On the Momentum Theorem for a Continuous System of Variable Mass

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(Received April 4, 1962)

Theorems of linear momentum are derived for a control system having an arbitrary motion in a continuous velocity field. The resulting equations are valid for a system of variable mass. By allowing the control system to be at rest, the usual equations of fluid mechanics are obtained as a special case; by allowing the control system to move with the fluid, the momentum equations for a system of constant mass are obtained as a special case. In this way, it is possible to discuss a common, but incorrect, conception of the momentum equation for a system of variable mass.

THE derivations of the theorem of linear momentum, which are found in most textbooks, are unnecessarily restrictive in that the results are limited either to systems of constant mass or to control systems[1] fixed in space. Furthermore, these derivations often result in a misconception of the law of momentum for a system of variable mass.

The purpose of the present paper is to derive momentum theorems for a control system having an arbitrary motion in a continuous velocity field. One of the theorems will be derived in terms of the motion of the center of mass of the system under consideration. In this way the correct laws of momentum for a system of variable mass will be obtained and the usual textbook theorems will be included as special cases.

In the development which follows, the very useful Leibnitz theorem (for differentiating a volume integral) will be used frequently. This theorem is

$$\frac{d}{dt}\int_\sigma f(\mathbf{r},t)d\sigma=\int_\sigma \frac{\partial f(\mathbf{r},t)}{\partial t}d\sigma+\int_s f(\mathbf{r},t)\mathbf{v}\cdot\mathbf{n}ds, \quad (1)$$

where f is in general a tensor function, σ is a control volume, s is the surface of σ, $\mathbf{n}$ is a unit outward normal vector at ds, $\mathbf{v}$ is the local velocity of s at point ds, $\mathbf{r}$ is the position vector, and t is time.

In order to illustrate the physical picture, assume f is a vector field. (See Fig. 1.) It is important to understand that the control volume does not necessarily move in the direction of the field $\mathbf{f}$, although, as a special case it might be allowed to do so. Another special case of interest is one in which the control volume σ is at rest ($\mathbf{v}=0$).

Suppose that f is the mass density field ρ of a fluid in motion. Then Eq. (1) has direct application in fluid mechanics. The mass M of the fluid which instantaneously occupies σ is then given by

$$M=\int_\sigma \rho d\sigma, \quad (2)$$

and from Eq. (1)

$$\frac{dM}{dt}=\frac{d}{dt}\int_\sigma \rho d\sigma=\int_\sigma \frac{\partial\rho}{\partial t}d\sigma+\int_s \rho\mathbf{v}\cdot\mathbf{n}ds. \quad (3)$$

Let $\mathbf{u}$ be the velocity of the fluid. The divergence theorem for the mass flux vector $\rho\mathbf{u}$ states that

$$\int_s \rho\mathbf{u}\cdot\mathbf{n}ds=\int_\sigma \nabla\cdot\rho\mathbf{u}d\sigma. \quad (4)$$

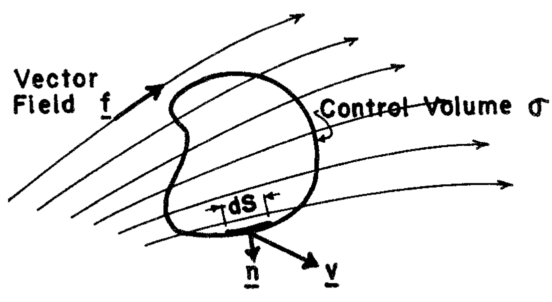

FIG. 1. Moving control surface σ in vector field $\mathbf{f}$. Underlined letters in figure correspond to boldface letters in text.

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[1] A control system (or control surface) is a hypothetical closed envelope, of arbitrary shape, through which fluxes of momentum and energy may be observed.

On adding Eqs. (3) and (4) we obtain

$$\frac{dM}{dt}=\int_\sigma\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{u}\right)d\sigma+\int_s\rho(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \quad (5)$$

Let the local velocity of the control surface be equal to the local fluid velocity ($\mathbf{v}=\mathbf{u}$). Then no mass can cross the boundaries of σ and it becomes a closed system of constant mass M (a fluid particle). Under these circumstances

$$\int_\sigma[(\partial\rho/\partial t)+\nabla\cdot\rho\mathbf{u}]d\sigma=0. \quad (6)$$

Now note that the first integral on the right side of Eq. (5) is independent of the control surface velocity $\mathbf{v}$ and in the second integral, the velocity $\mathbf{v}$ is arbitrary. This means that Eq. (6) is a general result. Since the size of the control volume σ is arbitrary, it follows that

$$(\partial\rho/\partial t)+\nabla\cdot\rho\mathbf{u}=0, \quad (7)$$

and Eq. (5) reduces to

$$dM/dt=\int_s\rho(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \quad (8)$$

Equation (8) is an integral equation of continuity for a control volume in motion in the fluid velocity field. For the special case in which the control volume is at rest ($\mathbf{v}=0$) it reduces to

$$dM/dt=-\int_s\rho\mathbf{u}\cdot\mathbf{n}ds. \quad (9)$$

Next, let $\mathbf{f}$ equal the momentum density of the fluid $\rho\mathbf{u}$. The momentum $\mathbf{P}$ of the fluid which instantaneously occupies σ is then given by

$$\mathbf{P}=\int_\sigma\rho\mathbf{u}d\sigma, \quad (10)$$

and from Eq. (1)

$$\frac{d\mathbf{P}}{dt}=\frac{d}{dt}\int_\sigma\rho\mathbf{u}d\sigma=\int_\sigma\frac{\partial\rho\mathbf{u}}{\partial t}d\sigma+\int_s\rho\mathbf{u}\mathbf{v}\cdot\mathbf{n}ds. \quad (11)$$

The divergence theorem for the momentum flux tensor $\rho\mathbf{uu}$ states that

$$\int_\sigma\rho\mathbf{uu}\cdot\mathbf{n}ds=\int_\sigma\nabla\cdot\rho\mathbf{uu}d\sigma. \quad (12)$$

On adding Eqs. (11) and (12), we obtain

$$\frac{d\mathbf{P}}{dt}=\int_\sigma\left(\frac{\partial\rho\mathbf{u}}{\partial t}+\nabla\cdot\rho\mathbf{uu}\right)d\sigma+\int_s\rho\mathbf{u}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \quad (13)$$

Expanding the integrand of the first integral and applying Eq. (7) reduces Eq. (13) to

$$\frac{d\mathbf{P}}{dt}=\int_\sigma\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)d\sigma+\int_s\rho\mathbf{u}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \quad (14)$$

The acceleration field $\mathbf{a}$ of the fluid is defined by

$$\mathbf{a}=d\mathbf{u}/dt=(\partial\mathbf{u}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{u}. \quad (15)$$

Thus Eq. (14) becomes

$$d\mathbf{P}/dt=\int_\sigma\rho\mathbf{a}d\sigma+\int_s\rho\mathbf{u}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \quad (16)$$

Again, let the local velocity of the control surface be equal to the local fluid velocity ($\mathbf{v}=\mathbf{u}$). The system σ is then a system of constant mass and the time rate of change of its momentum is equal to the external force $\mathbf{F}$ acting on it. Thus it is found that

$$\int_\sigma\rho\mathbf{a}d\sigma=\mathbf{F}. \quad (17)$$

Again, this result is general because the first integral on the right side of Eq. (16) is independent of $\mathbf{v}$; Eq. (17) is a statement of Newton's law of motion for the fluid which instantaneously occupies the control volume σ.

Equation (16) becomes

$$\mathbf{F}=(d\mathbf{P}/dt)+\int_s\rho\mathbf{u}(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}ds. \quad (18)$$

It can be seen that Eq. (18) reduces to the usual law for a system of constant mass M, namely,

$$\mathbf{F}=d\mathbf{P}/dt. \quad (19)$$

Another special case of interest is that in which the control volume σ is at rest ($\mathbf{v}=0$). For this case

$$\mathbf{F}=(d\mathbf{P}/dt)+\int_s\rho\mathbf{uu}\cdot\mathbf{n}ds. \quad (20)$$

If Eq. (11) is substituted into Eq. (18)

$$\mathbf{F}=\int_\sigma\frac{\partial\rho\mathbf{u}}{\partial t}d\sigma+\int_s\rho\mathbf{uu}\cdot\mathbf{n}ds. \quad (21)$$

This last equation is valid regardless of the motion of σ since $\mathbf{v}$ has been eliminated without making any special assumptions about the motion of σ. It is a statement of the instantaneous conditions and is valid if the control volume is fixed ($\mathbf{v}=0$) or if the control volume moves with the fluid ($\mathbf{v}=\mathbf{u}$).

Now the concept of center of mass will be introduced. The position $\mathbf{R}^*$ of the center of mass of the fluid which instantaneously occupies σ is defined by

$$M\mathbf{R}^*=\int_\sigma \rho\mathbf{r}d\sigma. \tag{22}$$

Hence, upon taking the derivative,

$$M\mathbf{V}^*=(d/dt)\int_\sigma \rho\mathbf{r}d\sigma-\mathbf{R}^*(dM/dt), \tag{23}$$

where

$$\mathbf{V}^*=d\mathbf{R}^*/dt. \tag{24}$$

From Eq. (1)

$$\frac{d}{dt}\int_\sigma \rho\mathbf{r}d\sigma=\int_\sigma \frac{\partial\rho\mathbf{r}}{\partial t}d\sigma+\int_s \rho\mathbf{r}\mathbf{v}\cdot\mathbf{n}ds. \tag{25}$$

The divergence theorem for the mass-moment flux tensor $\rho\mathbf{r}\mathbf{u}$ states that

$$\int_s \rho\mathbf{r}\mathbf{u}\cdot\mathbf{n}ds=\int_\sigma \nabla\cdot\rho\mathbf{r}\mathbf{u}d\sigma. \tag{26}$$

On adding Eqs. (25) and (26) we obtain

$$\frac{d}{dt}\int_\sigma \rho\mathbf{r}d\sigma=\int_\sigma\left(\frac{\partial\rho\mathbf{r}}{\partial t}+\nabla\cdot\rho\mathbf{r}\mathbf{u}\right)d\sigma+\int_s \rho\mathbf{r}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \tag{27}$$

Expanding the integrand of the first integral on the right side of Eq. (27) and applying Eq. (7) gives

$$\frac{d}{dt}\int_\sigma \rho\mathbf{r}d\sigma=\int_\sigma\left(\rho\frac{\partial\mathbf{r}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{r}\right)d\sigma+\int_s \rho\mathbf{r}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \tag{28}$$

The velocity field $\mathbf{u}$ of the fluid is identically equal to

$$\mathbf{u}=d\mathbf{r}/dt=(\partial\mathbf{r}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{r}. \tag{29}$$

On substituting into Eq. (28)

$$(d/dt)\int_\sigma \rho\mathbf{r}d\sigma=\int_\sigma \rho\mathbf{u}d\sigma+\int_s \rho\mathbf{r}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \tag{30}$$

The first integral on the right side of Eq. (30) is just the momentum $\mathbf{P}$ of the fluid which instantaneously occupies system σ. [See Eq. (10).] Thus,

$$(d/dt)\int_\sigma \rho\mathbf{r}d\sigma=\mathbf{P}+\int_s \rho\mathbf{r}(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds. \tag{31}$$

Substituting Eqs. (8) and (31) into Eq. (23) gives

$$M\mathbf{V}^*=\mathbf{P}+\int_s \rho(\mathbf{r}-\mathbf{R}^*)(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}ds, \tag{32}$$

from which

$$\frac{d\mathbf{P}}{dt}=\frac{d}{dt}(M\mathbf{V}^*)+\frac{d}{dt}\int_s \rho(\mathbf{r}-\mathbf{R}^*)(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}ds. \tag{33}$$

If the local velocity $\mathbf{v}$ of each point of the surface s of control volume σ is identical with the velocity field of the fluid $\mathbf{u}$, then σ is again a closed system of constant mass M. In this special case Eq. (33) reduces to

$$d\mathbf{P}/dt=(d/dt)(M\mathbf{V}^*)=M\mathbf{A}^*, \tag{34}$$

where

$$\mathbf{A}^*=d\mathbf{V}^*/dt. \tag{35}$$

Upon substituting Eq. (33) into Eq. (18) there is obtained

$$\mathbf{F}=\frac{d}{dt}(M\mathbf{V}^*)+\int_s \rho\mathbf{u}(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}ds+\frac{d}{dt}\int_s \rho(\mathbf{r}-\mathbf{R}^*)(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}ds. \tag{36}$$

Equation (36) is the momentum theorem for a continuous system of variable mass in terms of the motion of the center of mass. This equation shows that in the special case where the mass of the system is constant (in which case $\mathbf{v}=\mathbf{u}$), then

$$\mathbf{F}=(d/dt)(M\mathbf{V}^*). \tag{37}$$

A common conception of Eq. (37) is that it is a valid equation for a system of variable mass and reduces as a special case, when mass is constant, to

$$\mathbf{F}=M\mathbf{A}^*. \tag{38}$$

This conception is not correct. Equation (36) shows that Eq. (37) is valid only if $M=\text{constant}$ in which case it is entirely equivalent to Eq. (38).

The second special case of interest is that in which the control volume σ is at rest ($\mathbf{v}=0$). This gives

$$\mathbf{F}=\frac{d}{dt}(M\mathbf{V}^*)+\int_s \rho\mathbf{u}\mathbf{u}\cdot\mathbf{n}ds+\frac{d}{dt}\int_s \rho(\mathbf{r}-\mathbf{R}^*)\mathbf{u}\cdot\mathbf{n}ds. \tag{39}$$

In addition to the theorems derived above, various other theorems may be derived by using the same methods. For example; to derive the first law of thermodynamics, as it applies for a moving control volume, it is necessary only to let the field function f be equal to the total internal energy density ρe_0.