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ON THE THEORY OF POLYNOMIAL MATRIX INTERPOLATION AND ITS ROLE IN SYSTEMS AND CONTROL

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Introduction

In this paper, new theoretical results on polynomial and rational matrix interpolation are briefly outlined and their application to certain Systems and Control problems is summarized.

The use of interpolation type constraints in system and control theory is first discussed and a number of examples are presented. With this motivation, an appropriate formulation for polynomial matrix interpolation is introduced via its basic theorem, Theorem 1. It is shown that this generalization, of polynomial interpolation to polynomial matrix, is most general and it includes all other such interpolation schemes which have appeared in the literature. The basic theorem of rational matrix interpolation is also presented. A specialization of these results to involve characteristic values and vectors of polynomial matrices is shown; note that the eigenvalue, eigenvector problem of a real matrix can be seen as a special case. Equations involving polynomial and rational matrices are of great interest in systems and control. It is shown how to solve these equations using interpolation theory. Note that certain of these results have appeared in [1-4] and used in [5-6] and elsewhere. Here, for the first time, the overall approach is outlined and it is connected to interpolation conditions used by many in the system and control literature. This is in addition to new contributions in the polynomial matrix interpolation theory. It is hoped that this paper will provide some insight and will demonstrate the application of the polynomial matrix interpolation theory to the theory and practice of systems and control.

Motivation: Interpolation type constraints in Systems and Control theory

Many control system constraints and properties that are expressed in terms of conditions on a polynomial or rational matrix R(s), can be written in an easier to handle form in terms of $R(s_j)$, where $R(s_j)$ is R(s) evaluated at certain (complex) values $s = s_j$ $j = 1, \ell$. We shall call such conditions in terms of $R(s_j)$, interpolation (type) conditions on R(s). This is because in order to understand the exact implications of these constraints on the structure and properties of R(s), one needs to use results from polynomial interpolation theory. Note that although for the scalar polynomial case, interpolation is an old and very

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well studied problem, only recently polynomial matrix interpolation appears to have been addressed in any systematic way [1-3].

Next, a number of examples from Systems and Control theory where polynomial and polynomial matrix interpolation constraints are used, are outlined. This list is not complete, by far.

It is known that all the uncontrollable eigenvalues of $\dot{x} = Ax + Bu$ are given by the roots of the determinant of a greatest left divisor of the polynomial matrices $sI - A$ and B. An alternative, and perhaps easier to handle, form of this result is that s_j is an uncontrollable eigenvalue if and only if there exists real row vector v_j such that $v_j[s_jI - A, B] = 0$ (PBH controllability test). This is a more restrictive version of the previous result which involves left divisors, since it is not clear how to handle multiple eigenvalues when it is desirable to determine all uncontrollable eigenvalues. The results presented here can readily provide the solution to this problem.

Consider an example from control design: Given the plant P(s) and a controller G(s), the closed loop transfer function T(s) is given by $T(s) = (I + G(s)P(s))^{-1}G(s)P(s)$ assuming a unity feedback configuration. The design problem is now to find a controller G(s) if T(s) and P(s) are given. The method of Ragazzini (see description in [7] pp. 216-218) handles the scalar case. In particular, for stability, all unstable poles of P(s) must be zeros of 1 - T(s); and all unstable zeros of P(s) must be zeros of T(s) (note that T(s) is stable). Furthermore for causality, T(s) must have as its zeros at infinity all the zeros at infinity of P(s). Clearly, the condition $1 - T(s_j) = 0$ where s_j $j = 1, \ell$ are the unstable poles of P(s) is an alternative form of the first constraint on T(s); the case of multiple poles at s_j can be handled by differentiating. Similarly for the remaining constraints. The corresponding multivariable stability constraints were introduced in [8] and they are: $T(s) = N(s)X(s)$ and $X(s)$, $[I - X(s)N(s)]D^{-1}(s)$ stable where $P(s) = N(s)D^{-1}(s)$ a coprime polynomial factorization. These can also be expressed as interpolation constraints, however care should be taken when P(s) has multiple poles. Note that for a square polynomial matrix D(s) to have a zero of determinant at some value s_j, $D(s_j)a_j = 0$ where a_j is a nonzero (complex) vector. While in the polynomial case $d'(s_j) = 0$ is a necessary and sufficient condition for d(s) to have a second zero at s_j , a derivative of D(s) is not necessary in the matrix case.

The state feedback eigenvalue assignment problem has a rather natural formulation in terms of interpolation type constraints; similarly the output feedback problem[4-6].

More recently, stability constraints in the H^∞ formulation of the optimal control problem have been expressed in terms of interpolation type constraints[9-11]. It is rather

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interesting that [9-10] discuss a "directional" approach which is in the same spirit of the approach we take here (and in [1-7]).

The above are just a few of the many examples of the strong presence of interpolation type conditions in the systems and control literature; this is because they represent a convenient way to handle certain types of constraints. However, a closer look reveals that the relationships between conditions on $R(s_j)$ and properties of the matrix $R(s)$ are not clear at all and this needs to be explained. Only in this way one can take full advantage of the method and develop new approaches to handle control problems. Our research on matrix interpolation and its applications attempts to address this need.

Results from Polynomial Matrix Interpolation Theory

In the following some of the basic results of the polynomial and rational matrix interpolation theory are outlined. Note that this method, in addition to offering a systematic approach to solve certain system control problems, perhaps more importantly, it is a completely new theory developed from the bottom up and it opens a new field of study.

<u>Basic Theorem of Polynomial Matrix Interpolation</u>

The basic theorem of polynomial interpolation can be stated as follows: Given ℓ distinct complex points s_j $j = 1, \ell$ and ℓ complex values b_j there exists a unique $(\ell-1)$th degree polynomial $q(s)$ for which

$$q(s_j) = b_j \qquad j = 1, \ell \tag{1}$$

A generalization of this result to polynomial matrices is as follows:

Let $S(s) :=$ blk diag $[\, 1, s, \ldots, s^{d_i}]'$ where d_i $i = 1, m$ are non-negative integers; let $a_j \neq 0$ and b_j denote (mx1) and (px1) complex vectors respectively and s_j complex scalars.

<u>Theorem 1</u>: Given (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d_i with $\ell = \Sigma\, d_i + m$ such that the square $(\Sigma\, d_i + m)\mathrm{x}\ell$ matrix

$$S_\ell = [S(s_1)\, a_1, \ldots, S(s_\ell)a_\ell] \tag{2}$$

has full rank, there exists a unique (pxm) polynomial matrix $Q(s)$, with ith column degree equal to d_i, $i = 1, m$ for which

$$Q(s_j)a_j = b_j \qquad j = 1, \ell \tag{3}$$

It should be noted that when $p = m = 1$ and $d_1 = \ell-1$ this theorem reduces to the polynomial interpolation theorem. To see this, note that in this case the nonzero scalars a_j $j = 1, \ell$, can be taken to be equal to 1, in which case S_ℓ is exactly the well known Vandermonde matrix which is nonsingular if and only if s_j are distinct.

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Proof: Since the column degrees of Q(s) are d_i, Q(s) can be written as

$$Q(s) = QS(s) \tag{4}$$

where $Q(px(\Sigma d_i + m))$ contains the coefficients of the polynomial entries. Substituting in (3), Q must satisfy

$$QS_\ell = B_\ell \tag{5}$$

where $B_\ell = [b_1, ..., b_\ell]$. Since S_ℓ is nonsingular, Q and therefore Q(s) are uniquely determined. □

Example 1: Let Q(s) be 1x2 (=pxm) and let $\ell = 3$ interpolation points be specified: $\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{ (-1,[1, 0]', 0), (0,[-1, 1]', 0), (1, [0, 1]', 1)\}$. Q(s) is uniquely specified when d_1 and d_2 are chosen so that $\ell(=3) = \Sigma d_i + m = (d_1 + d_2) + 2$ or $d_1 + d_2 = 1$ and S_3 has full rank. Clearly the resulting Q(s) depends on the particular choice for the column degrees d_i and different combinations of d_i will result to different matrices Q(s):

(i) Let $d_1 = 1$, and $d_2 = 0$. Then S(s) = blk diag[[1,s]',1] and (5) becomes:

$$Q\,S_3 = Q\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0,\ 0,\ 1]$$

from which Q = [1, 1, 1] and Q(s) = QS(s) = [s+1, 1].

(ii) Let $d_1 = 0$, $d_2 = 1$. Then S(s) = blk diag[1, [1, s]'] and (2.5) gives Q = [0, 0, 1] from which Q(s) = [0, s], clearly different from (i) above. □

Note that alternative polynomial bases, other than $[1, s, s^2, ...]'$, which might offer computational advantages in determining Q(s) from interpolation equations (5) can of course be used (and they have); for example, Tschebyscheff polynomials.

The relation between (4) and an alternative, also commonly used representation of Q(s), is now shown, namely:

$$Q(s) = Q_d\, S_d(s) \tag{6}$$

where $S_d(s) := [I, ..., Is^d]'$ an m(d+1)xm matrix with $d = \max(d_i)\ \ i = 1, m$. Notice that $S(s) = KS_d(s)$ where K $((\Sigma d_i + m) \times m(d + 1))$ describes the appropriate interchanges of rows in S_d (s) needed to extract S(s). Representation (6) can be used in matrix interpolation as the following corollary shows:

Corollary 1: Given $(s_j, a_j, b_j)\ j = 1, \ell$ and nonnegative integer d with $\ell = m(d+1)$ such that the $m(d+1)x\ell$ matrix $S_{d\ell} = [S_d(s_1)\, a_1, ..., S_d(s_\ell) a_\ell]$ has full rank, there exists a unique (pxm) polynomial matrix Q(s) with highest polynomial entry degree d which satisfies (3).

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Example 2 Let Q(s) be 1x2 (= pxm), d = 1 and let the $\ell = m(d+1) = 4$ interpolation points (s_j, a_j, b_j) be as follows: let the first 3 be the same as in Example 1 and the fourth be (2, [0, 1]', 1). The equation $Q_d S_{d\ell} = B_\ell$ now becomes

$$Q_d S_{d\ell} = Q_d \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [0, 0, 1, 1] = B_\ell$$

from which $Q_d = [1, 1, 1, 0]$ and $Q(s) = Q_d S_d(s) = [\, s+1, 1]$ as in Example 2.1 (i). If the fourth interpolation point is taken to be equal to (2, [0, 1]', 2) then $B_\ell = [0, 0, 1, 2]$ while $S_{d\ell}$ remains the same. Then $Q_d = [0, 0, 0, 1]$ and $Q(s) = Q_d S_d(s) = [\, 0, s]$ as in Example 1(ii). □

Interpolation constraints of the form

$$Q(z_k) = Q_k \quad k = 1, q \tag{7}$$

have also appeared in the literature. These conditions are but a special case of (3). In fact for each k, (7) represents m special conditions of the form (3). To see this, consider (3) and blocks of m interpolation points where $s_i = z_1$ $i = 1, m$ with $a_i = e_i$, $s_{m+i} = z_2$ $i = 1, m$ with $a_{m+i} = e_i$ and so on, where the entries of e_i are zero except the ith entry which is 1; then Q_1 of (7) above is $Q_1 = [b_1, ..., b_m]$, $Q_2 = [b_{m+1}, ..., b_{2m}]$ and so on. In this case s_j are not distinct but the value is repeated m times. A simple comparison of the constraints (7) to the polynomial constraints (1) seems to suggest that this is an attempt to directly generalize the scalar results to the matrix case. As in the polynomial case, z_k $k = 1, q$ therefore should perhaps be distinct for Q(s) to be uniquely determined. Indeed this is the case as the following corollary shows:

Corollary 2: Given (z_k, Q_k) $k = 1, q$ with $q = d + 1$, and Q_k (pxm), such that the m(d+1)xmq matrix $S_{dk} := [S_d(z_1), ..., S_d(z_k)]$ has full rank, there exists a unique (pxm) polynomial matrix Q(s) with highest polynomial entry degree d which satisfies (7).

Example 3 Let Q(s) be 1x2 (=pxm), d = 1 and let the q = d+1 = 2 interpolation points be $\{z_k, R_k \;\; k = 1, 2\} = \{(0, [1, 1]), (1, [2, 1])\}$. In view of $Q(s) = Q_d S(s)$, $Q_d[S_d(z_1), S_d(z_2)] = [R_1, R_2]$ or

$$Q_d \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1, 1, 2, 1]$$

from which $Q_d = [1, 1, 1, 0]$ and $Q(s) = Q_d S_d(s) = [s+1, 1]$ as in Examples 1 and 2. □

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Characteristic Values and Vectors

If a complex number z satisfies q(z) = 0, where q(s) is a polynomial, then z is a root of q(s). In view of this, let us apply the basic matrix interpolation Theorem 2.1 with $b_j = 0$; that is the pxm matrix Q(s) must satisfy

$$Q(s_j)\, a_j = 0 \qquad j = 1, \ell \tag{8}$$

Definition 1: A complex scalar s_j is a characteristic value of Q(s) if it is a zero of an invariant polynomial of Q(s). The mx1 complex vector a_j which satisfies $Q(s_j)a_j = 0$ is the corresponding characteristic vector of Q(s).

Q(s) may have repeated characteristic values so s_j typically has an algebraic and a geometric multiplicity defined below for Q(s) nonsingular; it is straightforward to extend the definitions to a general pxm Q(s). In the case of a real matrix A, if some of the eigenvalues are repeated one may have to use generalized eigenvectors. Here we also define generalized characteristic vectors of Q(s). In the results below, only characteristic vectors which satisfy relation (8), which does not contain derivatives of Q(s), are considered for simplicity and clarity. The general results are not included here; as perhaps expected, they involve derivatives of Q(s)[1].

Let Q(s) be an (mxm) nonsingular matrix. If s_j is a zero of $|Q(s)|$ repeated n_j times, define n_j to be the *algebraic multiplicity* of s_j; define also the *geometric multiplicity* of s_j as the quantity (m - rank $Q(s_j)$).

Theorem 2: There exist a complex scalar s_j and ℓ_j linearly independent (mx1) vectors a_{ij} with i = 1, ℓ_j which satisfy

$$Q(s_j)\, a_{ij} = 0 \tag{9}$$

if and only if s_j is a zero of $|Q(s)|$ with algebraic multiplicity $(=n_j) \geq \ell_j$ and geometric multiplicity $(=(m - \text{rank}Q(s_j)) \geq \ell_j$.

The complex values s_j and vectors a_{ij} are characteristic values and vectors of Q(s). In the case when $\ell_j = 1$, the theorem simply states that s_j is a zero of $|Q(s)|$ if and only if rank$Q(s_j) < m$, an obvious and well known result. The conditions of Theorem 2 imply certain structure for the Smith form of Q(s). In particular (9) implies that the Smith form of Q(s) has factors $(s - s_j)$, raised to some power, in ℓ_j locations on the diagonal[1].

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In the following, matrices Q(s) with given characteristic values and vectors s_j and a_{ij} are characterized.

Theorem 3: Let $n = \deg|Q(s)|$. There exist σ distinct scalars s_j and (mx1) nonzero vectors a_{ij} $i = 1, \ell_j$ $j = 1, \sigma$ with $\sum_1^{\sigma} \ell_j = n$ and a_{ij} $i = 1, \ell_j$ linearly independent which satisfy (9) if and only if the zeros of $|Q(s)|$ have σ distinct values s_j $j = 1, \sigma$, each with algebraic multiplicity $n_j = \ell_j$ and geometric multiplicity $(m - \text{rank}\, Q(s_j)) = \ell_j$.

Note that the independence condition on the mx1 vectors a_{ij} implies that $\ell_j \leq m$, that is no characteristic value is repeated more that m times in Theorem 3; derivatives must be used if s_j is repeated more than m times[1]. The following corollary of Theorem 3 formalizes the most familiar case:

Corollary 3: Let $n = \deg|Q(s)|$. There exist n distinct complex scalars s_j and (mx1) nonzero vectors a_j $j = 1, n$ which satisfy (8) if and only if the zeros of $|Q(s)|$ have n distinct values s_j.

Rational Matrix Interpolation :

The polynomial matrix interpolation Theorem 1 directly leads to rational matrix interpolation theorem:

Theorem 4: Assume that (s_j, c_j, b_j) $j = 1, \ell$ $c_j \neq 0$ with $\ell = \sum d_i + m$ and a polynomial matrix D(s) with $|D(s_j)| \neq 0$ are given, such that the S_ℓ matrix in (2) with $a_j = [D(s_j)]^{-1} c_j$ has full rank. There exists a unique rational matrix H(s) of the form $H(s) = N(s)D(s)^{-1}$, where the polynomial matrix N(s) has column degrees $\deg_{ci}[N(s)] = d_i$, for which

$$H(s_j)\, c_j = b_j \qquad j = 1, \ell \tag{10}$$

Rational matrix interpolation results can be directly derived from corresponding polynomial matrix interpolation results. All results of polynomial matrix interpolation can therefore be extended to the rational matrix case. Notice that if D(s), in Theorem 4, is taken to be I, Theorem 1 is obtained. One could also use the results of Corollary 1 and 2 to obtain alternative rational matrix interpolation results.

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Solutions of Polynomial and Rational Matrix Equations

Consider the equation

$$M(s)L(s) = Q(s) \tag{11}$$

where L(s) (txm) and Q(s) (kxm) are given polynomial matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the polynomial matrix solutions M(s) (kxt).

First consider the left hand side of equation (11). Let $M(s) := M_0 + ... + M_r s^r$ where r is a non-negative integer, and let $d_i := \deg_{ci}[L(s)]$ i = 1, m. If $\hat{Q}(s) := M(s)L(s)$, then $\deg_{ci}[\hat{Q}(s)] = d_i + r$. According to the basic polynomial matrix interpolation Theorem 2.1, the matrix $\hat{Q}(s)$ can be uniquely specified using $\sum (d_i+r) + m = \sum d_i + m(r+1)$ interpolation points. Therefore consider ℓ interpolation points (s_j, a_j, b_j) j = 1, ℓ where

$$\ell = \sum d_i + m(r+1) \tag{12}$$

and such that the theorem assumptions are satisfied; that is, if $S_r(s) :=$ blk diag$[1, s, ..., s^{d_i+r}]'$, the $(\sum d_i + m(r+1))x\ell$ matrix $S_{r\ell} := [S_r(s_1)\, a_1, ..., S_r(s_\ell)a_\ell]$ has full rank. Note that for distinct s_j, $S_{r\ell}$ will have full column rank for almost any set of nonzero a_j (this can be formally shown). Now in view of Theorem 1 $\hat{Q}(s)$ which satisfies

$$\hat{Q}(s_j)a_j = b_j \tag{13}$$

is uniquely specified given these ℓ interpolation points (s_j, a_j, b_j). To solve (11), these interpolation points must be appropriately chosen so that the equation $\hat{Q}(s)$ (= M(s)L(s)) = Q(s) is satisfied:

Write (11) as

$$ML_r(s) = Q(s) \tag{14}$$

where $M := [M_0, ..., M_r]$ (kxt(r+1)) and $L_r(s) := [L(s)', ..., s^r L(s)']'$ (t(r+1)xm). Let $s = s_j$ and postmultiply (14) by a_j j = 1, ℓ. Define

$$b_j := Q(s_j)a_j \quad j = 1, \ell \tag{15}$$

and combine the equations to obtain

$$ML_{r\ell} = B_\ell \tag{16}$$

where $L_{r\ell} := [L_r(s_1)\, a_1, ..., L_r(s_\ell)a_\ell]$ $(t(r+1)x\ell)$ and $B_\ell := [b_1, ..., b_\ell](kx\ell)$.

<u>Theorem 5</u>: Given L(s), Q(s) in (11), let $d_i := \deg_{ci}[L(s)]$ i = 1, m and select r to satisfy

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \tag{17}$$

Then a solution M(s) of degree r exists if and only if a solution M of (16) does exist; furthermore, $M(s) = M[I, sI, ..., s^r I]'$.

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It is not difficult to show that solving (16) is equivalent to solving

$$M(s_j)c_j = b_j \quad j = 1, \ell \tag{18}$$

where

$$c_j := L(s_j)a_j,\ b_j := Q(s_j)a_j \quad j = 1, \ell \tag{19}$$

In view now of Corollary 1, the matrices M(s) which satisfy (18) are obtained by solving

$$MS_\ell = B_\ell \tag{20}$$

where $S_\ell := [S(s_1)\, c_1, \ldots, S(s_\ell)c_\ell]$ $(t(r+1)x\ell)$, with $S(s) := [I, sI, \ldots, s^rI]'$ $(t(r+1)xt)$ and $B_\ell := [b_1, \ldots, b_\ell]$ $(kx\ell)$; M(s) is then $M(s) = M[I, sI, \ldots, s^rI]'$ where M $(kxt(r+1))$ satisfies (20). Solving (20) is an alternative to solving (16).

Theorem 5 shows that there is a one-to-one mapping between the solutions of degree r of the polynomial matrix equation (11) and the solutions of the linear system of equations (16) (or of (20)). In other words, using (16) (or (20)), we can characterize all solutions of degree r of (11). Note that the conditions (17) of the theorem are not restrictive as they are necessary conditions for a solution M(s) in (11) of degree r to exist; that is, all solutions of M(s)L(s) = Q(s) of any degree can be found using Theorem 5. Also note that no assumptions were made regarding the polynomial matrices in (11), that is Theorem 5 is valid for any matrices L(s), Q(s) of appropriate dimensions. Solving (16) (or (20)) is equivalent to solving (11) for solutions M(s) of degree $\leq$ r. When applying this approach, it is not necessary to determine in advance a lower bound for r; it suffices to use a large enough r. Theorem 5 provides the theoretical guarantee that in this way all solutions of (11) can be obtained. Searching for solutions is straightforward in view of the availability of computer software packages to solve linear system of equations. Even when an exact solution does not exist, it can be approximated using, for example, least squares.

Now let's consider the <u>rational matrix equation</u>:

$$M(s)L(s) = Q(s) \tag{21}$$

where L(s) (txm) and Q(s) (kxm) are given rational matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the rational matrix solutions M(s) (kxt). Let $M = D^{-1}(s)N(s)$, a polynomial fraction form of M(s) to be determined; then equation (21) can be written as:

$$[N(s) \quad D(s)]\begin{bmatrix} L(s) \\ -Q(s) \end{bmatrix} = 0 \tag{22}$$

Let $s = s_j$ and postmultiply by a_j $j = 1, \ell$ with a_j and ℓ chosen properly (see below). Let

$$c_j := [\, L(s_j)', , -Q(s_j)']'\, a_j \quad j = 1, \ell \tag{23}$$

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The problem now is to find a polynomial matrix [N(s) D(s)] which satisfies

$$[N(s_j)\quad D(s_j)]\, c_j = 0 \qquad j = 1, \ell \tag{24}$$

which is a polynomial interpolation problem that can be solved using the approaches proposed above. In fact (24) is of the form of (18). The question really is how to select a_j $j = 1, \ell$ and ℓ so that by solving (24) all solutions to (21) are obtained:

Note that instead of solving (22) one could equivalently solve [N(s) D(s)][Lp(s)' -Qp(s)']' = 0, where [Lp(s)' -Qp(s)']' is a polynomial numerator of [L(s)' -Q(s)']'; so the problem to be solved is now (11), a polynomial matrix equation, where L(s) = [Lp(s)' -Qp(s)']' and Q(s) = 0. Therefore, Theorem 5 does apply and all solutions [N(s) D(s)] of degree r can be determined by solving (16) or (18) which in this case is (24) above.

Note that restrictions can be easily imposed to guarantee that $D^{-1}(s)$ exists and that $M(s) = D^{-1}(s)N(s)$ is proper. The existence of solutions of (21) and their causality depends on the given rational matrices L(s) and Q(s) (see for example [12]). Our approach here will find a proper rational matrix of order r when such solution exists. Additional interpolation type constraints can be added so the solution satisfies additional specifications.

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