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NEW BOUNDS ON PARAMETER UNCERTAINTIES FOR ROBUST STABILITY

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Abstract - The problem of robust stability in linear systems with parametric uncertainties is considered. A new method is developed to determine bounds on uncertain parameters in the state-space model of the system, so that stability is preserved. Both continuous- and discrete-time systems are considered. Unlike previous results, the stability bounds are derived in terms of actual parameters instead of their absolute values and consequently, the stable region in the parameter space is not necessarily symmetric with respect to the origin; furthermore, the uncertain parameters do not have to be linearly independent to each other.

I. Introduction

We are interested in the stability of systems with parameter uncertainties in the state-space model and in particular in obtaining bounds on the uncertain parameters to guarantee the stability of the system. This problem is related to the robust stability problem of interval matrices, which has been studied by many researchers; note that [1] contains a review of this subject including recent research results. Using these results one can determine if a matrix with entries varying over some interval remains stable; such results however do not generally provide the range of parameters for stability, which is the problem of interest here. Most of previous results on robust stability that provide bounds on the parameter uncertainties in the state-space model to preserve stability [2-6] are restricted to bounds on the absolute values of the uncertain parameters; that is the corresponding stable region in the parameter space is always symmetric with respect to the origin. Clearly, this may introduce conservatism in the results and in fact, as it is shown later in the paper, such results can sometimes be very conservative indeed.

Progress has been made recently in obtaining less conservative parameter bounds for robust stability using Lyapunov approach [5,9,10]. In particular, the bounds developed in [10] are not necessarily symmetric with respect to the origin in the parameter space as in the previous results, and this reduces the conservatism significantly. The stability bounds presented in this paper are similar to those in [10] for continuous-time systems and independent uncertain parameters, except that they are expressed explicitly as a set of inequalities, instead of a convex hull over a set of intervals in the parameter space. This is where, however, the similarities with [10] end. The approach taken here, which uses Lyapunov stability and explicit inequalities, not only offers additional insight but it allows the derivation of corresponding bounds for the discrete-time case for the first time; furthermore, as it will be shown, these results enable us to investigate robust stability in problems with nonlinearly dependent uncertainties.

Consider the state-space model for continuous-time systems with perturbation E

$$\dot{x} = (A + E)x \tag{1.1}$$

where A is a nxn real Hurwitz matrix. Assume that the perturbation matrix, E, takes the form

$$E = \sum_{i=1}^{m} k_i E_i \tag{1.2}$$

where E_i are given real constant matrices and k_i are real uncertain parameters. The upper and lower bounds on k_i i = 1, m, are to be found such that if k_i i = 1, m are within these bounds, the system in (1.1) remains stable; that is the eigenvalues of (A + E) have negative real parts. For discrete-time systems, the state-space model has the form

$$x(k+1) = (A+E)x(k) \tag{1.3}$$

with E defined again as in (1.2). In this case, the bounds on k_i are to be found so that the eigenvalues of (A + E) have magnitude less than one.

II. Stability Bounds for Continuous and Discrete-Time Systems

Since it is assumed that A in (1.1) is Hurwitz, there exists a symmetric positive definite matrix P which is the unique solution of the Lyapunov equation (see for example [8])

$$PA + A^TP + 2I = 0. \tag{2.1}$$

Define P_i as

$$P_i := (E_i^TP + PE_i)/2, \quad i = 1, m; \tag{2.2}$$

Note that P_i are real and symmetric (Hermitian) matrices. The following theorem establishes the stability constraints on the actual uncertain parameters, k_i i = 1, m. It is derived using results from Lyapunov stability theory, via an approach similar to the one used in [5]. Let $\lambda(X)$ denote any eigenvalue of matrix X, and $\lambda_{max}(X)$ and $\lambda_{min}(X)$ the largest and smallest eigenvalues of X, respectively.

<u>Theorem 1</u>[9]: The system in (1.1) is asymptotically stable if

$$\sum_{i=1}^{m} k_i \lambda_i < 1 \tag{2.3}$$

with λ_i i = 1, m defined by

$$\lambda_i = \begin{cases} \lambda_{max}(P_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(P_i) & \text{for } k_i < 0 \end{cases} \quad i = 1, m \tag{2.4}$$

The significance of this theorem is that it takes into consideration the directional information which is often available in practice, thus reducing the conservatism found in earlier literature results. To demonstrate this, it is shown below that the stability bound obtained here is always less than or equal to one of the bounds proposed in [5], namely

$$\sum_{i=1}^{m} |k_i| \sigma_{max}(P_i) < 1 \tag{2.12}$$

where $\sigma_{max}(\cdot)$ denotes the largest singular value; see also Example 1.

Since P_i is a Hermitian matrix, $\sigma_{max}(P_i) = \max\{ |\lambda|, \lambda \in \lambda(P_i)\}$. Hence for λ_i defined in (2.2), we have $|\lambda_i| \leq \sigma_{max}(P_i)$. Therefore,

$$\sum_{i=1}^{m} k_i \lambda_i \leq \sum_{i=1}^{m} |k_i| |\lambda_i| \leq \sum_{i=1}^{m} |k_i| \sigma_{max}(P_i).$$

In other words, if (2.12) is satisfied, then (2.3) is satisfied. That is, the stability bound found in Theorem 1 is always less conservative than the one in (2.12). Clearly the reason the new stability bound is less conservative is that it takes the directional information into consideration. This can be explained by the fact that as a parameter varies in different directions, it affects the system stability differently. This can be easily shown using, for example, the root locus technique where it is well known that, for different signs of the parameter, the root locus is completely different; that is the effect of a single parameter k in A on its eigenvalues can be completely different for the same |k| and opposite sign. Any tests therefore which ignore the sign are bound to be conservative in typical cases.

Furthermore, from the stability conditions (2.3), if for some k_j we have

$$k_j \lambda_j \leq 0, \tag{2.13}$$

then such uncertain parameters will not affect the system stability. This is because for k_j that satisfy (2.13),

$$\sum_{i=1}^{m} k_i \lambda_i \leq \sum_{i=1, i\neq j}^{m} k_i \lambda_i,$$

and therefore the stability criteria in (2.3) becomes

$$\sum_{i=1, i\neq j}^{m} k_i \lambda_i < 1. \tag{2.14}$$

In (2.14) the conservatism is further reduced since there are fewer parameters to be considered. Furthermore, if the lower bounds, a_j (≥ 0) of the absolute values of such k_j

$$|k_j| \geq a_j, \tag{2.15}$$

are known, then the uncertainties in k_j can be actually used in offsetting the destabilizing effect of other uncertain parameters. This is formalized in the corollary below:

<u>Corollary 1</u>[9]: Assume there exist k_j which satisfy (2.13) and (2.15) for some j. Then the system in (1.1) is stable if

$$\sum_{i=1, i\neq j}^{m} k_i \lambda_i < 1 + \sum_{j} a_j |\lambda_j|. \tag{2.16}$$

Theorem 1 gives a stability region in the parameter space and this region is defined by the inequality in (2.3). From this inequality, it can be seen that the stability bound on one uncertain parameter is also dependent on the size of the uncertainties in other parameters. From (2.3), if there is a large uncertainty in one of the parameters, then in general we cannot allow large uncertainties in the rest of the uncertain parameters. The size of λ_i can be viewed as a weighting factor which decides to what degree the parameter k_i can vary. Clearly, any method which gives a single stability bound for all uncertain parameters, will introduce significant conservatism.

<u>Example 1</u> Let m = 2 with A, E_1 and E_2 given as:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \quad E_1 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of P_i defined in (2.2) are

$\lambda(P_1) = \{-1, 0\}$ and $\lambda(P_2) = \{1, 0\}$

and therefore the stability bounds given by (2.14) are:

$k_2 < 1$	for	$k_1 > 0, k_2 > 0$
$\forall k_1, k_2$	for	$k_1 > 0, k_2 < 0$
$k_2 - k_1 < 1$	for	$k_1 < 0, k_2 > 0$
$k_1 > -1$	for	$k_1 < 0, k_2 < 0$

The corresponding stability bound obtained in [5] is

$|k_1| + |k_2| < 1$ for any k_1, k_2

and the actual stability bound in this case is $k_2 - k_1 < 2$. Note that the stability region obtained using the new method is open to infinity.

Example 2 Let $m = 2$ with A, E_1 and E_2 given as:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \quad E_1 = \begin{bmatrix} -5 & 1 \\ 1 & -1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 7.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}$$

Also assume that k_1 has a lower bound, $k_1 \geq 2$. The eigenvalues of P_i are

$\lambda(P_1) = \{-2, -2\}$ and $\lambda(P_2) = \{3, 3\}$,

therefore, from (2.4), $\lambda_1 = -2$ and $\lambda_2 = 3$. Note that since $\lambda_1 k_1 < 0$, k_1 will not affect the system stability. By Corollary 1, the stability bound is

$$k_2\lambda_2 < 1 + 2{*}2 = 5, \text{ or}$$
$$k_2 < 5/3.$$

This example shows that some uncertainties not only do not destabilize the system, but also play a role of offsetting the destabilizing effect of other uncertainties. Here the presence of the uncertainty, k_1, actually increases the stability bound of k_2 from $k_2 < 1/3$ to $k_2 < 5/3$, where $k_2 < 1/3$ is the stability bound obtained without taking k_1 into consideration.

The above results were derived for continuous-time linear systems. A similar approach can be used for discrete-time linear systems with parametric uncertainties in the state-space model (1.3). This is briefly discussed below and corresponding results for the discrete-time case are outlined.

Define the Lyapunov function as $V(x) = x^TPx$, where P is the solution of the Lyapunov equation for discrete-time system (see for example [8]).

$$A^TPA - P + 2I = 0 \quad (2.19)$$

Then it can be shown that

$$\Delta V = V(x(k+1)) - V(x(k))$$
$$= 2x^T\Big[\sum_i k_iP_i + \sum_{i,j} k_ik_jF_{ij} - I\Big]x \tag{2.20}$$

where P_i is defined as

$$P_i := (E_i^TPA + A^TPE_i)/2, \quad i = 1, ..., m \tag{2.21}$$

and

$$F_{ij} := E_i^TPE_j/2. \tag{2.22}$$

Note that ΔV in (2.20) has a similar form as its counterpart, dV/dt, in the case of continuous-time systems and a similar approach can be used here to derive the stability bounds. The following result which is applicable to the discrete-time system (1.3) is the counterpart of Theorem 1 and it can be proved in a similar way:

Theorem 2[9]: The system in (1.3) is asymptotically stable if

$$\sum_i k_i\lambda_i + \sum_{i,j} k_ik_jf_{ij} < 1 \tag{2.23}$$

with λ_i defined in (2.4), P_i defined in (2.21) above and f_{ij} defined as,

$$f_{ij} = \begin{cases} \lambda_{max}(F_{ij}) & \text{for } k_ik_j \geq 0 \\ \lambda_{min}(F_{ij}) & \text{for } k_ik_j < 0 \end{cases} \quad i, j = 1, m \tag{2.24}$$

Example 3 Let the nominal discrete-time system be

$$x(k+1) = \begin{bmatrix} .5 & 0 \\ 0 & -.5 \end{bmatrix} x(k)$$

and the perturbation be

$$E = k_1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then from (2.21) to (2.24), we have

$P = diag(8/3, 8/3)$, $P_1 = diag(-4/3, -4/3)$, $P_2 = -P_1$, $F_{11} = F_{22} = P_2$, $F_{12} = F_{21} = P_1$, $\lambda_1 = -4/3, \lambda_2 = 4/3$, $f_{11} = f_{22} = 4/3$, and $f_{12} = f_{21} = -4/3$. Applying (2.23), the stability constraint on k_1 and k_2 is

$$(k_2 - k_1)^2 + (k_2 - k_1) < 3/4$$

or equivalently

$$k_2 - k_1 < 1/2$$
$$k_2 - k_1 > -3/2$$

Interestingly, this stability region, derived by applying the new stability bound, is exactly the same as the actual one; of course in other examples this may not be the case.

III. Systems with Nonlinearly Dependent Uncertain Parameters

It is shown in the following how the above results can be used to solve more complicated problems in robust stability of dynamic systems. Consider the following problem: given the uncertain system

$$\dot{x} = (A + E(r))x, \tag{3.1}$$

and

$$E(r) = \sum_{i=1}^{m} k_i(r)E_i \tag{3.2}$$

where $A \in R^{nxn}$ is Hurwitz, $k_i(r)$ $i = 1,m$ are given continuous functions of $r \in R$, and $E_i \in R^{nxn}$ $i = 1,m$ are given constant matrices, determine the stability region $\Psi \subset R$ such that for $r \in \Psi$, (3.1) remains stable.

Note that here the uncertain parameters are functions of one parameter r. Similar approach can be taken when they depend on more than one parameters, however this will not be discussed in this note. It is can be easily shown that for the more complicated perturbation matrix E(r) in (3.2), Theorem 1 still holds and the corresponding stability constraints are, in this case,

$$\sum_{i=1}^{m} k_i(r)\,\lambda_i < 1. \tag{3.3}$$

Inequality (3.3) serves as a starting point in the stability analysis of system (3.1) and (3.2). It is significant because it enables us to study the effect of r on the system stability. Such problems can not be solved directly by using existing methods since the uncertain parameters $k_i(r)$ are, in general, nonlinearly dependent to each other via r.

There are two possible methods to obtain the stability region Ψ. One is an analytical method by which the bounds for r are explicitly derived from (3.3). However, this is not always possible due to the arbitrariness of the functions E(r) and $k_i(r)$. The other method is a graphical approach where with the help of computer software packages, such as Matlab, we can easily plot $f(r) = \sum_{i=1}^{m} k_i(r)\,\lambda_i$ as a function of r and therefore determine the stability region Y, which is the region that satisfies $f(r) < 1$.

Example 4 Consider the stability of the system

$$\dot{x} = \left(\begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + k_1(r)\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} + k_2(r)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)x$$

where the system matrix is affected by the uncertainty r through the nonlinear functions $k_1(r) = e^r$ and $k_2(r) = r^3$. By Theorem 1, we first calculate the eigenvalues of P_i defined in (2.1)

$\lambda(P_1) = \{-1, 0\}$ and $\lambda(P_2) = \{1, 0\}$

and then the stability bounds given by (3.3) can be found as:

$k_2(r) < 1$	for	$k_1(r) > 0, k_2(r) > 0$
$\forall\, k_1(r), k_2(r)$	for	$k_1(r) > 0, k_2(r) < 0$
$k_2(r) - k_1(r) < 1$	for	$k_1(r) < 0, k_2(r) > 0$
$k_1(r) > -1$	for	$k_1(r) < 0, k_2(r) < 0$

Substituting $k_1(r) = e^r$ and $k_2(r) = r^3$ in the above inequalities, the equivalent stability constraints in terms of r are

$r^3 < 1$ for $e^r > 0, r^3 > 0$

and by simple manipulation the stable region for the uncertainty r is found to be $-\infty < r < 1$, which is rather close to the exact stability bound, $-\infty < r < 1.25$.

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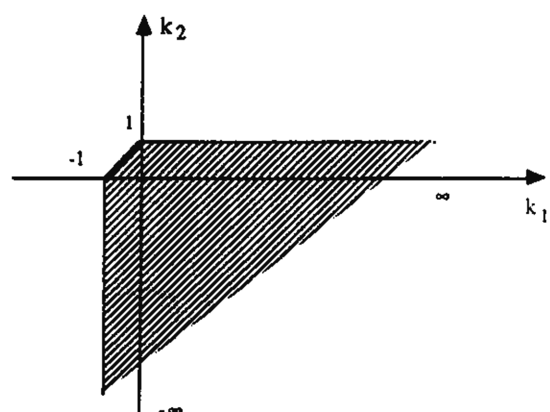

Figure 1: Example 1, stability bounds of k_1 and k_2.