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# An Attachment Theory for Microsphere Adhesion

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A theory of the mechanics of adhesion between a microsphere and substrate is presented. When a force is applied to an elastic body, the deformation depends not only on the magnitude of the force but also its location and distribution. Molecular adhesion between bodies is a surface force localized to the contact area. In contrast, applied forces such as from gravity, flow fields, inertia, *etc.*, are distributed over the volume (body forces) and/or surface areas. Effects of different types of force systems on deformation, particularly when these forces are combined, can influence adhesion. The Hertzian structural stiffness parameter $K$ does not reflect the effects of differently distributed multiple forces. A theory is developed that takes into account simultaneous application of the adhesion force and applied forces through the development of a reduced stiffness, $K_R$. The paper also develops an equivalent Hertzian process for the condition of adhesion forces alone so that the mechanics of adhesion can be modeled completely by Hertzian theory. Illustrations of how adhesion alone is handled and how the reduced stiffness behaves are provided using experimental data from compressed, crossed rods and from hard particles in static equilibrium with both relatively hard and soft substrates.

*Keywords:* Microparticles; adhesion; contact and applied forces; contact deformation

## INTRODUCTION

The first theory of the adhesion of contacting deformable microspheres to gain wide acceptance was the JKR theory [1], also referred

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to as the JKRS theory [2]. It is an elegant theory that follows a clever derivation and is based on the concept of surface energy. Fundamentally, it relates force and deformation to surface adhesion energy. The DMT theory [3] followed shortly after; it takes a somewhat different view of how and where the adhesion force acts. More recently, Maugis [4], in a more rigorous fashion, applied fracture mechanics to the adhesion problem. Assuming that the tensile adhesion stresses act in an annulus around the contact area, Maugis obtained a more general theory, with the JKR and DMT theories as special cases. A view of the relationships between these theories is presented by Johnson [5]. All of these theories apply Hertzian mechanics of contacting spherical bodies. All are for static loading and elastic deformation (without dissipation) and address the interrelationship between the deformation due to an externally applied force (such as a pull-off or separation force) and its effect on adhesion. When dealing with adhesion loading alone, these theories have been reasonably successful. But when externally applied forces (such as inertia forces during impact) act in combination with adhesion, agreement with experiment is lacking. Horn *et al.* [6] show that when an applied force compresses a contact area, the JKR theory significantly under-predicts the contact radius. A reason for the lack of agreement is that Hertzian theory has an inherent limitation that arises when applied to the adhesion problem. For two spheres in contact or a sphere in contact with a substrate, the theory determines stresses and deformations (deflection and contact radius) due to a single equivalent resultant force, $P$. However, the theory is not sensitive to the nature, distribution or location of the actual force or forces that constitute $P$. $P$ could be a point force, the resultant of a body force, the resultant of a surface force or it could be the resultant of a combination of forces (such as adhesion, a surface force, and weight, a body force). In particular, during impact, adhesion acts in combination with an inertial force distributed throughout the sphere, a combination of a surface force and body force. The mass center deformation of a sphere during impact in the presence of adhesion is different from the mass center deformation in the absence of adhesion. The nature of such differences is illustrated in this paper through the use of an analogy from simple beam theory. To rectify the deficiency of Hertzian theory for application to adhesion, a sensitivity to combined adhesion and body forces is introduced artificially using the concept of reduced stiffness.

The model developed in this paper is based on Hertzian mechanics to apply to the attachment process of microspheres under the combined action of adhesion and an externally applied force. In some respects the derivation of the model is patterned after the JKR theory but with some specific differences. It leads to a concept of reduced stiffness which is explored in detail. The incentive behind this work is to explore the problem of attachment in the presence of an external force such as an inertial force for applications to microsphere impact. Another purpose is to complement and expand on existing static models and to add to the body of knowledge of adhesion mechanics by taking an unconventional approach. This paper treats the attachment phase of particle surface adhesion interaction. Research on the problem of removal or separation is ongoing. Removal forces can differ from forces that accompany attachment (it is easy to push against a flat surface but not so easy to pull on one); removal is considered to be a separate problem and not treated here directly. Finally, some of the equations and concepts developed in the paper are compared with existing experimental data.

## THE CONCEPT OF REDUCED STIFFNESS

### Linear Beam Theory Analogy

To explore the effects of load distribution and location and to examine the concepts of stiffness and flexibility of a microsphere in more tangible terms, consider an analogy using static, elementary beam theory. Although the analogy is imperfect, it is informative. Figure 1 shows three loading conditions on cantilever beams. Condition (a) is analogous to adhesion alone, (b) to an externally applied force alone and (c) is analogous to combined loading. The resultant force on beam (a) is $F$ at a distance $\xi\ell$ from the fixed end. The constant $\xi$ is such that $0 \leq \xi \leq 1$ and the tip deflection $y$ is:

$$y_a = \frac{Fl^3}{2EI}\xi^2\left(1 - \frac{\xi}{3}\right) \tag{1}$$

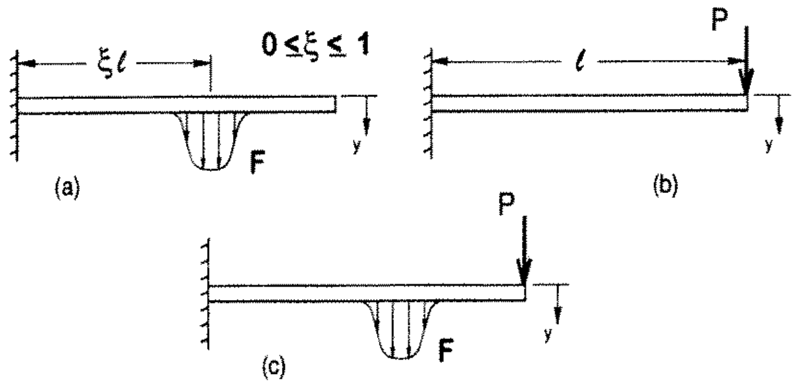


FIGURE 1 Different loading systems on a cantilever, (a) distributed force alone, (b) point force alone and (c) combination of forces.

$E$ is Young's modulus and $I$ is the second area moment of the cross section. For loading systems (b) and (c):

$$y_b = \frac{P\ell^3}{3EI} \tag{2}$$

$$y_c = \frac{P\ell^3}{3EI}\left[1 + \frac{3F}{2P}\xi^2\left(1 - \frac{\xi}{3}\right)\right] \tag{3}$$

For convenience, let $K_b = EI/\ell^3$ be a stiffness parameter for the beams where beam stiffness, $k$, is defined as force per unit deflection. In case (b), for example, the stiffness is $k = 3K_b$. This is different from the stiffness for loading condition (a) and certainly differs for the loading system (c) where

$$k = 3K_b \bigg/ \left[1 + \frac{3F}{2P}\xi^2\left(1 - \frac{\xi}{3}\right)\right] \tag{4}$$

Equation (4) shows that the stiffness for load $P$ depends upon the force ratio $F/P$. If $F$ is proportional to $P$ then the stiffness, $k$, is a constant. Note also that the presence of $F$ (with the same direction as $P$) *reduces* the stiffness, $k$, by increasing the denominator of Eq. (4). Furthermore, the stiffness, $k$, is a *structural* parameter that depends on the beam's material through the modulus $E$, the beam's geometry through $I$ and $L$ and the loads' positions through $L$ and $\xi$. It will be seen later that the equations of Hertzian theory contain neither a load position parameter nor a load ratio parameter such as $F/P$. In fact, this is the deficiency or limitation of Hertzian theory that is being rectified through the work of this paper. Finally, it is important to recognize that stiffness can be computed taking by the ratio of the applied force to the static deflection. This plays an important role in interpreting the reduced Hertzian stiffness.

Before proceding, these concepts can be examined further using the JKR theory [1].

## JKR Reduced Stiffness

One of the basic equations of the JKR theory can be examined to show conformance with the above concept of reduced stiffness. Solution for the contact radius, a, from JKR theory provides the following equation:

$$a^3 = \frac{R}{K}\left[P + \frac{3w\pi R}{2} + \sqrt{3w\pi RP + \left(\frac{3w\pi R}{2}\right)^2}\right] = \frac{R}{K_R^*}P \qquad (5)$$

where the "stiffness" comparable with Eq. (4) can be expressed as

$$K_R^* = K\Big/\left[1 + \frac{3w\pi R}{2P} + \sqrt{\frac{3w\pi R}{P} + \left(\frac{3w\pi R}{2P}\right)^2}\right] \qquad (6)$$

Here, $P$ is an externally applied force, $w$, is the Dupré surface energy constant (specific surface energy) and $R$ is the microsphere radius. The quantity $w\pi R$ has the units of force and so $w\pi R/P$ can be considered as the ratio of an adhesion force to an externally applied force. Consequently, $K_R^*$ is a reduced stiffness corresponding to the JKR theory. Although the JKR theory demonstrates conformance with the concept of reduced stiffness, the theory has been found to provide stiffness values well above what is measured experimentally. A somewhat different approach is presented in the following.

## THEORY

Consider a microsphere in static equilibrium under the action of adhesion alone against a flat rigid surface as shown in Figure 2a.

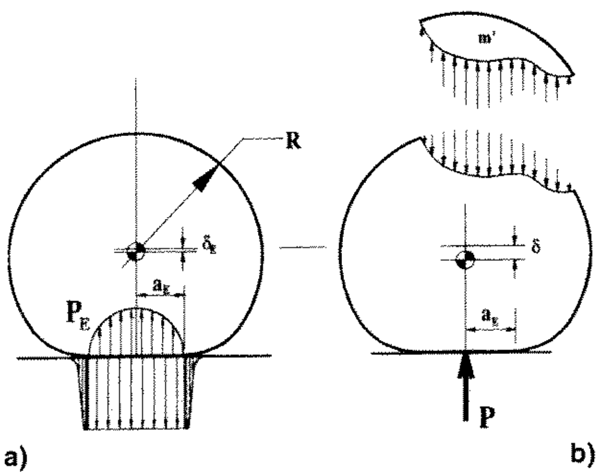


FIGURE 2 Different loadings and static mass center deflections with the same contact radius, $a_E$, for a microsphere. The one on the left represents adhesion alone, where the sphere is practically stress free except for the contact region. On the right is for an inertial load $P$, showing an arbitrary cut portion $m'$.

Under ideal conditions it has a contact area with radius $a_E$, a deflection, $\delta_E$ and resultant tensile adhesion force $P_E$, distributed over the periphery of the contact circle. It is balanced by a compressive force of the same magnitude. Now consider the same microsphere in the absence of adhesion and with an externally applied load distributed throughout the sphere with an equivalent point force, $P$, that develops the same contact radius, $a_E$ and a Hertzian deflection [1], $\delta$, as depicted in Figure 2b. The adhesion force is a localized tensile contact *surface* force and does not see the same resistance to the deformation as seen by a force, $P$, distributed throughout the body of the sphere. The downward deformation, $\delta_E$, is likely to be significantly smaller for adhesion loading than $\delta$ for distributed loading since for the former most of the sphere above the contact region is stress free. In general, equal forces with different load application points and/or distributions will produce different deformations.

Hertzian theory can be summarized by the following equations:

$$P = K\sqrt{R}\delta^{3/2} = \frac{K}{R}a^3 \tag{7}$$

[1] The mass center is chosen here as representative of those points in the body away from the contact region with deflection $\delta$, corresponding to Hertzian theory [7] and also is chosen because it is a convenient point of reference for impact studies.

where $a^2 = R\delta$ and $K$ (a material constant) is given by

$$K = \frac{4}{3\pi(k_1 + k_2)} \tag{8}$$

$$k_i = \frac{1 - \nu_i^2}{\pi E_i} \tag{9}$$

$$R = \frac{r_1 r_2}{r_1 + r_2} \tag{10}$$

with $r_1$ and $r_2$, the radii of the particle and substrate (for a rigid flat substrate surface, $R = r_1$ of the microsphere). Equation (7) shows a nonlinear, 3/2-power relationship between the external force, $P$, and its resulting deflection, $\delta$. In such cases stiffness often is represented by a tangent modulus,

$$\frac{dP}{d\delta} = \frac{3}{2} K\sqrt{R}\,\delta^{1/2} \tag{11}$$

The tangent modulus is not a constant but is proportional to the parameter $K$. It is clear from all of the above equations that Hertzian theory is insensitive to the nature, location and distribution of the force(s) making up the resultant $P$. To establish such a dependence for applications to adhesion, two assumptions are made for the conditions of combined loading: (a), that a representative, reduced value of the stiffness parameter, $K$, can be found that is a consequence of the presence of adhesion and that (b), the process remains Hertzian. The reduced value is called the *reduced stiffness*, $K_R$.

## Reduced Stiffness

Figure 3 shows three Hertzian deformation curves. The one indicated as $P_1 = P(K)$ with origin $O$, has stiffness $K$ and represents the force-deformation characteristics of an elastic sphere under the action of a compressive, externally applied force in the *absence* of adhesion. The curve indicated as $P(K_R)$ has a reduced stiffness $K_R$ and represents the force-deformation characteristics of a sphere under the *combination* of

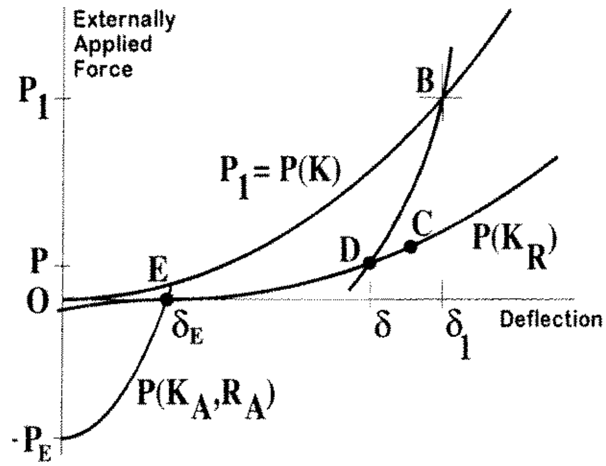


FIGURE 3 $P(K_i)$ curves represent Hertzian deformation for stiffness $K_i$ under externally applied forces. Positive $P$ is compression; negative $P$ is tension. Point $E$ is static equilibrium.

an external load and adhesion. A third is $P(K_A, R_A)$, due to adhesion alone[2] and is discussed more fully later. Although this paper does not explicitly cover particle dynamics, one of its goals is application to impact. A view here is that the curve $P(K_R)$ can be used to represent behavior during microsphere impact, namely, under uniformly distributed inertial loading in the presence of adhesion. The use of a body force combined with adhesion permits applications to dynamic loads such as from impact, where the equation of motion can be written as

$$m\ddot{\delta} + K_R\sqrt{R}\,\delta^{3/2} = -f_A(a, a_E, \delta, \delta_E) \tag{12}$$

where $f_A$ is a function of the problem's variables, including velocities, and is stated here in a general, unspecified form. (See [8] for an example of modeling with such an equation). A normal impact with an intial velocity less than the capture velocity will be followed by damped oscillations (possibly overdamped motion) along the $P(K_R)$ curve with dissipation eventually leading to static equilibrium at point $E$. This is the same point reached by adhesion alone along $P(K_A, R_A)$ under static conditions. The same force, $P(K_R)$, for example, could

[2] Strictly speaking, the force of adhesion alone should not appear on a plot of externally applied forces, but it should be understood that $P(K_A,R_A)$ represents the resultant force of adhesion.

also represent the combination of adhesion and a static gravity load in such cases where weight is significant. It is assumed that attachment takes place when the deflection, $\delta$, and radius, $a$, reach zero simultaneously. The effects of snap-on [2] are neglected in this paper.

The curve $P(K_R)$ is now investigated and a derivation is presented that leads to an expression for the reduced stiffness, $K_R$. The derivation has some similarities to that of the derivation of the JKR equation but also some essential differences, discussed later. With reference to Figure 3, a continuously varying external load is hypothesized to be applied in sequence over a cycle (points $O$ to $E$) such that:

1. an external force, $P_1$, in the absence of adhesion compresses the microsphere against a substrate reaching a deflection $\delta_1$ (path $OB$) and a contact area with radius $a$,
2. then the force $P_1$ is relaxed and the effects of adhesion simultaneously are introduced, taking the path $BD$ in such a way that the contact area and radius, a, remain constant but the center deflection drops to $\delta$ at load $P$ and the stiffness changes to $K_R$ from $K$ and,
3. the force $P$ is removed to complete a cycle of loading and unloading, following $P(K_R)$ along the curve from point $D$ to $E$, reaching the state of static equilibrium under adhesion alone (since $P = 0$).

During the relaxation phase, Step 2, the adhesion force is introduced as the deflection decreases from $\delta_1$ to $\delta$, so work is done by the adhesion force, referred here to as $W_A$, the work of adhesion. From conservation of energy, the work done of static forces [3] over the cycle $O-B-D-E$ can be expressed as:

$$W_{OB} + W_{BD} + W_A + W_{DE} = 0 \tag{13}$$

The first work term is:

$$W_{OB} = \int_0^{P_1} P d\delta = \frac{2}{5(K\sqrt{R})^{2/3}} P_1^{5/3} \tag{14}$$

[3] The Hertzian process itself is conservative. It is tempting here to include energy losses in the process cycle to represent material dissipation, but only work terms that affect an idealized process can be considered.

The work done over the path DE depends on the function $P(K_R)$. This function is to be Hertzian but a choice exists on how it is translated to the equilibirum point. That is, $P = K_R\sqrt{R}(\delta - \delta_E)^{3/2}$ could be used but so can $P = K_R\sqrt{R}(\delta^{3/2} - \delta_E^{3/2})$; both accomplish making the external force, $P$, be zero at the equilibrium point. The latter form is chosen. It will be seen later that this results in a better match of the theory with experiment. The work relative to an arbitrary point with force $P$ and contact radius $a$ is:

$$W_{DE} = -\frac{3}{5}K_R\sqrt{R}\,\delta_E^{5/2} - \frac{2}{5}\frac{1}{\left(K_R\sqrt{R}\right)^{2/3}}\left(P + K_R\sqrt{R}\,\delta_E^{3/2}\right)^{5/3} + \left(K_R\sqrt{R}\right)^{1/3}\delta_E^{3/2}\left(P + K_R\sqrt{R}\,\delta_E^{3/2}\right)^{2/3} \quad (15)$$

Path $BD$ is under the conditions where the external force is relaxed and adhesion is introduced. Specifically, the conditions are that the force changes from $P_1$ to $P$, the deflection changes from $\delta_1$ to $\delta$, the area and contact radius remain constant and the stiffness *changes* from $K$ to $K_R$. The work of the force $P$ is:

$$W_{BD} = \int_{P_1}^{P} P d\delta \quad (16)$$

where, from Eq. (7), d$\delta$ can be expressed as

$$d\delta = \frac{2a^2}{3RP}dP - \frac{2a^2}{3RK}dK \quad (17)$$

The process from $B$ to $D$ is based on Hertzian theory and Eq. (17) maintains the conditions of a body or bodies with spherical contact geometry under the influence of compressive forces. Substituting this into Eq. (16), using $P/K = a^3/R$ and integrating gives:

$$W_{BD} = \frac{2a^2}{3R}(P - P_1) - \frac{2a^5}{3R^2}(K_R - K) \quad (18)$$

The surface energy is used as the work of adhesion, $W_A$. To an arbitrary point on ED it is $-w\pi a^2$ and over the full cycle:

$$W_A = -w\pi a_E^2 \quad (19)$$

where $w$ is the Dupré surface energy constant (specific surface energy). The work is negative since the adhesion force has a positive sense and the deflection, $\delta$, is decreasing over path $BD$. Substitution of the above into Eq. (13) gives:

$$\begin{aligned}\frac{2}{5(K\sqrt{R})^{2/3}}P_1^{5/3}+\frac{2a^2}{3R}(P-P_1)-\frac{2a^5}{3R^2}(K_R-K)\\ -\frac{3}{5}K_R\sqrt{R}\,\delta_E^{5/2}-w\pi a_E^2\\ -\frac{2}{5}\frac{1}{\left(K_R\sqrt{R}\right)^{2/3}}\left(P+K_R\sqrt{R}\,\delta_E^{3/2}\right)^{5/3}\\ +\left(K_R\sqrt{R}\right)^{1/3}\delta_E^{3/2}\left(P+K_R\sqrt{R}\,\delta_E^{3/2}\right)^{2/3}=0\end{aligned} \tag{20}$$

Recall that the process from $O$ to $B$ is such that $P_1 = Ka^3/R$ and that from $E$ to $D$, $P = K_R\sqrt{R}(\delta^{3/2} - \delta_E^{3/2})$. Placing these into Eq. (20) and imposing the condition that $K_R = K$ when $w = 0$ determines a relationship for $K_R/K$:

$$\frac{K_R}{K}=1-\frac{5}{2}\left(\frac{5}{9}\right)^{5/2}\frac{w\pi R^2}{Ka_E^3} \tag{21}$$

This equation provides values of $K_R$, reduced stiffness, for corresponding values of $K, R$ and $a_E$. Equation (21) can be multiplied by $K$ to provide a direct solution for the reduced stiffness. Then, other than the constant, the second term on the right hand side can be written as $(w\pi R)/(a_E^3/R)$ and has the units of stiffness, $\mathrm{N/m^2}$. With reference to the earlier discussion of a structural analogy, Eq. (21) does not explicitly contain a load position parameter yet it does fulfill the condition of being a ratio of a static (adhesion) load to a deflection. As such, it does represent the effect of adhesion in reducing the stiffness to an applied load. It is conceivable that the right hand side of Eq. (21) could go to zero indicating that the reduced stiffness is zero. A way of looking at this that as $K_R \to 0$, the externally applied force had diminishing influence over the particle. Reaching such a limit appears unrealistic; if it occurs, it may represent a deficiency in the theory. This remains to be examined.

As a result of the above assumptions and equations, a microsphere that begins contact with a surface at $\delta = 0$ under the combined action of a uniformly distributed force and adhesion follows the Hertzian process $P(K_R)$ with a force of:

$$P = K_R\sqrt{R}\,\delta^{3/2} - K_R\sqrt{R}\,\delta_E^{3/2} = \frac{K_R}{R}a^3 - \frac{K_R}{R}a_E^3 \qquad (22)$$

with $K_R$ given by Eq. (21). Although Eq. (22) has the 3/2 power relationship of a Hertzian process and a cubic relationship between force and contact radius it is not identical (by choice) to a translated Hertzian equation. However, as will be seen by comparisons of the above with experimental data, this appears to be an advantage rather than a deficiency.

When applied to impact, the above equations indicate that when a particle approaches and makes contact, the force will follow the $P(K_R)$ curve to some arbitrary point $C$ when the velocity becomes zero. Being compressed, it reverses direction and if it has enough energy, it will rebound from the surface. If it does not leave the surface the microsphere will remain attached and oscillate elastically (and nonlinearly) while the force varies along the $P(K_R)$ curve until the kinetic energy is totally damped. It remains attached to the surface in static equilibrium, point $E$ on the diagram.

A question can be raised concerning whether or not the stiffness, $K_R$, can be measured or computed directly. One way of computation is to use a finite element analysis (see [9] for such an approach on a related problem) to determine ($K_R$) for specific force distributions. It may be possible to use the period of a dynamic oscillation to recover the stiffness, $K_R$. However, such an approach may not be feasible because this is a nonlinear oscillation problem. To provide a more comprehensive theory, the deformation process under adhesion alone is now considered.

## Adhesion Alone

An additional assumption is now made, that the process of static attachment to a surface by adhesion alone can be represented by an *equivalent* tensile Hertzian process $P(K_A, R_A)$ shown in Figure 3 with equivalent stiffness, $K_A$, and equivalent radius, $R_A$. It is necessary to use equivalency since the Hertzian process does not adequately represent the behavior of a sphere under adhesion loading alone [1]. According to the equivalent model, a microsphere proximate to a substrate is attracted by adhesion and reaches its deformed static condition according to:

$$P - P_E = K_A \sqrt{R_A}\, \delta^{3/2} = \frac{K_A}{R_A} a^3 \tag{23}$$

From Hertzian mechanics the work done by a force, $P$, in going from $P = 0$ (point $E$) to $P = P_E$ (see Fig. 3) must equal the surface adhesion energy. This determines that $P_E = -(5/3)w\pi R_A$ where $w$ is the Dupré surface energy constant. Consequently,

$$\frac{K_A}{R_A} = \frac{P_E}{a_E^3} = \frac{5}{3}\frac{w\pi R_A}{a_E^3} \tag{24}$$

If $a_E$ and $\delta_E$ are measured independently, then from Hertzian theory the equivalent radius can be found from $R_A = a_E^2/\delta_E$ and $K_A$ can be determined from Eq. (24). Using Eq. (24), Eq. (21) can be rewritten as

$$\frac{K_R}{K} = 1 - \frac{3}{2}\left(\frac{5}{9}\right)^{5/2} \frac{R^2 K_A}{K R_A} \tag{25}$$

Equations (21) and (25) are expressions for the reduced stiffness.

JKR theory can be examined for its corresponding values of $K_A$ and $R_A$. Equilibrium under adhesion alone occurs when $a_E^3 = 3w\pi R^2/K$. Equilibrium occurs under the condition of zero external force; substitution into Eq. (24) indicates that this corresponds to $K_A/R_A^2 = (5/9)K/R^2$.

$K_A$ and $R_A$ both can be determined using experimental measurements of both $a_E$ and $\delta_E$. It would be nice to be able to have an analytical method. One way is to use the work of Maugis [10], who extended the Hertzian theory to include large contact radii. He gives an expression for the equilibrium deformation, $\delta_E$:

$$\delta_E = \frac{a_E}{2} \ell n \frac{R + a_E}{R - a_E} \pm \left(\frac{8w\pi a_E}{3K}\right)^{1/2} \tag{26}$$

From Eq. (23),

$$\delta_E = \left(\frac{w\pi R_A}{K_A\sqrt{R_A}}\right)^{2/3} \tag{27}$$

Equating these and using Eq. (24) provides individual values of $R_A$ and $K_A$ without the need for experimental values of $\delta_E$. In fact, a value of $\delta_E$ results. Maugis also developed an equation for the static contact radius that can be used to provide a value of $a_E$ for use in Eq. (26) rather than an experimental value. It remains to be verified if Maugis' equations provide accurate values for applications to adhesion alone. They should be an improvement over direct application of Hertzian theory, but this topic is not pursued here.

## COMPARISONS WITH EXPERIMENTAL DATA

### Applied Force, Crossed-Rod Experiments

Horn *et al.* [6] carried out experiments where contacting crossed rods were loaded compressively by an external force, beginning at the static equilibrium position. The contact radius was measured as a function of force. Although the loads in Horn's experiments were reversed and the rods were pulled apart, only the force and contact radius values up the maximum force are applicable and will be used here. First, the equations are nondimensionalized both for convenience and because the Horn data are provided in such a form. Nondimensional forces are $Q = 4RP/Ka_E^3$ and $Q_1 = 4RP_1/Ka_E^3$. The nondimensional contact radius is $\alpha = a/a_E$ and the displacement is $\partial = \delta/(a_E^2/R)$. Because the process is Hertzian, $a^2 = R\delta$ and so $a_E^2 = R\delta_E$. Using these, Eq. (5) gives for the JKR theory (for comparison):

$$\alpha_{\mathrm{JKR}}^3 = \frac{1}{4}\left[Q + 2 + 2\sqrt{Q+1}\right] \tag{28}$$

From JKR theory, since $a_E^3 = 3w\pi R^2/K$ and the pull-off (separation) force $|P_S| = 3w\pi R/4$, then $Q_{\mathrm{JKR}} = P/|P_S|$. Nondimensionalizing Eq. (22) gives:

$$\alpha^3 = 1 + \frac{K}{4K_R}Q \tag{29}$$

Figure 4 shows $Q(\alpha)$ plotted for JKR theory and for reduced stiffness theory with $K_R = K$ (straight Hertzian theory), $K/2$ and $K/5$ (chosen arbitrarily). In addition, the data of Horn *et al.* [6], consisting of $P/|P_S|$ plotted against $\alpha$ for increasing loads, is plotted. It seems clear that Hertzian theory ($K_R = K$) and JKR theory do not follow the data but it does appear that almost all of the data lie between $K_R = K/2$ and $K/5$. Note that the form [$\alpha^3 - 1$ rather than $(\alpha - 1)^3$] chosen for Eq. (22) and its nondimensional counterpart, Eq. (29), retains the 3/2 power relationship, satisfies the equilibrium position condition and follow the data quite well.

A beneficial feature of reduced stiffness theory for microsphere contact problems is not only that it can be made to "follow the data" but that through $K_R$ it includes a dependence on the Dupré surface energy constant. So, for applications such as impact simulation, as materials change $w$ and $K_R$ change, permitting more accurate modeling. Both the reduced stiffness, $K_R$, and the equivalent stiffness and radius, $K_A$ and $R_A$, depend on the static equilibrium value of the contact radius. Experimental data of others and the expression for $K_R$ now are examined to get an understanding of some of the trends for $K_R/R$.

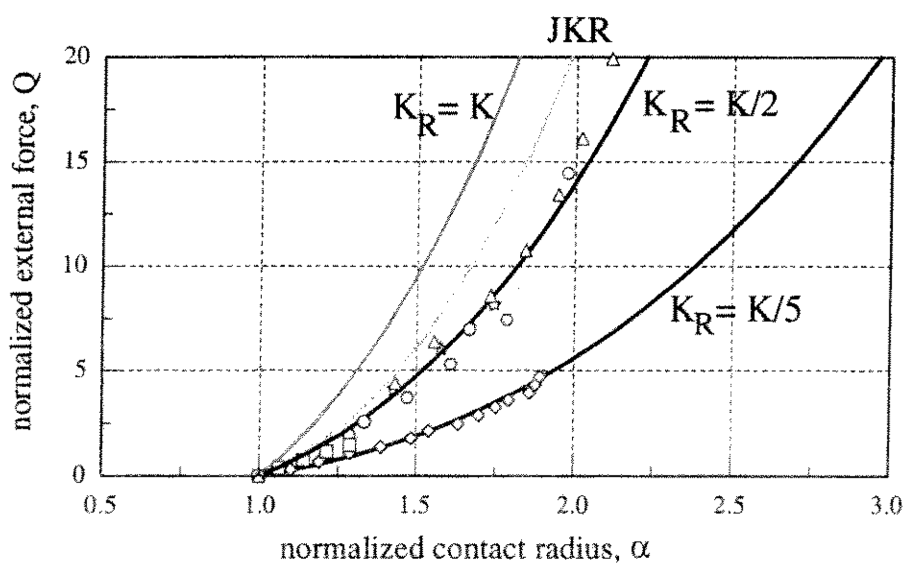


FIGURE 4 Comparison of Hertzian, JKR and the reduced stiffness theories with the experimental results of Horn *et al.* [6].

## Equilibrium, Hard Particle-Soft Substrate

Rimai and Busnaina [11] report an extensive set of measurements of the static equilibrium conditions of microspheres. They include contact radius and particle radius measurements of (soda-lime) glass microspheres on polyurethane substrates having different values of Young's modulus. The particles ranged in size from 1 μm to 60 μm. One of their findings is that the cube of a $a_E$ is proportional to the square of $R$. This observation and Eq. (21) imply that $K_R$ is a constant for these materials[4]. A set of values from their data is chosen for examination here because it represents a hard particle in equilibrium with a relatively soft substrate. Data are displayed in the Table where Eq. (21) was used to determine the values of reduced stiffness.

Since these experiments involve static equilibrium and adhesion alone, the results and Eq. (24) can be used to shed some light on the values of the equivalent stiffness, $K_A$ and radius, $R_A$. The data indicate that $K_A/R_A^2 = 2.7 \times 10^6/R^2$, an inverse relationship for these materials.

## Equilibrium, Hard Particle-Hard Substrate

Bowen *et al.* [12] measured the static contact radius of glass spheres adhering to a silicon substrate for a range of radii. These results indicated that the static equilibrium value of $a_E^3$ was proportional to $R^{3/2}$, that is, $R^2/a_E^3 = c\sqrt{R}$. Their data can be used to determine the constant $c$; using this in Eq. (21) gives:

$$K_R = K - 1.76 \times 10^{11}\sqrt{R} \tag{30}$$

This indicates that for glass spheres on a silicon substrate, $K_R$ depends on $R$. A typical value of $K_R$, for $R = 11.7$ μm, is given in the Table.

As above, information about $K_A$ and $R_A$ can be learned from the experimental data. Equation (24) indicates that $K_A/R_A^2 = 1.03\times$

[4] In fact, this raises the question about whether $K_R$ should be constant. The equation from the linear analogy shows that the reduced stiffness can depend on position of the load and so it is not inconceivable that $K_R$ may depend on $R$ for spheres.

TABLE Typical values of reduced stiffness

| Dupré energy $w$, $J/m^2$ | Substrate material | Glass microspheres Contact radius $a_E$, $\mu m$ | Particle radius $R$, $\mu m$ | Hertzian stiffness $K$, $N/m^2$ | Reduced stiffness $K_R$, $N/m^2$ |
|---|---|---|---|---|---|
| 0.17 | soft[1] | 5.5 | – | $6.8 \times 10^6$ | $5.9 \times 10^6$ |
| 0.17 | firm[1] | 5.5 | – | $74.1 \times 10^6$ | $73.2 \times 10^6$ |
| 0.62 | Si | 0.5 | 11.7 | $64.0 \times 10^9$ | $62.8 \times 10^9$ |

[1] polyurethane.

$10^{12}/R^{3/2}$. These results indicate that not only can $K_R$ vary with particle size but $K_A$ and $R_A$ can as well the equivalent stiffness and radius.

## DISCUSSION/CONCLUSIONS

Fundamental differences exist between the derivation of the attachment model developed above and the JKR model. In both derivations an approach is followed that views adhesion being introduced to the microparticle-substrate system in the presence of an external compressive force. However, in the JKR approach, the potential energy of the system is minimized to determine the static equilibrium conditions. Here, the work over a cycle is equated to the work done by the adhesion force. In the JKR derivation, a linear relationship between the external force and deflection is used during the introduction of adhesion (path $BD$, Fig. 3) whereas here the curve is determined by a simultaneous variation in the stiffness parameter, $K$, and deformation, $\delta$, according to Hertzian theory (since, along this path, contact of spherical bodies is sustained). The JKR theory predicts a pull-off (removal) force independent of the material properties. Without additional study, the reduced stiffness theory does not cover separation or detachment.

The analysis and modeling of contact adhesion is inextricably tied to Hertzian theory, because of theory's simplicity (and consequent convenience to the analyst) and its suitability to applications (local spherical body contact geometry). An externally applied force together with adhesion displays a harmony between the forces, acting in unison.

When the applied force compresses a microsphere against a surface, the contact area increases and the adhesion force increases; when the applied force eases the particle away from the surface, the contact area is reduced and the adhesion force reduces. This is the essence of the interaction of an applied force and adhesion force through the deformation process of the microsphere. To a compressive force the presence of adhesion makes a sphere seem more flexible than it actually is – establishing the need for using a reduced stiffness when studying elastic behavior of microspheres. The JKR theory has a *built-in* reduced stiffness. But when it is nondimensionalized (see Eq. (30)) it loses this property and does not follow experimental data (see Fig. 4). Calculation of reduced stiffness can require either experimental measurement of the contact radius and the normal deflection, the ability to calculate these quantities or some combination. Or it can be done using finite element methods.

If it can be assumed that the analysis of reduced stiffness presented above is applicable to uniformly-distributed body forces, then its use with dynamic problems such as impact and with static problems including body forces such as gravity is appropriate. If so, the use of the reduced stiffness should permit more accurate simulation of microsphere adhesion in combination with gravity (when particle diameters become large enough for gravity to be significant) and when rebound and attachment occur for particle impact. In fact, the impact problem provided motivation for the development of the reduced stiffness theory.

Removal or separation of a particle from a substrate using an applied force is another area where the effects of simultaneous loading must be considered. This topic needs to be examined and such research is ongoing.

The above work does not deal with the details of the beginning and end of contact, when $\delta$ and $a$ are near zero. It certainly is possible that particles may not begin or end contact with a zero contact radius, $a = 0$, exactly when $\delta = 0$. Johnson and Pollock [2] discuss the phenomenon of snap-on and snap-off where the adhesion force causes the proximate surfaces of the microsphere and substrate to "reach out" into or from contact. The exact nature of this phenomenon is likely to be material dependent; little theory and few experimental results are available. In fact, an answer to these questions may even demand a more rigorous definition of the meaning of *contact*. The relationship of reduced stiffness to this problem remains to be examined.

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## NOTATION

*Variables*

- $a$ contact radius, m
- E Young's modulus (modulus of elasticity), N/m$^2$
- $F$ force
- $f$ mathematical function
- $I$ second area moment of cantilever beam cross section, m$^4$
- $K$ Hertzian stiffness parameter, defined in Eq. (2) N/m$^2$
- $K_R^*$ stiffness constant corresponding to JKR theory
- $K_b$ cantilever beam stiffness parameter, N/m
- $k$ beam stiffness (force per unit deflection), N/m
- $\ell$ length of cantilever beam, m
- $m$ mass of microsphere, kg
- $P$ single equivalent, external point force, N
- $Q$ nondimensional single equivalent, external point force
- $R$ effective radius, defined in Eq. (10), m
- $r_i$ radius of body $i$ in contact region, m
- $W$ work, J
- $W_A$ work of surface adhesion force, J
- $w$ Dupré surface energy constant (also, specific work of adhesion), J/m$^2$
- $\alpha$ nondimensional contact radius
- $\partial$ nondimensional mass center deflection normal to contact surface

$\delta$ relative mass center deflection normal to contact suface, m
$\nu_i$ Poisson's ratio for material of body $i$
$\xi$ nondimensional constant defined in Figure 1

*Subscripts*

$A$ conditions of adhesion (alone)
$B$ point on Hertzian process curve $P(K)$, external force alone
$b$ cantilever beam
$C, D$ point on Hertzian process curve $P(K_R)$, combined adhesion and external force
$E$ equilibrium point, adhesion alone
$O$ origin of force-deflection coordinate system
$R$ used with $K$ to indicate reduced stiffness
$S$ separation

## References

[1] Johnson, K. L., Kendall, K. and Roberts, A. D., "Surface energy and the contact of elastic solids", *Proc. R. Soc. Lond.* A.**324**, 301–313 (1971).
[2] Johnson, K. L. and Pollock, H. M., "The Role of Adhesion in the Impact of Elastic Spheres", *J. Adhesion Sci.Technol.* **8**(11), 1323–1332 (1994).
[3] Derjaguin, B. V., Muller, V. M. and Tuporov, Y. P. T., "Effect of contact deformation on the adhesion of particles", *J. Colloid Interface Sci.* **53**, 314–326 (1975).
[4] Maugis, D., "Adhesion of Spheres: the JKR-DMT transition using a Dugdale model", *J. Colloid Interface Sci.* **150**, 243 (1992).
[5] Johnson, K. L., "Mechanics Modeling of Adhesion of Spherical Surfaces", *Proc., 20th Annual Meeting, Adhesion Soc.*, Hilton Head, SC, USA, pp. 155–156 (1997).
[6] Horn, R. G., Israelachvili, J. N. and Pribac, F., "Measurement of the Deformation and Adhesion of Solids in Contact", *J. of Colloid and Interface Sci.* **115**, 492 (1987).
[7] Timoshenko, S. and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill, New York (1951).
[8] Brach, R. M. and Dunn, P. F., "Macrodynamics of Microparticles", *Aerosol Science and Technology* **23**(1), 51–71 (1995).
[9] Quesnel, D. J., Rimai, D. S., Gady, B. and DeMejo, L. P., "Exploring the JKR Formalism with Finite Element Analysis", *Proc., 21st Annual Meeting Adhesion Society* Savannah, GA, pp. 290–292 (1998).
[10] Maugis, D., "The JKR-DMT Transition in the Presence of a Liquid Meniscus and the Extension of the JKR Theory to Large Contact Radii", In: *Contact Mechanics* Raous, M. *et al.*, Eds. (Plenum Press, New York 1995).
[11] Rimai, D. S. and Busnaina, A. A., "The adhesion and removal of particles from surfaces", *Particulate Sci. and Technol.* **13**, 249–270(1995).
[12] Bowen, R. C., DeMejo, L. P. and Rimai, D. S., "A Method of Determining the Contact Area Between a Particle and Substrate Using Scanning Electron Microscopy", *J. Adhesion* **51**, 191–199 (1995).