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Gottlob Frege

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Abstract

Gottlob Frege (1848-1925) made significant contributions to both pure mathematics and philosophy. His most important technical contribution, of both mathematical and philosophical significance, is the introduction of a formal system of quantified logic. His work of a more purely-philosophical kind includes the articulation and persuasive defense of anti-psychologism in mathematics and logic, the rigorous pursuit of the thesis that arithmetic is reducible to logic, and the introduction of the distinction between sense and reference in the philosophy of language. Frege's work has gone on to influence contemporary work across a broad spectrum, including the philosophy of mathematics and logic, the philosophy of language, and the philosophy of mind. This essay describes the historical development of Frege's central views, and the connections between those views.

I. Introduction

Friedrich Ludwig Gottlob Frege was born on November 8, 1848 in the Hanseatic town of Wismar. He was educated in mathematics at the University of Jena and at the University of Göttingen, from which latter he received his doctorate in 1873. He defended his *Habilitation* the next year in Jena, and took up a position immediately at the University of Jena. Here he spent his entire academic career, lecturing in mathematics and logic, retiring in 1918. His death came on July 26, 1925 in the nearby town of Bad Kleinen.[i]

Frege is best known for three significant contributions to philosophy. The first is his development of modern quantified logic, a contribution as much to mathematics as to philosophy. The second is his pursuit of the thesis of *logicism*, the thesis that arithmetic (including the classical theory of the real numbers) is part of pure logic. The third is Frege's account of the nature of language, including his noteworthy claim that there are two kinds of meaning had by virtually all significant pieces of language, commonly known in English as the *sense* and the *reference* of those pieces of language. The three contributions are closely connected, and can best be understood by following their parallel development throughout the course of Frege's work. Following a brief overview of the three contributions, this essay will proceed roughly chronologically to explore their interaction over the course of Frege's major works.

I.1 Logicism.

Frege's logicism is the thesis that arithmetic (by which Frege means the usual theories of natural numbers, integers, and real numbers, but not including geometry) is part of logic. As

Frege understands it, the truth of his version of logicism would imply that arithmetical truths are objective, in the sense of having no dependence on human mathematical (or other) activities, and that they are analytic, in a sense close enough to Kant's that the truth of logicism so understood would entail that Kant is wrong about arithmetic.

Though the logicist project was later taken up by modern empiricists, Frege's reasons for favoring it were not empiricist. His view was that arithmetical truth is clearly more deeply grounded than are any forms of synthetic knowledge, and that this depth, and an associated certainty and unrevisability, give good reason to take arithmetic to be grounded directly in logic. Motivation aside, Frege's central view is that his logicist thesis is susceptible to direct demonstration, of a rigorous kind. This demonstration was to have consisted in (a) a thorough *analysis* of fundamental arithmetical truths and of their components, and (b) a *proof* of those fundamental truths, so analyzed, from purely-logical premises. The attempt to carry out this demonstration occupies a substantial part of Frege's research from 1879 through at least 1903, and involves as a crucial component the development of a formal system of quantified logic. That system embodies for the first time the principles that form the heart of modern logic today.

Frege's attempt to demonstrate the truth of logicism was a failure. As he learned via a letter from Bertrand Russell in 1902, one of the principles that he, Frege, had taken to be a basic principle of logic was in fact paradoxical. This meant, for reasons that we'll detail below, both that Frege's analyses of arithmetical truths were flawed, and that his attempts to prove arithmetic from logic were unworkable. It also meant that the formal system of logic he presented in his mature work was inconsistent. The inconsistency in the formal system is easily remedied by removing a single problematic principle, but this remedy makes it impossible to use the system to prove the truths of arithmetic. That is to say: while the system salvaged from Frege's own via the removal of the problematic parts is of considerable importance in its own right as a forerunner of modern logical systems, it is not sufficient for its original purpose, the defense of logicism.

1.2 Quantified Logic

Frege's development of a *formal system* of logic was, as above, motivated by the attempt to provide extremely rigorous proofs of the fundamental truths of arithmetic. By a "system" of logic we mean: (i) a language in which all of the statements appearing in proofs are to be expressed, together with (ii) a collection of fundamental truths to be taken as axioms in any proofs, and (iii) a collection of inference-rules, i.e. rules by means of which truths can be inferred from other truths in order to generate proofs. To say that the system is "formal" is to say that components (i) – (iii) are specified entirely syntactically: what counts as a well-formed sentence, or an axiom, or an instance of an inference-rule, is determined entirely by the symbols and the order in which they appear; one need make no appeal to the meanings of the linguistic items in question in order to determine their status in the system. Frege's own reason for insisting on the formal presentation of his deductive system was that of rigor: his idea was that a syntactic specification of proof would remove all ambiguity and unclarity, and would ensure that all steps taken in a proof were explicitly acknowledged. This is not to say that the formulas of Frege's formal system were meaningless: on the contrary, each such formula expressed a determinate claim. Frege's idea was that a syntactically-specified system of proof would give a rigorous way of demonstrating the logical grounding of the claims expressed by its sentences. This fundamental idea, that proofs can be carried out via principles that are specified entirely by means of their syntactic form, has since become a defining characteristic of modern logic.

Frege's system is a system of *quantified* logic. The fundamental *quantifiers* are those notions expressed by the terms "all" and "there exist," and it is the interaction between these notions, and those expressed "and," "not," and "if … then," that explains the validity of a huge swath of valid arguments, both in ordinary reasoning and in mathematics. What's new in Frege's logic, in addition to its explicit syntactic implementation, is the accurate treatment of the interaction between quantifiers and the other structural features of arguments. The inference, e.g., from "Every even number is less than some odd number" and "2 is even" to "There is an odd number such that 2 is less than it" is a kind of inference handled smoothly by Frege's and not by prior treatments of logic.

The axioms and inference-rules Frege introduces include all of those axioms and rules now familiar as the principles of classical first-order logic, plus two further kinds of principles, in virtue of which Frege's system is importantly richer. The first is the inclusion in Frege's system of higher-order quantifiers. Though for various reasons the most popular systems of logic in use today do not include these quantifiers (but include only their first-order versions), logics including Frege-style higher-order quantifiers play an important role in both philosophical and mathematical arenas today. The second additional feature of Frege's formal system in its mature version is that it includes notation for and principles governing what he called *value-ranges*, a version of the modern idea of sets. On this point Frege's fundamental idea was flawed; this is the mistake, noted above and described in detail below, that makes the formal system inconsistent. Formal systems of logic after Frege do not include this principle; his insights concerning value-ranges, subsequently cleaned up so as to avoid paradox, are now pursued via systems of set theory.

1.3 Philosophy of Language

Though the topic was arguably of secondary interest to Frege himself, he is perhaps best known for his views about language and meaning. Frege's mature theory of language involves the view that words and sentences typically have two important semantic features, called by Frege the *Sinn* ("sense") and the *Bedeutung* ("reference") of those linguistic items. Consider an ordinary sentence, say,

(A) Alice likes geraniums.

One of the important things about this string of symbols is that it can be used, and indeed is typically used, to *say* something, to make a claim that's either true or false. Frege's view is that the claim expressed by this sentence, a claim that's also expressed by other sentences (e.g. by sentences of German and of French) is the important thing to focus on when we're talking about truth or falsehood. If Alice does in fact like geraniums, then the claim – or, as Frege puts it, the *Gedanke* ("thought") expressed by (A) - is true.

Thoughts, for Frege, are not mental entities (despite the terminology). The thought that happens to be expressed by (A) is, as he sees it, true; and this does not depend on any person's having either entertained that idea, or produced that sentence. In Frege's view, this independence between thoughts and people's activities is immediately evident once we notice that, for example, Mount Aetna would have been covered in snow even if nobody had ever noticed that it was. That is to say: the thought *Mount Aetna is covered in snow*, a thought expressible by sentences in many different languages but not dependent on any particular language, is in fact true, and would have been true even if no person had ever seen or imagined that mountain.

Thoughts are, in this sense, the primary bearers of truth and falsehood. For similar reasons, they are also the primary bearers of such logical relations as entailment, consistency, inconsistency, and so on. When, having uttered (A), I go on to say "Therefore, someone likes geraniums," I have noticed, according to Frege, that the thought expressed by

(B) Someone likes geraniums

follows logically from the thought expressed by (A). Similarly in the domain of arithmetic: when we set about to *prove* e.g. that 7+5=12, the thing we prove is not the sentence itself, but the thought it expresses. This forms a crucial part of Frege's explanation of how researchers who speak different languages can nevertheless be engaged in the same pursuits: they can prove the same theorems, investigate the same questions, and agree or disagree about the same claims. In such cases, the items each investigator is concerned with are thoughts, each of which is expressible via various sentences in different languages.

This view of language-meaning extends to parts of sentences. Each part of a sentence (each word or phrase) makes a particular contribution to the thought expressed by that sentence. This contribution is called the "*Sinn*" ("sense") of that part. Because the thought expressed by

(i) The morning star is bright

is a different thought from that expressed by

(ii) The evening star is bright,

Frege concludes that the phrases "the morning star" and "the evening star" have different senses, even though they happen to pick out the same object, the planet Venus. This object, the planet, is what Frege calls the "*Bedeutung*" ("reference") of the phrase "the morning star." (It is also, of course, the reference of "the evening star.") The pattern established here – in which two singular terms can have the same reference while having different senses – forms a crucial part of Frege's semantic theory. Both sense and reference are essential to the role of words and phrases in sentences: the sense of a word or phrase is what it contributes to the thought expressed, while the reference is what it contributes to determining the truth-value of that thought. We investigate in more detail below the nature of senses and references, and their interaction.

II. The Historical Development of Frege's Work

II.1 Early Work: *Begriffsschrift* and *Grundlagen*

The work of Frege's that has been most important to philosophers begins with his 1879 monograph entitled *"Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens,"* familiarly known as "Begriffsschrift."[ii] In that work, Frege presents the first version of his formal system of logic. That system contains the formalism for first- and higher-order quantification, but does not contain the problematic notation or axioms for value-ranges. The system is elegant, powerful, and consistent.

Begriffsschrift announces the first step in Frege's logicist program. It raises the question of whether the truths of arithmetic are provable by means of pure logic alone, and points out that the only way to settle this question is to provide clear analyses and rigorous proofs of those truths. The formal system is introduced as the means of presenting the proofs. Frege does not take a position in Begriffsschrift on the answer to the announced question, but makes some important steps towards its resolution. In addition to introducing the formal system for proofs, Frege provides here several crucial analyses of proto-arithmetical notions, including that of the ancestral of a binary relation, and of the notion of *following in a series*. Having given the analyses, Frege demonstrates that some surprisingly-rich claims regarding series, claims that will later prove pivotal in his analysis of arithmetic, can be proven by means of pure logic.

Frege's views about the nature of logic begin to appear in this early work. Especially significant here is the early appearance of his anti-psychologism, a thesis that animates all of Frege's further work. The view is that logic is a normative discipline, in the sense that it provides norms for valid reasoning, but that it has no grounding in psychological (or other) facts about human beings. It is, as Frege later puts it, the science of truth: its principles are principles governing the entailment relation that obtains between truths, whether or not people actually manage to reason in accordance with them. The principles of logic are, in short, conceived by Frege here and henceforth as principles against which one might measure the rationality of individual reasoning-processes, but not as descriptions of those processes.

Frege's semantic theory is, in 1879, relatively rudimentary. He holds, as he will continue to hold, that logical entailment is a relation that obtains not between sentences, but between the things expressed by sentences. But in this early work, Frege has not yet introduced the two-tiered semantic theory in which *sense* is distinguished from *reference*, and does not yet employ the notion of *thought*. Here, the things expressed by sentences are known as *contents of possible judgment*, which contents are determined by the contents of the parts of the sentences in question. As to the nature of the contents of sub-sentential pieces of language, Frege does not have much to say in this period, but appears to take the contents of singular terms to be essentially what will later become their references, i.e. the objects (e.g. geometric points or numbers) that they stand for.

Frege's next major work is his *Grundlagen der Arithmetik* (*Foundations of Arithmetic*), published in 1884.[iii] The primary purpose of this work is to present, in ordinary language (i.e. in German, as opposed to his formal language), Frege's own analyses of fundamental arithmetical truths, and to sketch the proofs of those truths from logical principles. Here Frege comes down clearly on the side of logicism, and takes this small book to provide the outline of the decisive demonstration of that thesis. *En route* to presenting his own account of the nature of arithmetic, Frege presents biting criticisms of a number of other accounts, and is particularly interested in undermining both empiricist (Millian) and what he takes to be overly "psychological" accounts of the subject. Frege's own account of arithmetic turns on the idea that numbers are objects, and that their existence and fundamental properties are independent of our thoughts about them. They are, in that sense, *objective*: we do not create, but we discover, numbers and arithmetical truths. Frege also argues in *Grundlagen* that arithmetical facts are essentially facts about the sizes of collections: to say e.g. that 2+3=5 is to say something that logically entails various facts about the possibility of matching up unions of 2- and 3-membered collections with 5-membered collections.

The reconciliation of these two apparently-conflicting claims, i.e. that arithmetical truths are about objects (the numbers), and that they are equivalent to facts about arbitrary collections, is at the heart of Frege's view of the nature of arithmetic and of arithmetical discourse. The first central point in this account is Frege's view of what we mean when we say something like "There are four pens on the desk." His view is that in a case like this, we have attributed a property (roughly, the property of *having four things falling under it*) to the concept *pen on the desk*. (A *concept* (*Begriff*), here is not a mental entity, but is the kind of thing that a predicate stands for.) We have also, equivalently (in a way that needs explaining; see below) affirmed a relation between a particular object, the number *four*, and this concept (*pen on the desk*). As Frege puts it, "the content of a statement of number is an assertion about a concept."[iv]

The connection between the object *four* and the property of *having four things falling under it* turns critically on the kind of object that a number is, for Frege. To explain this, we need first to explain the notion of an *extension*. What we mean when we talk about *extensions of concepts* is explained via the following equivalence: to say "the extension of the concept F = the extension of the concept G" is to say something equivalent to : "Everything that falls under F falls under G, and vice-versa." This equivalence, the *principle of extensionality*, underwrites Frege's novel view that we can obtain knowledge of objects without the aid of intuition: if we have analytic knowledge of some claim of the form "∀x(Fx iff Gx)", then (because the principle of extensionality is itself analytic, as Frege sees it), we can conclude via purely analytic principles the corresponding statement: "the extension of F = the extension of G." To obtain the numbers themselves, we need the notion of a *cardinality concept*. A cardinality-concept is a second-level concept (so, one under which concepts, rather than objects, fall), and it's one under which a first-level concept falls if and only if that first-level concept has a specific number of objects falling under it. For example: the concept *having exactly two objects falling under it*, a concept under which fall all and only those first-level concepts under which fall exactly two objects, is a cardinality concept. Notice that for any first-level concept F, there is a cardinality-concept *equinumerous with F*, under which fall all and only those concepts equinumerous with F. Finally: numbers, for Frege, are the extensions of cardinality concepts. Specifically, where F is a first-level concept, *the number of F's* is the extension of the cardinality-concept *equinumerous with F*.

The number zero is, as Frege understands it, the number of the concept *not-self-identical*. That is: it's the extension of that second-level concept under which fall exactly those first-level concepts that are equinumerous with the concept *not-self-identical*. Since there are, as a matter of logic, no non-self-identical objects, a concept is equinumerous with the concept *not-self-identical* iff nothing falls under it. Consider now a statement of the form "There are zero G's", which Frege understands to mean "0 = the number of G's." Given Frege's analysis of "0" and of "the number of G's," this statement means:

> "The extension of that concept under which fall all and only those concepts equinumerous with the concept *not self-identical* = the extension of that concept under which fall all and only those concepts equinumerous with the concept *G*."

And *this* statement is straightforwardly equivalent (via the principle of extensionality) with the statement:

> "The concept *not-self-identical* is equinumerous with the concept *G,*"

which itself is logically equivalent with the statement

> "∀x~Gx"

It's worth pausing to notice just how important this point is for Frege. On the analysis in question, a statement about a particular object, 0, and its relation to a concept G, is logically equivalent to a claim about how many things fall under G. This is exactly what Frege needs in order to reconcile his two fundamental views about cardinal number: namely, that a statement of number is a statement about the cardinality of a concept, and that a statement of number is a statement about a particular object, that number. At least, this is what he needs for this reconciliation in the case of the number zero. In order to carry out this analysis for the rest of the finite cardinal numbers, Frege needs to provide canonical concepts guaranteed (via principles of pure logic) to have exactly the right number of objects falling under them. This is done as follows: the number one is, as Frege understands it, the number that belongs to the concept *identical with zero*. The number two is the number that belongs to the concept *identical with zero or identical with one*. And so on. In this way, Frege demonstrates how we can so understand the numbers that they are individual objects, and objects whose relations with arbitrary concepts "encode," as it were, facts about the cardinalities of those concepts.

Having provided accounts of the individual numbers, Frege next sketches proofs, from (what he takes to be) purely logical principles, of a core of fundamental claims about number (essentially, variants of what we now know as the Dedekind-Peano axioms). The underlying idea was that from these fundamental claims we would be able in a routine way to prove the whole of the arithmetic of the finite cardinals, so that the proofs of these fundamental claims, once the details were filled in, would provide the decisive demonstration of the logicist thesis. Frege promises that the final demonstration will be provided in future work, work that will give the detailed and rigorous proofs of those fundamental arithmetical truths.

The semantic theory found in *Grundlagen* is similar to that of *Begriffsschrift*: Frege does not yet have the distinction between sense and reference, but does clearly hold that the important properties and relations in which he's interested – i.e., truth, logical entailment, and relations inter-definable with these – apply in the first instance not to sentences, but to the contents thereof. One important thesis insisted upon in *Grundlagen*, and one that proves highly influential throughout the semantic tradition that follows Frege, is what has come to be called the *context principle*. The principle is the thesis that in order to understand what the meaning of a word is, we should not take the word in isolation, but should consider the contribution made by that word to the sentences in which it appears. Frege takes it that this thesis is especially important when we investigate the meanings of numerals. The idea is that instead of asking, in isolation, what the numeral "3" stands for, i.e. what the number three *is*, we should ask how that numeral contributes to the meanings of whole sentences in which it occurs. As Frege sees it, the failure to appreciate this point has been responsible for the mistaken idea that numerals refer to ideas, or to mental constructions of some kind: when people have looked for an object that can be identified in isolation, the only candidates have seemed to be some such psychological objects. Frege's view is that the mistake lies in the initial focus on a search for objects that are identifiable independently of context (e.g. specific *ideas*), rather than on an investigation into the contribution made by numerals to whole statements. Adherence to the context principle, says Frege, enables us to make sense of a kind of object that's identifiable in a quite different way than are, say, the objects of the material world. As we might put it, the context principle is essential to Frege's account of our apprehension of abstract objects.

During the next decade, a good deal of Frege's attention is presumably taken up with his monumental *Grundgesetze der Arithmetik*[v], whose first volume was published in 1893. This is the work in which Frege provides the remarkably rigorous proofs, promised in *Grundlagen*, of the fundamental truths about numbers. Prior to the publication of *Grundgesetze*, however, Frege writes and publishes three articles that have become enormously influential in the philosophy of language. These are "*Über Sinn und Bedeutung*" ("On Sense and Reference")[vi], "*Funktion und Begriff*,"[vii], and "*Über Begriff und Gegenstand*."[viii] In these essays, Frege introduces the distinction between sense and reference discussed above. Here we turn to some of the details of that mature semantic theory.

<u>II.2 The 1891/92 Essays</u>

Frege says very little about why, in the course of working out the very-demanding proofs that were to establish his logicist thesis, he pauses to develop an account of the nature of language. Part of the explanation would seem, however, to be as follows. Frege's view of arithmetic is that it is an *objective* science, in the sense that its truths hold independently of human activities. On this view, the fact that 13 is prime, for example, is in no way dependent on any ideas that anyone has ever had about 13; this thought would have been true whether anyone had ever entertained it or not. This means that the objects that arithmetic deals with, principally including the numbers, cannot be, and cannot be in any way dependent on, ideas or other human productions. Consider now the question of what we're trying to prove when we prove that

(D) 2+2=4.

It would seem quite evident that what we're trying to do here is quite different from what we're trying to do when we attempt to prove that

(E) 4 = 4.

This fact, that to prove (D) is not the same thing as to prove (E), might be taken to provide an argument against Frege's objectivist conception of arithmetic. The argument is as follows:

> If arithmetic is about independently-existing objects, then, because the object *2+2* is the same as the object 4, (D) says exactly what (E) says, and hence a proof of (D) can be given just by proving the trivial (E). But this is absurd. So arithmetic is not about independently-existing objects.

The lesson to be drawn from the difference between (D) and (E), according to this line of thought, would be that (D) is about something like a 'way of constructing' a number, or a pair of 'ideas,' or some such thing. Because it's plausible to claim that the idea, or the 'way of constructing' a number given by the phrase '2+2' is different from the idea or construction given by '4,' one can make sense, on this line of thought, of the clear difference between (D) and (E).

For Frege, it is important to resist this line of argument, and to maintain that arithmetic is really about the independently-existing arithmetical objects, despite the difference between e.g. (D) and (E). His response is that the difference between (D) and (E) is a matter of a difference in the *senses*, i.e. the *thoughts* expressed by those sentences, despite the fact that their singular terms, i.e. '2+2' and '4' have the same *reference*. The resulting two-tiered theory thus straightforwardly allows Frege to hold both that (D) and (E) express different items of knowledge (one of them being trivial, the other not), and that the sentences, and the thoughts themselves, are true in virtue of purely objective facts about independently-existing objects.

On the mature semantic theory as it's developed in this period, the thought expressed by a sentence is also known as the *sense* of that sentence, and is determined entirely by the senses of the sentence-parts and their order of composition. Thoughts are, as above, the primary bearers of truth and falsehood, and of the logical relations. They are also the things with which one must be appropriately related in order to be said to understand the sentences that express them. When I am engaged in lucid conversation with someone, I *grasp* the thoughts expressed by the sentences used by that other person. This I do largely in virtue of my understanding of her words, i.e., in virtue of my grasp of the senses associated with those words. Thoughts are also, finally, the contents of belief, knowledge, and the other attitudes. To believe or to doubt something is to bear a particular relation to a specific thought.

Thoughts therefore play a unifying role in the account of assertions, sentences, beliefs, and the truth-values thereof. The fact that one person can doubt just what another believes, and that this can be the content of another's assertion or the claim expressed by a given sentence, is explained by Frege in terms of the fact that in each case, the content in question is a thought. That these apparently-disparate entities – beliefs, assertions, sentences – can all be true or false is in turn explained by the fact that each is related in a straightforward way to a thought, and that this thought is, at bottom, the thing that is true or false.

The reference of a singular term (roughly, a noun phrase - e.g. a definite description, name, or pronoun) is an object (a person, place, physical object, number, etc.). The sense of a singular term is again what that term contributes to the senses of the sentences in which it appears, and is also, typically, a way of presenting the reference of that term. The other parts of language, for example predicate phrases, relation terms, and so on, also have senses and references, according to Frege. The reference of a one-place predicate (e.g. "… is blue") is what Frege now calls a *concept*. Concepts as conceived in this later period are similar to the concepts of *Grundlagen* in their close relationship to predicates, but they are essentially richer in two ways: First, they are clearly now the references of predicative phrases, and hence are not the entities grasped in an act of understanding. Secondly, concepts in the mature period are importantly different from objects in that they are essentially predicative. To understand this notion, consider the difference between the sentence (A) above and the list of terms

(AL) Alice, the property of liking geraniums

While (A) expresses a thought, (AL) doesn't; the series of terms in the latter doesn't form the kind of unified whole sufficient for expressing a truth or falsehood. Frege's account of the difference is that the concept-phrase "… likes geraniums" in (A), unlike the phrase "the property of liking geraniums" in (AL), refers to something essentially predicative: the concept referred to is of such a nature that to refer to it is to predicate. Another way Frege has of putting the point is the metaphorical one that concepts are "unsaturated" or "incomplete," unlike objects, which are "saturated" or "complete." To use a concept-phrase in a sentence is to predicate, because concepts are just the kinds of things reference to which amounts to predication.

Concepts are, for Frege, a particular kind of *function*. Functions are unsaturated, i.e., again, essentially predicative. Any part of a sentence that doesn't refer to an object refers to a function. For example, in the sentence "2 + 2 = 4," the part "… + … ", i.e. the plus-sign accompanied by two gaps, refers to a function. Specifically, it is the function that takes as arguments pairs of numbers, and returns as value the sum of those numbers. Similarly, "the paternal grandfather of …" takes as argument a person, and returns as value the father of the father of that person. A concept (e.g. that referred to by "… likes geraniums" or "… is blue") also gives values for arguments: in this case, the arguments are objects of appropriate kinds (something that does or doesn't like geraniums, something that is or isn't blue), while the value is what Frege calls a *truth-value*, i.e. the value *true* or the value *false*. Concepts, in short, are one-place functions from objects to truth-values.[ix]

In a complex term, like "Alice's paternal grandfather," the reference of the whole term is determined in the obvious way by function-argument application: the function referred to by "…'s paternal grandfather," when applied to Alice, gives as value the person Robert (i.e., Alice's paternal grandfather). Hence the reference of "Alice's paternal grandfather" is simply the person Robert. (Notice that though we can now conclude that the two phrases "Alice's paternal grandfather" and "Robert" have the same reference, they do not of course have the same sense.) Similarly, the reference of "the positive square root of nine" is the result of applying the function referred to by "the positive square root of …" to the argument 9, to yield 3.

Frege's view is that the truth-value of a sentence is determined in just the same way as is the reference of a complex singular term, via the application of function to argument. The truth-value of (A) is determined by applying the function referred to by "… likes geraniums" to the argument Alice, to deliver the value *true*. Because of the obvious parallels here, Frege uses the word "reference" to include not just the object referred to by a singular term and the function referred to by a function-expression, but also the truth-value of a sentence. Sentences, in short, are said to *refer* to truth-values.

Finally, function-expressions, including concept-expressions, have a sense as well as a reference. The senses are, predictably, simply the contributions made by those expressions to the thoughts expressed by the sentences that embed them. The phrases "… is a prime number between 12 and 16" and "… is a positive square root of 169," though they refer to concepts that deliver the same value for every argument, nevertheless have different senses. This follows from the fact that a person might e.g. know that "There is exactly one positive square root of 169" is true while doubting whether "There is exactly one prime number between 12 and 16" is true.

To sum up Frege's views about the senses and references of different parts of language: The reference of a sentence is a truth-value, while its sense is a thought. The reference of a singular term is an object, while its sense is the contribution made by that term to the thoughts expressed by sentences in which that term occurs. This sense is often usefully thought of as a "mode of presentation" of the object referred to. The reference of a function-expression is a function, an essentially predicative entity. The sense of such an expression is, again, just what that expression contributes to the thoughts expressed by sentences in which it appears.

There are also, as Frege sees it, terms that, though in some sense meaningful, lack reference. The phrase "the greatest prime number" is one such. Though the phrase clearly lacks a reference (since there's no greatest prime), it is also not a nonsense phrase. We can, for example, prove rigorously that there is no greatest prime number, which we couldn't do if the phrase in question were nonsense. Frege's description of this situation is that the phrase "the greatest prime number" has a sense but no reference. Because the reference of a sentence is determined by function-argument application in the way described above, no sentence involving this term can have a reference either. That is to say, sentences like "The greatest prime number is even," on Frege's view, have no truth-value. They do, however, have a sense.

The essays published in 1891/92 also develop the application of the theory of sense and reference to various natural-language phenomena. The fundamental idea that the reference of a sentence is determined by the references of its parts, for example, is seen to face a potential difficulty when applied to sentences involving constructions like "Aristotle believed that …" , "Jürgen said that…", and so forth. The difficulty is as follows: The ellipsis in the above examples is typically filled by a complete sentence, resulting in a complex sentence like

(S) Aristotle believed that humans are rational.

That embedded sentence ("humans are rational") would normally, on Frege's view, have a truth-value as its reference - in this case, the value *true*. But now something has gone wrong, since the truth-value of (S) is not determined just by the reference of "Aristotle believed that …" and the value *true*. We can see this by noticing that the sentence

(S*) Aristotle believed that the earth orbits the sun

has parts with the same ordinary references as do the parts of (S), but has a different truth-value. In short, the truth-value of the whole sentence, in the case of (S) and (S*), is not determined by what one would ordinarily take to be the references of its parts. Frege's response to this difficulty is to say that in contexts involving the relation "… believed that…", the sentence embedded to the right of that phrase has as its reference not its *ordinary* reference (i.e. its truth-value), but its *ordinary* sense (i.e. a thought). The sentence (S) is true iff Aristotle bears the relation of *belief* to the thought ordinarily expressed by "humans are mortal," and accordingly, the phrase "humans are mortal," in this context, refers to that thought. Similarly, in appropriate contexts, an embedded sentence might refer to itself (e.g. in quotation-mark contexts). In such cases, the embedded sentences have what Frege calls their "indirect" reference. The difficulty sketched above is thus averted, with the result that, uniformly, the reference of a sentence is determined by the references of its parts.

II.3 Mature Logicism: The *Grundgesetze*

In 1893, Frege publishes the first volume of his *magnum opus*, the *Grundgesetze der Arithmetik*. Much of what became the second volume was presumably completed by this time as well, though the second volume was not published until 1903[x]. In *Grundgesetze*, Frege provides the rigorous development of arithmetic whose groundwork was laid in the *Begriffsschrift* of 1879 and the *Grundlagen* of 1884. Specifically, *Grundgesetze* introduces a formal system of logic very like that of *Begriffsschrift*, and demonstrates (a) how to analyze fundamental concepts and truths of arithmetic in terms of surprisingly-simple constituent concepts, and (b) how to prove, with exacting rigor, the thus-analyzed arithmetical truths from (what he took to be) purely-logical principles, using purely-logical methods of inference. The analyses and proofs themselves exhibit an extraordinary degree of precision and rigor, reflecting Frege's view that for his purposes, it is not enough that each step in a proof be obviously correct; it is required that the most fundamental principles on which each step is based be made evident to anyone who works step-by-step through the proofs. The proofs are, in this sense, a monument to analytic care and precision.

Frege took himself to have completed that part of the project that deals with the natural numbers (i.e. the finite cardinal numbers 0 1, 2, 3, and so on). That is to say, he provides here the promised purely-logical proofs of highly-analyzed versions of a handful of fundamental truths about the arithmetic of the natural numbers. The idea, once again, was that these truths would in turn suffice for the proof of the whole of the arithmetic of the natural numbers. The completed part of the *Grundgesetze* project also includes the beginnings of a theory of real numbers.

A critical part of the formal system of *Grundgesetze* is its inclusion of a term-forming operator which, when attached to a predicate-phrase, gives a name of the *value-range* (*Wertverlauf*) of the associated function. Value-ranges are the modernized, formalized version of the *extensions* of *Grundlagen*, and obey a similar extensionality principle. Specifically, the principle is that for functions F and G, the value-range of F = the value-range of G iff $(\forall x)(Fx = Gx)$. Where F and G are concepts (first-level functions of one argument whose value is always a truth-value), the identity-conditions of the associated value-ranges are exactly those of the earlier extensions. Value-ranges play in *Grundgesetze* a role similar to the central role played by extensions in *Grundlagen*: numbers are now, in *Grundgesetze*, understood as value-ranges of concepts, and the extensionality-principle governing these items (Basic Law V) is essential to the most-important results regarding numbers.

Frege's work on the project was stopped short by the letter from Bertrand Russell in 1902 noted above, in which Russell points out the inconsistency in Frege's system.[xi] The inconsistency arises as follows. Consider the predicate R(x), a predicate satisfied by exactly those objects **o** such that: **o** is the value-range of some concept C such that ~C(**o**). Now we consider the value-range of R, which we'll call "**r**." We ask whether that object **r** falls under the concept R, i.e. whether R(**r**). If R(**r**), then (given the definition of R) **r** is the value-range of some concept under which it, **r**, does not fall. From this it follows (given extensionality) that **r** does not fall under *any* concept of which it is a value-range. But this contradicts our supposition that R(**r**), since **r** is of course the value-range of R. Thus far, we have shown that ~R(**r**). But this too leads to contradiction: If ~R(**r**), then **r** is the value-range of a concept (namely, R) under which it does not fall. So, by definition, R(**r**). At this point, we have deduced both that R(**r**), and that ~R(**r**), which is a contradiction. Since we did this using just Frege's fundamental principles about value-ranges, we see that something is very badly wrong with those principles.

The problem this raises for Frege's logicist project is twofold. First of all, it means that the purportedly-logical axioms on which Frege meant to found arithmetic are not in fact truths of logic. That this is a serious difficulty follows from the fact that there is no straightforward way to replace the problematic parts of the system by principles that are both purely logical and strong enough to develop arithmetic as Frege understands arithmetic. The second difficulty raised by the contradiction stems from the fact that Frege's all-important analyses of arithmetical truths turn crucially on the notion of value-ranges, that notion that is shown by Russell's contradiction to be incoherent. Both the analytic stage and the proof-theoretic stage of Frege's logicism are therefore undermined by the difficulty over value-ranges.

Frege's immediate reaction to the paradox was to attempt to modify his principles regarding value-ranges so as to maintain the general structure of his proofs while regaining consistency.[xii] He eventually came to realize, however, that no such "fix" would work, and that no principles governing value-ranges could count both as 'purely logical' by his own lights, and sufficiently rich to ground the existence of an infinite collection of objects, i.e. the numbers. His conclusion was that the paradox, and the consequent failure of his fundamental principles, shows that there is no way to give a purely-logical grounding of arithmetic.

This pessimistic conclusion was not universally shared in the light of Russell's paradox. In section III.1 below, we outline the central ways in which logicist programs are still being pursued, ways which share a good deal with Frege's original program, but depart from it in some significant ways. Frege's own conclusion was that the combination he had envisioned, a conception of arithmetic in which arithmetic is about distinctive objects and yet grounded in pure logic, is untenable. Judging by some late manuscript notes, he appears to have held at the end of his life that the most promising account of arithmetic would maintain the view of arithmetic as concerned with distinctive objects, but would ground the truths about those objects in facts given by pure intuition.[xiii]

II.4 Later Work Outside of *Grundgesetze*

To return to Frege's chronology: In the final few years of the nineteenth century, and into the twentieth, the mathematical world in Germany and much of Europe saw an explosion of interest in logic, and in the nature of mathematical theories. An important part of this development involved increasing sophistication regarding the nature of axioms, and in the understanding of the nature of, and of proof-techniques regarding, the relations of dependence and independence that obtain between parts of theories. In 1900, David Hilbert published a monograph entitled *"Foundations of Geometry*," in which the newly-emerging techniques for demonstrating consistency- and independence-results, and the new, modern conception of axioms, are presented.[xiv] In the years 1899-1900, Frege engaged in a correspondence with Hilbert, in which Frege presents an opposing view of the nature of axioms, and of the fundamental logical relations of consistency and independence.[xv] Frege continues to develop his position in two series of essays, each entitled "On the Foundations of Geometry," published in 1903 and 1906[xvi]. At the heart of Frege's argumentative strategy is a defense of his view that mathematical theories, and hence their axioms, must be understood as collections of thoughts. This view is in direct conflict with the view defended by Hilbert, in accordance with which the axioms of a theory ought to be understood essentially as partially-interpreted sentences. On Frege's view, each mathematical theory is a set of truths about a determinate subject-matter, while on Hilbert's view, a theory is a set of general structural requirements, applicable to a wide variety of different domains. Hilbert's conception of theories is geared towards the investigation of the structural properties of those various domains, and of demonstrating such results as the consistency and independence of sets of axioms, the categoricity of defining conditions, and so on. Frege's conception, on the other hand, is geared toward the investigation of the fundamental truths of a given subject-matter, as these concern a specific collection of objects, functions, and relations. In the debate between the two logicians, and in Frege's follow-up essays, we find a rich articulation of a fundamental cleavage between two ways of conceiving of the nature of logic and of its role in mathematics. The issue remains alive today, with both Frege's and Hilbert's sides defended in various ways as part of ongoing investigations into the nature of mathematical and logical knowledge.

In the years 1918 to 1925, Frege returns to the philosophy of language, writing a series of essays on sense and reference that expand upon the conception of language articulated in the period 1891-92.[xvii] In these later years, he treats for example the problem of "indexicals," i.e. words like "I" and "yesterday" whose reference depends in part on the context in which they are used. Frege's fundamental idea here is that the context of use is part of what determines the sense expressed by the use of a sentence, so that e.g. two utterances of "It's raining today" will express different senses if they occur on different days. Frege also pursues in more detail than previously the nature of thoughts (i.e., as sketched above, the senses of sentences). As against the worry that thoughts are somehow too ephemeral to be real, Frege replies that thoughts are indeed real, and that (in keeping with his general anti-psychologism), they are to be distinguished from anything subjective, like ideas. Thoughts exist in what he calls in 1918 a 'third realm,' a realm of objects that differ from material objects in not being concrete, but that differ from ideas in not being subjective.[xviii]

Frege died in Bad Kleinen in 1925, and is buried in the *Friedhof* at Wismar, the city of his birth. Though he judged his logicist project to have been a failure, he seems to have recognized nevertheless the importance of his logical investigations. Six months before his death, Frege expressed to his son Alfred, regarding a number of unpublished essays, a sentiment that applies as well to the published work:

> "Even if not all is gold, there is gold in them. I believe there are things here which will one day be prized much more highly than they are now."[xix]

III. Aftermath: Frege's Legacy

III.1 Logic and Mathematics

Frege's idea that logic can be pursued via the use of formal systems is so commonplace now as to be worth hardly a second thought, but it was an entirely new idea at the time, one that Frege introduced and pressed into extraordinarily fruitful service. Though our means of employing formal systems now incorporates some elements that Frege would have rejected, as he explains in his controversy with Hilbert, nevertheless the fundamental idea of encoding logical inferences via syntactic transformations remains at the heart of modern logic. Frege's introduction of the quantifier, similarly, brought logic into the modern era.[xx] His analyses of fundamental arithmetical concepts in terms of simpler logical ones came out of fruitful interaction with the mathematics of his day and cannot in all respects claim originality, but Frege's single-minded and rigorous employment of those analyses in pursuit of an epistemologically-significant goal has left a permanent mark on the philosophy of mathematics.

The logicist project itself, despite its clear failure (indeed, to some extent because of that failure) has been an especially fruitful influence on subsequent investigations. The central reason for continued interest in Frege's logicism stems from a continued interest in the nature of mathematical, specifically arithmetical, truth. In providing his analyses of arithmetical truths, Frege argues for a number of theses about arithmetic, including the claims that arithmetic is not based on anything psychological, that it is not grounded in intuition, and that arithmetic deals with a specifically arithmetical collection of objects, the numbers. The arguments Frege provides for these theses continue to play a significant role in the development of competing philosophical views about the nature of mathematical truth and mathematical knowledge. Because Frege's views taken together require the truth of logicism (though not the success of his particular means of demonstrating it), any real engagement with Frege's important arguments about the nature of mathematical truth and knowledge must come to grips with his logicism. Progress on these issues requires progress on the question of whether the logicism is, in the end, a viable thesis, and if not, then which parts of Frege's edifice one must give up.

Bertrand Russell himself did not take the paradox he discovered in Frege's system to be a reason to reject the logicist thesis. After the paradox, he pursues together with Alfred North Whitehead a revised version of logicism that is intended to be essentially in the spirit of Frege's project.[xxi] One difficulty with the Russell-Whitehead project, and a difficulty that would presumably have led Frege to view it as not quite "logicist," is that the fundamental principles, the axioms, from which Russell and Whitehead attempt to derive much of mathematics include principles (especially the axiom of infinity and the axiom of reducibility) that Frege would not have regarded as purely logical. The difficulty here is that they lack the kind of self-evidence that Frege demanded of fundamental logical truths. On this question, subsequent scholars have primarily sided with Frege.

More problematic for the Russell-Whitehead version of logicism, together with that of Frege, are Kurt Gödel's incompleteness theorems.[xxii] These theorems were proven after Frege's death, so did not directly affect Frege's own research, but provide yet another blow to the kind of program envisioned both by Frege himself and by those, e.g. Russell and Whitehead, who worked on resuscitating a Frege-style logicist project in the aftermath of Russell's paradox. Gödel's first incompleteness theorem shows that, as long as the truths of arithmetic are expressible, as both Frege and Russell would have agreed, via a syntactically complete set of sentences of a formal language of arithmetic, then there is no way to axiomatize arithmetic. That is to say, Frege's assumption that there is a manageable "core" of arithmetical truths from which the rest of arithmetic is provable is simply false. No decidable set, and certainly no finite set, of truths can serve as the proof-theoretic basis for all of arithmetic. This result by itself does not show that logicism is false, but it does undermine the straightforward Fregean way of attempting to demonstrate its truth, namely the strategy of proving the (purported) core arithmetical truths from truths of logic. In addition, it means that logicism can be true only if the truths of logic themselves form a quite unmanageable collection (specifically, one that's not recursively enumerable).

The concept of *value-range* at the core of Frege's difficulties with logicism is one of a handful of similarly compelling, and similarly problematic, concepts that have been at the heart of much mathematical work since the middle of the nineteenth century. The notions of the *graph* of a function, and more familiarly of a *set* of numbers or of functions, are of the same ilk as Frege's value-ranges, and are similarly slippery: their role is sufficiently ubiquitous and foundational that it appears that, as Frege puts it, we "cannot get on without them."[xxiii] But the attempt to lay down basic principles governing these entities has revealed that the most natural way of understanding them, i.e. Frege's way, in terms of a purportedly-analytic principle of extensionality, is not workable. It is in this sense that Frege's clarity about the importance of value-ranges, and the precision of his attempt to lay down their fundamental principles, has been salutary for the development of modern foundational theories and for an understanding of their philosophical significance. Current foundational work, following the lead of Cantor and Zermelo, focuses on the axiomatic presentation of theories of sets, dropping Frege's claim to the analyticity of those axioms. The philosophical question of what such foundational strategies can tell us about the nature of mathematical truth, of mathematical objects, and of mathematical knowledge is one that continues to be a subject of fruitful debate, and one that turns in part on the question of which of Frege's aims, in attempting to found arithmetic on value-ranges, must be given up in the modern setting.

Finally: an especially important thread in the influence of Frege's logicism concerns the impact of that thesis on modern empiricism. Here the influence is twofold. First of all, Frege's logicist notion of the *reduction* of truths about a given subject-matter to truths about something arguably more simple serves as a model for later attempts to reduce the empirical sciences to something closely linked to the immediate contents of experience. In this regard, Frege's influence on both Bertrand Russell and Rudolf Carnap is direct, and through them the Fregean conception of theoretical reduction, or descendants of it, continues to influence empiricists and their critics today. The second central influence of Frege's logicism on this movement was that the logicist thesis offered hope to early 20th-century empiricists that mathematical knowledge might be subsumed under the umbrella of logic, and hence no longer stand as a *prima facie* obstacle to the empiricists' denial of synthetic *a priori* knowledge. In this, the tradition that follows Frege goes further than Frege would have gone himself: Frege is no empiricist, and holds that geometry (unlike arithmetic) offers a clear instance of the synthetic *a priori*. Though Frege's own attempt to demonstrate the truth of his logicist thesis was, as above, a clear failure, the question of whether that logicist thesis, or some near relative of it, is in fact defensible remains open.[xxiv]

III.2 Language and Mind

Frege's theory of sense and reference has had an enormous impact on 20th- and early 21st-century philosophy of language and mind. Most significant here have been his views that each significant piece of language has a *sense* that's grasped by competent speakers, and that it is in virtue of expressing this sense that a piece of language refers to the item that it in fact refers to. Put together, these two views about sense give rise to a thesis about the connection between mental states and language that Frege himself did not dwell on: the thesis that what our words refer to is determined by what we, the speakers of those words, understand, or represent, when we use them. So stated, the thesis is not entirely clear, depending as it does on exactly what one means by “understand” or “represent,” and also on what it is for such a represented entity to *determine* the reference of a word. While different accounts of exactly what Frege has in mind when he speaks of “grasping” the sense of a sentence will yield different verdicts on the degree to which Frege can be said to explicitly agree with different variations of the theme here, the fundamental idea is relatively straightforward: the Fregean view, not-unreasonably so-called, is the view that our terms refer to entities in virtue of those entities’ satisfaction of properties that we speakers explicitly associate with those words.

The view, thus vaguely stated, has a natural appeal, and might seem to be simply part of the obvious view that language is conventional: our words refer to what we take them to refer to, and our thoughts are about various things in virtue of our ability to, as Russell puts it, “describe” those things. A good deal of modern philosophy of mind and of language revolves around the question of whether, and if so in what sense, this basic idea can be right. The central line of argument against the Fregean view turns on the claim that reference is not determined exclusively by what’s explicitly represented by competent speakers, but is determined (in part) by other, non-represented relationships between speakers and those references. With respect to proper names, for example, it has been argued that speakers can refer via the use of names without knowing uniquely-identifying qualities of the names’ bearers; what’s required instead is intentional participation in a community-wide practice, one whose details need not be represented.[xxv] A similar argument applies to natural-kind terms like ‘water’ and ‘gold;’ here the idea is that successful reference does not require explicit representation of features of the kinds in question, but something more like an appropriate spatial or causal relationship to instances of the kind.[xxvi] With respect to indexical terms like “I” and “yesterday,” it has been argued that the explicitly-represented information is insufficient to determine reference, and that non-represented features of the speaker’s context play an essential role in the determination of reference.[xxvii] In all of these cases, the argument against the central Fregean thesis is that if the *sense* of a singular term is just what the competent speaker understands, then sense is insufficient to determine reference.

Similar reasons have prompted some to disagree with Frege’s idea that there are two kinds of semantic value for each piece of language. Here the argument has been most pronounced with respect to proper names: the claim is that proper names have only a single semantic role, which is to refer to their bearers. Other features of names, it is argued, for example the collection of things that the user knows about the bearer, are taken on this view to form merely collateral information, and not to serve as part of the semantics of the name. Definite descriptions, on the other hand, have on this line of argument just the relevant descriptive properties as semantic value, while the reference is merely collateral.[xxviii]

Arguments in favor of something like the Fregean position turn on the kinds of considerations originally raised by Frege himself. That proper names have senses is argued for by noting that two co-referring proper names can play different semantic roles, as can be seen by the possibility of turning an informative statement into a tautology by substituting an instance of a proper name for an instance of a co-referring one. That sense (or: something explicitly represented by competent speakers) determines reference is argued for by noting for example that contextual features and the speaker's relationship to them are part of what is, in the relevant cases, explicitly represented by speakers.[xxix] And so on.

Current debates regarding the role of descriptive mental representation in successful reference, and the necessity of a two-tiered semantic theory, go well beyond Frege's own relatively rudimentary views about mental content and semantics. But the power of those original views is still felt in these debates, with Frege's central questions still very much alive, and his fundamental ideas about language and thought forming an important theoretical stronghold.

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[i] See Lothar Kreiser**,** *Gottlob Frege: Leben, Werk, Zeit*, (Hamburg: Felix Meiner Verlag, 2001). Also see: Nikolay Milkov, "Frege in Context," *British Journal for the History of Philosophy* 9 (3) 2001: 557-570; Gottfried Gabriel and Wolfgang Kienzler (eds) *Frege in Jena. Beiträge zur Spurensicherung* (*Kritisches Jahrbuch der Philosophie,* Band 2) (Würzburg: Königshausen & Neumann, 1997); Gottfried Gabriel and Uwe Dathe (eds), *Gottlob Frege. Werk und Wirkung* (Paderborn: Mentis, 2000); Christian Thiel and Michael Beaney, "Frege's Life and Work" in Michael Beaney and Erich Reck (eds) *Gottlob Frege: Critical Assessments of Leading Philosophers* Vol I, Routledge 2005

[ii] Halle: L. Nebert, 1879. English translation by Stefan Bauer-Mengelberg as *Begriffsschrift, A Formula Language, Modeled Upon That of Arithmetic, for Pure Thought* in van Heijenoort (ed), *From Frege to Gödel A Source Book in Mathematical Logic, 1879-1931* (Cambridge, MA: Harvard University Press 1967), pp 5-82.

[iii] *Die Grundlagen der Arithmetik*, *Eine logisch mathematische Untersuchung über den Begriff der Zahl* (Breslau: Wilhelm Koebner, 1884). English translation by J. L. Austin as *The Foundations of Arithmetic, A logio-mathematical enquiry into the concept of number* (Oxford: Blackwell, 1953). (Hereafter, "*Grundlagen*.")

[iv] *Grundlagen* §55.

[v] *Grundgesetze der Arithmetik Band I* (Jena: Hermann Pohle, 1893). Partial English translation by M. Furth as *The Basic Laws of Arithmetic* (Berkeley and Los Angeles: University of California Press, 1967). (Hereafter: "*Grundgesetze I"*)

[vi] "*Über Sinn und Bedeutung*," *Zeitschrift für Philosophie und philosophische Kritik* 100, pp. 25-50. Reprinted in I. Angelelli (ed), *Kleine Schriften* (Hildesheim: Georg Olms Verlag, 1990) (2nd edn) (hereafter, *"KS")*, pp 143-162. English translation by M. Black as "On Sense and Meaning" in B. McGuinness (ed) *Collected Papers on Mathematics, Logic and Philosophy* (Oxford and New York: Blackwell Press, 1984) (hereafter, *"CP"*) pp 157-177.

[vii] *Funktion und Begriff*, (Jena: Hermann Pohle, 1891). Reprinted in *KS* pp 125-142. English translation as "Function and Concept" in *CP* pp 137-156.

[viii] "*Über Begriff und Gegenstand," Vierteljahrsschrift für wissenschaftliche Philosophie 16 (1892) 192-205*. Reprinted in *KS* pp 167-178. English translation as "On Concept and Object" in *CP pp 182-194.*

[ix] This is somewhat over-simple. For Frege also acknowledges higher-level concepts; these are functions not from *objects* to truth-values, but from functions to truth-values.

[x] *Grundgesetze der Arithmetik Band II* (Jena: Hermann Pohle 1893).

[xi] Russell to Frege 16 June 1902 English translation in *Philosophical and Mathematical Correspondence*, Gabriel et al (eds) (Chicago: University of Chicago Press, 1980) (hereafter, PMC) pp 130-131.
[xii] See the Appendix to *Grundgesetze I*
[xiii] See “Neuer Versuch der Grundlegung der Arithmetik,” dated by the editors at 1924/25, in Hermes et al (eds), *Nachgelassene Schriften* (2nd revised edition) (Hamburg: Felix Meiner Verlag, 1983) pp 298-302. English translation in Hermes et al (eds), *Posthumous Writings* (Chicago: University of Chicago Press, 1979) pp 278-281.
[xiv] David Hilbert, *Grundlegung der Geometrie* (Stuttgart: Tuebner, 1900). English translation of the 10th edition: *Foundations of Geometry*, L. Unger (trans), P. Bernays (ed) (La Salle, IL: Open Court, 1971).
[xv] Letters between Frege and Hilbert 1895-1903. English translation in PMC pp 32-52.
[xvi] “Über die Grundlagen der Geometrie,” *Jahresbericht der Deutschen Mathematiker-Vereinigung 12, 1903, pp 319-324, 368-365.* Reprinted in *KS* pp 262-272. English translation as “Foundations of Geometry: First Series” in *CP* pp 272-284. “Über die Grundlagen der Geometrie,” *Jahresbericht der Deutschen Mathematiker-Vereinigung* 15, 1906, pp 293-309, 377-403, 423-430. Reprinted in *KS* pp 281-323. English translation as “Foundations of Geometry: Second Series” in *CP* pp 293-340.
[xvii] “Der Gedanke,” *Beiträge zur Philosophie des deutschen Idelismus I*, 1918, pp 58-77, reprinted in *KS* pp 342-361; English translation in *CP* pp 351-372. (Hereafter, “Der Gedanke.”) “Die Verneinung,” *Beiträge zur Philosophie des deutschen Idelismus I*, 1918, pp 143-157, reprinted in *KS* pp 362-377; English translation in *CP* pp 372-389. “Gedankengefüge,” *Beiträge zur Philosophie des deutschen Idealismus III*, 1923, pp 36-51, reprinted in *KS* pp 378-394; English translation in *CP* pp 378-394.
[xviii] “Der Gedanke” p 69 (p. 353 of *KS,* p. 363 of *CP*).
[xix] Letter to Alfred Frege of 26 July 1925, quoted by Hermes, Kambartel and Kaulbach in their historical introduction to Frege [1979], p ix.
[xx] That is not to say that here Frege was alone: the quantifier was also introduced at roughly the same time, though independently, by Giuseppe Peano and by C. S. Peirce.
[xxi] Bertrand Russell and Alfred North Whitehead, *Principia Mathematica* (Cambridge, England; Cambridge University Press, 1910-1913).
[xxii] Kurt Gödel, “Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I” (“On formally undecidable propositions of *Principia Mathematica* and related systems I”) in Feferman et al (eds), *Kurt Gödel Collected Works Volume I* (New York and Oxford: Oxford University Press, Clarendon Press, 1986) pp 144-195
[xxiii] *Grundgesetze* I, p. x (Furth p. 6)
[xxiv] See for example Crispin Wright, *Frege’s Conception of Numbers as Objects* (Aberdeen University Press, 1983) for a defense of what has come to be known as the “neo-Fregean” approach to logicism.
[xxv] See Saul Kripke, *Naming and Necessity*, Cambridge, MA: Harvard University Press 1980 (hereafter, “Kripke”).
[xxvi] See Hilary Putnam, “The Meaning of Meaning,” *Minnesota Studies in the Philosophy of Science* 7:131-193 (1975); also Kripke.
[xxvii] See John Perry, “Frege on Demonstratives,” *Philosophical Review* 86 (4) 1977 474-497 and John Perry, “The Problem of the Essential Indexical,” *Nous* 13 (1979) 3-21.
[xxviii] See Bertrand Russell, “On Denoting,” *Mind* 14 (56) (1905) 479-493.
[xxix] See John Searle, *Intentionality, an Essay in the Philosophy of Mind*, New York: Cambridge University Press 1983.

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