\underbrace{\begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}}_{R(s)}\underbrace{\begin{bmatrix} s^2 & 0 \\ 1 & -s+1 \end{bmatrix}^{-1}}_{P^{-1}(s)}.$$

We note that $R(s)$ and $P(s)$ are relatively right prime and that $P(s)$ is column proper. Furthermore, since $T_S(s)$ is strictly proper, $R(s)$ is of lower column degree than $P(s)$. As is shown in [2], if

$$F(s)=\begin{bmatrix} -2s-2 & 0 \\ s+2 & 2 \end{bmatrix},$$

then $|P(s)-F(s)|=(s+1)(s^2+2s+2)$; i.e., all 3 (controllable and observable) poles of the system can be arbitrarily positioned in $\mathrm{Re}(s)<0$, at $s=-1$ and $-1\pm j$, by employment of the Luenberger observer given in [2] with (see Theorem 1)

$$Q(s)=\begin{bmatrix} s & 5 \\ -1 & s+2 \end{bmatrix},\quad H(s)=\begin{bmatrix} 3s & 10 \\ 2s+2 & 2s+4 \end{bmatrix},$$

and

$$K(s)=\begin{bmatrix} -5 & 0 \\ -1 & 0 \end{bmatrix}.$$

In view of Theorem 2, therefore, if we employ the proper feedforward compensator

$$T_{\hat{C}}(s)=[Q(s)-K(s)]^{-1}H(s)=\frac{\begin{bmatrix} 3s^2+6s & 0 \\ 2s^2+12s+10 & 2s^2+14s+20 \end{bmatrix}}{s^2+7s+10}$$

in series with $T_S(s)$ in the unity feedback configuration of Fig. 2, the overall closed-loop system $S\hat{C}$ will be asymptotically stable with poles given by the zeros of $|Q(s)|=s^2+2s+5$ (i.e., $s=-1\pm j2$) as well as those of $|P(s)-F(s)|$.

IV. Remarks

In conclusion, it might again be noted that the stabilization scheme of Fig. 2 is based directly on the observation that linear variable feedback, as implemented by a Luenberger observer, can be used to arbitrarily position all of the unstable poles of a closed-loop stabilizable system, and we have merely shown that a Luenberger observer ($\underset{\sim}{C}$) has an "equivalent" feedforward/unity feedback representation ($\hat{C}$). Strictly speaking, the two configurations are not equivalent since the zeros of $|Q(s)|$ in the $S\hat{C}$, Fig. 2 configuration are not "canceled" as they are in the SC, Fig. 1 configuration.

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Stable Proper *n*th-Order Inverses

P. J. ANTSAKLIS

***Abstract*—A stable proper right (left) *n*th-order inverse of a given linear time-invariant system of order *n* can always be constructed, via a simple algorithm, if a proper right (left) inverse exists and the zeros of the given system are stable. Furthermore, it is shown that all of the poles of this inverse can be arbitrarily assigned except those which equal the zeros of the given system.**

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The author is with the Department of Computing and Control, Imperial College, London, England, on leave from the Division of Engineering, Brown University, Providence, RI 02912.

I. Introduction

This technical note deals with the important problem of obtaining a stable proper right (left) inverse of a given nth-order system. As it was shown in [1], there is no guarantee (it rather depends on the particular system) that a minimum-order stable proper inverse exists, even when all the zeros of the given system are stable. Furthermore, the problem of finding a stable proper inverse of the lowest possible order generally requires extensive searching (increasing the order of the proper inverse and testing for stabilizability), and therefore, the need for a simple method which guarantees a stable inverse if one exists. Although recent work [1] has provided a technique to obtain a stable proper solution to the more general exact model matching problem, this method involves extensive manipulation of polynomial matrices and can be quite cumbersome even in the special case of the right (left) inverse, where, in addition, an inverse of order generally higher than n is obtained.

A simple algorithm in the state-space is presented in this note which guarantees a stable nth-order proper right (left) inverse if a proper right (left) inverse of a given nth-order system exists and the zeros of the given system are stable. In particular, a method is presented to construct a proper right inverse using an equivalent to linear state feedback feedforward compensation scheme (Lemma 3); using now a property of the zeros of the given system (Lemma 4), it is shown (Theorem 1) that, if a proper inverse of a system with stable zeros exists, then an nth-order stable proper inverse can always be found with all of its poles arbitrarily assignable except those which equal the zeros of the given system (Algorithm).

II. Preliminaries

Assume that a controllable and observable[1] linear time-invariant system $\{A,B,C,E\}$ is given, where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, $E\in R^{p\times m}$, and let the $(p\times m)$ rational matrix $T(s)$ be its proper transfer matrix; i.e.,

$$T(s)=C(sI-A)^{-1}B+E. \tag{1}$$

A system is a right (left) inverse of $\{A,B,C,E\}$ if its transfer matrix $T_R(s)(T_L(s))$ satisfies

$$T(s)T_R(s)=I_p \quad (T_L(s)T(s)=I_m). \tag{2}$$

It should be noted at this point that since the left inverse can be written as $T^T(s)T_L^T(s)=I_m$, i.e., as a right inverse of the dual of the given system, only the right inverse problem needs to be studied.

The linear state-variable feedback (lsvf) compensation plays an important role in the following and it is defined as the control law:

$$u(t)=Fx(t)+Gv(t) \tag{3}$$

where $F\in R^{m\times n}$, $G\in R^{m\times q}$, $u(t)$ and $x(t)$ the vectors of the input and the state of the given system (1), and $v(t)$ an external vector input. Note that if (1) is compensated by (3), the closed-loop system is $\{A+BF, BG, C+EF, EG\}$ with transfer matrix

$$T_{F,G}(s)=(C+EF)[sI-(A+BF)]^{-1}BG+EG. \tag{4}$$

III. Main Results

Before establishing the main theorem of this paper, some important lemmas are in order. Namely:

Lemma 1: Given system (1), a right inverse exists if and only if rank $T(s)=p$; a proper right inverse exists if and only if rank $E=p$; furthermore, an inverse system can be stable only if the zeros of (1) are stable.

[1]These assumptions, although not essential to the method, are being made to ensure minimum order n of the given system.

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Proof: These results are well known and their proofs can be found in a number of references, e.g., [1]–[4].

The following lemma provides a way of representing any linear state-variable feedback (lsvf) compensation scheme via a feedforward compensator. In particular:

Lemma 2: Any lsvf compensation scheme can be represented via feedforward compensation through a system with a proper transfer matrix. This feedforward compensator will be called "the equivalent to lsvf feedforward compensator $T_{ec}(s)$."

Proof: Let

$$T_{ec}(s)=F[sI-(A+BF)]^{-1}BG+G. \tag{5}$$

Then, calculations show that

$$\begin{aligned}T(s)T_{ec}(s)&=[C(sI-A)^{-1}B+E][F(sI-A-BF)^{-1}BG+G]\\&=(C+EF)[sI-(A+BF)]^{-1}BG+EG=T_{F,G}(s).\end{aligned} \tag{6}$$ [2]

In view of Lemma 2, it is now clear that the system

$$\{A+BF,BG,F,G\} \tag{7}$$

which is a state-space representation of $T_{ec}(s)$ of (5), represents the equivalent to lsvf feedforward compensation scheme.

The importance of $T_{ec}(s)$ in the right inverse problem is illustrated by the following lemma:

Lemma 3: If a proper right inverse exists, then it can always be constructed using the equivalent to lsvf feedforward compensation scheme.

Proof: Choose the gain matrices F and G so they satisfy

$$EF=-C;\ EG=I_p. \tag{8}$$

Then, in view of (6), $T_{F,G}(s)=I_p$ and the right inverse system is given by (5) or (7), where F and G satisfy (8). Note that there always exists such a pair (F,G), since rank $E=p(p\leqslant m)$ as it is implied by Lemma 1 and the assumption that a proper right inverse exists.

A convenient method to solve (8) with respect to (F,G) and obtain a proper right inverse of (1) is now presented; note that this particular technique will be used later on to construct a stable proper right inverse.

Find an $m\times m$ nonsingular matrix M such that $EM=[I_p:0]$. Then (8) becomes: $-C=EMM^{-1}F=[I_p:0]\begin{bmatrix}\tilde{F}_1\\ \tilde{F}_2\end{bmatrix}=\tilde{F}_1$ and $I_p=EMM^{-1}G=[I_p:0]\begin{bmatrix}\tilde{G}_1\\ \tilde{G}_2\end{bmatrix}=\tilde{G}_1$, which implies that

$$F=M\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix};G=M\begin{bmatrix}I_p\\ \tilde{G}_2\end{bmatrix} \tag{9}$$

satisfy (8) for any $\tilde{F}_2$ and $\tilde{G}_2$. Consequently, in view of the proof of Lemma 3, it is clear that the system

$$\left\{A+BM\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix},\quad BM\begin{bmatrix}I_p\\ \tilde{G}_2\end{bmatrix},\quad M\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix},\quad M\begin{bmatrix}I_p\\ \tilde{G}_2\end{bmatrix}\right\} \tag{10}$$

is a proper right inverse of (1) for any $\tilde{F}_2\in R^{(m-p)\times n}$ and $\tilde{G}_2\in R^{(m-p)\times p}$. Note that the poles of system (10) are the eigenvalues of $A+BF=A+BMM^{-1}F=A+[\tilde{B}_1\tilde{B}_2]\begin{bmatrix}\tilde{F}_1\\ \tilde{F}_2\end{bmatrix}=A-\tilde{B}_1C+\tilde{B}_2\tilde{F}_2$, which shows that $\tilde{F}_2$ can be used to alter the poles of the proper right inverse system (10). This of course depends on the controllability of $\{A-\tilde{B}_1C,\tilde{B}_2\}$, an observation which leads to the following lemma.

Lemma 4: The uncontrollable modes of $\{A-\tilde{B}_1C,\tilde{B}_2\}$ are the zeros of $\{A,B,C,E\}$.

Proof: The zeros of $\{A,B,C,E\}$ are those z_i [5] for which

$$\operatorname{rank}\begin{bmatrix}z_iI_n-A, & B\\ -C, & E\end{bmatrix}<n+\min(p,m)=n+p. \tag{11}$$

[2] Note that these calculations can be avoided if the differential operator representation and the structure theorem [2] are used.

If this matrix is postmultiplied first by $\begin{bmatrix}I_n & 0\\ 0 & M\end{bmatrix}$, where the $m\times m$ nonsingular matrix M satisfies $EM=[I_p:0]$, and then by

$$\begin{bmatrix}I_n & 0\\ -\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix} & I_m\end{bmatrix},$$

its rank will not be affected. That is, the matrix

$$\begin{bmatrix}z_iI_n-A, & B\\ -C, & E\end{bmatrix}\begin{bmatrix}I_n & 0\\ 0 & M\end{bmatrix}\begin{bmatrix}I_n & 0\\ -\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix} & I_m\end{bmatrix}$$

$$=\begin{bmatrix}z_iI_n-A+\tilde{B}_1C-\tilde{B}_2\tilde{F}_2, & \tilde{B}_1,\tilde{B}_2\\ 0, & I_p,0\end{bmatrix}$$

will have reduced rank (less than $n+p$) only if z_i is a zero of $\{A,B,C,E\}$. Note, however, that the z_i which reduce the rank of the last matrix are exactly those which reduce the rank of $[sI_n-[A-\tilde{B}_1C+\tilde{B}_2\tilde{F}_2],\tilde{B}_2]$ or of $[sI_n-(A-\tilde{B}_1C),\tilde{B}_2]$ which are exactly the uncontrollable poles of $\{A-\tilde{B}_1C,\tilde{B}_2\}$ [5].

Remark 1: In view of the comments following (10), it is clear that $\tilde{F}_2$ can be used to arbitrarily assign all of the poles of the proper right inverse (10) only if the system $\{A,B,C,E\}$ does not have any zeros. Generally, the poles of the proper right inverse (10) will consist of 1) a set of k poles equal to the zeros of $\{A,B,C,E\}$, and 2) a set of $n-k$ poles arbitrarily assignable via $\tilde{F}_2$. This is because the eigenvalues of $A+BF=A-\tilde{B}_1C+\tilde{B}_2\tilde{F}_2$ consist, as it was shown above, of two sets; one unaffected by $\tilde{F}_2$, i.e., the set of the k uncontrollable modes of $\{A-\tilde{B}_1C,\tilde{B}_2\}$ which equal the zeros of (1), and one arbitrarily assignable via $\tilde{F}_2$, i.e., the set of the $n-k$ controllable modes of $\{A-\tilde{B}_1C,\tilde{B}_2\}$.

The main result of this paper can now be stated.

Assume that system (1) is given where rank $E=p$, i.e., a proper right inverse exists. Then:

Theorem 1: If the zeros of (1) are stable, a stable nth-order right inverse can always be found. Specifically, a proper nth-order right inverse can always be constructed with k of its poles equal to the (k) zeros of (1) and the remaining $(n-k)$ poles arbitrarily assignable.

Proof: This result has already been established via Lemma 3, Lemma 4, and Remark 1.

An algorithm is now presented which realizes the claim of Theorem 1. Note that this algorithm is completely justified in view of the previously developed results.

Algorithm

Step 1: Find an $m\times m$ nonsingular matrix M such that $EM=[I_p:0]$.

Step 2: Calculate $[\tilde{B}_1,\tilde{B}_2]\triangleq BM$ and $A-\tilde{B}_1C$.

Step 3: Find a lsvf matrix $\tilde{F}_2$ which arbitrarily assigns the $n-k$ ($k=$number of zeros of (1)) controllable poles of $\{A-\tilde{B}_1C,\tilde{B}_2\}$.

Step 4: The desired stable proper right inverse is

$$\left\{A+BM\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix},\quad BM\begin{bmatrix}I_p\\ \tilde{G}_2\end{bmatrix},\quad M\begin{bmatrix}-C\\ \tilde{F}_2\end{bmatrix},\quad M\begin{bmatrix}I_p\\ \tilde{G}_2\end{bmatrix}\right\} \tag{12}$$

where $\tilde{F}_2$ was determined in Step 3 and $\tilde{G}_2$ any $(m-p)\times p$ real matrix (which can be taken 0 for convenience).

Remark 2: The pair (F,G) used in the above satisfies (8). Note, however, that $T_{F,G}(s)=I_p$, iff there exists (F,G) such that $EG=I_p$ and $(C+EF)[BG,(A+BF)BG,\cdots]=0$, as it can be easily seen if $T_{F,G}(s)$ is written as

$$T_{F,G}(s)=EG+\frac{1}{s}[(C+EF)BG]+\frac{1}{s^2}[(C+EF)(A+BF)BG]+\cdots.$$

This implies that (8) are only sufficient conditions; they are necessary and sufficient for $T_{F,G}(s)=I$ only if $\{A+BF,BG\}$ is completely controllable. Note that such a case is when $p=m$, where, in addition, the pair (F,G) is determined uniquely from (8) and the inverse (12) is unique. In view of the above, it is clear that there might exist stable proper nth order inverses of the form (7), which are not given by the algorithm.

IV. Conclusion

It has been shown in this note that, if a given nth-order system (1) has a stable proper right (left) inverse, then a stable proper right (left) inverse of order n can always be constructed using an equivalent to linear state-variable feedback feedforward compensation scheme. In particular, a simple algorithm was given to construct a proper (rank $E=p$) right inverse of order n with k of its poles equal to the (k) zeros of (1) and the remaining $(n-k)$ poles arbitrarily assignable. Finally, note that the zeros of system (1), which are being found in the process, are the uncontrollable modes of a new system (Lemma 4); this clearly suggests a method to evaluate the zeros of a given system.

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A Note on Selecting Low-Order Discrete-Time Dynamic Systems via Modal Methods

G. B. MAHAPATRA

Abstract—**A method is presented in this note to select the dimension of low-order discrete-time dynamic systems via the modal methods of Marshall and Davison.**

I. Introduction

Model reduction techniques for continuous-time state-space models have been the subject of many recent investigations with modal approaches of Davison [1], Chidambara and Davison [2], and Marshall [3]. These concepts have been applied recently by Wilson *et al.* [4] to reduce high-order discrete-time dynamic systems. An important problem in this area of research is to decide how small a large system can safely be simplified to before causing excessive error. Mahapatra [5] established these conditions for continuous-time state-space models using modal reduction techniques of Davison [1]. The criterion was expressed in terms of the lowest eigenvalue neglected, and the sizes of the high- and low-order systems. But it is not yet clear about the smallness of the simplified high-order discrete-time dynamic systems. An attempt is made in this note to establish this criteria for stable discrete-time dynamic systems employing modal reduction concepts of both Davison [1] and Marshall [3].

II. Development of Criterion via Davison's Approach

Consider an nth-order discrete-time dynamic system with zero initial condition:

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+2) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} u(k). \tag{1}$$

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The author is with the Department of Electrical Engineering, Roorkee University, Roorkee (U.P), India.

The m-state variables $X_1(k)$ are to be retained in the low-order model. Equation (1) can be written as

$$X(k+1) = GX(k) + Hu(k) \tag{2}$$

where G is a constant matrix of order $n\times n$, $u(k)$ is of order $r\times 1$. Let P be the modal matrix and D be the diagonal matrix such that the eigenvalues λ_i, $i=1,2,\cdots,n$ occupying the unit circle are arranged in descending order of magnitude.

$$D = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_m & & \\ & & & \lambda_{m+1} & \\ & & & & \lambda_n \end{bmatrix}. \tag{3}$$

Then

$$\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Y_1(k) \\ Y_2(k) \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} Y_1(k+1) \\ Y_2(k+1) \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} Y_1(k) \\ Y_2(k) \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} u(k) \tag{5}$$

where D_1 is the dominant eigenvalue matrix of order m, D_2 is the eigenvalue matrix of order $(n-m)$ to be neglected during simplification. δ_1 and δ_2 are the partitioned matrices of matrix δ, given by

$$\delta = P^{-1}H. \tag{6}$$

When eigenvalue matrix D_2 and the corresponding eigenvector for matrix are neglected to approximate solution of $X_1(k)$ in (4), the reduced solution is given by

$$X_1^*(k) = P_{11}Y_1(k). \tag{7}$$

From (4) and (7) the error is given by

$$\begin{aligned} E(k) &= X_1(k) - X_1^*(k) \\ &= P_{12}Y_2(k). \end{aligned} \tag{8}$$

Assuming u to be unit step function, from (5)

$$Y_2(k) = \sum_{i=0}^{k-1} (D_2)^{k-i-1}\delta_2. \tag{9}$$

From (8) and (9)

$$\|E(k)\| \leqslant \|P_{12}\|\,\|\delta_2\|\,\Big\|\sum_{i=0}^{k-1}(D_2)^{k-i-1}\Big\| \tag{10}$$

$$\leqslant \beta \sum_{i=0}^{k-1} \|(D_2)^{k-i-1}\| \tag{11}$$

where

$$\beta = \|P\|\,\|P^{-1}\|\,\|H\| = \text{Const. for the given plant.} \tag{12}$$

Retaining only the significant eigenvalue $\lambda_{m+1}, \|D_2^i\|$ can be approximated to

$$\|D_2^i\| \simeq |\lambda_{m+1}|^i; \qquad i=1,2,\cdots,(k-1). \tag{13}$$

From (11) and (13)

$$\|E(k)\| \leqslant \beta\left[\sqrt{n-m} + \sum_{i=1}^{k-1}|\lambda_{m+1}|^i\right]. \tag{14}$$

Expressing (14) in geometric series,

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