The air wave surrounding an expanding sphere

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[For summary see p. 292.]

Introduction

When the surface of a sphere vibrates in any assigned manner the spherical sound waves which are propagated outwards can be represented by well-known formulae provided that the motion is such that only small changes in air density occur. When the motion of the spherical surface is radial the velocity potential of the sound wave is

$$\phi = r^{-1} f(r-at), \tag{1}$$

where a is the velocity of sound and r is the radial co-ordinate. The velocity, u, and the excess, $p-p_0$, of pressure over the atmospheric pressure p_0 are

$$u = r^{-2} f(r-at) - r^{-1} f'(r-at), \tag{2}$$

$$p-p_0 = -\rho a r^{-1} f'(r-at). \tag{3}$$

If R is the radius of the sphere which, by its expansion, is producing waves, R is a function of t and the surface condition is

$$\dot{R} = R^{-2} f(R-at) - R^{-1} f'(R-at). \tag{4}$$

Equation (4) is an equation for finding the function f. A simple case in which equation (4) can be solved is when $\dot{R}$ is constant so that the sphere is expanding at a uniform velocity. Taking $t=0$ when $R=0$ the radius at time t can be expressed in the form

$$R = \alpha a t, \tag{5}$$

where α is a non-dimensional constant. The limitation that the changes in density are small implies that equations (1)–(3) are true only where α is small compared with 1.

Writing $w = R - at = (\alpha-1)at$ equation (5) becomes

$$\frac{\alpha-1}{\alpha w} f'(w) - \left(\frac{\alpha-1}{\alpha w}\right)^2 f(w) + a\alpha = 0. \tag{6}$$

The solution of equation (6) which is valid for negative values of w is

$$f(w) = \frac{a\alpha^3}{1-\alpha^2} w^2 + c(-w)^{(\alpha-1)/\alpha}. \tag{7}$$

The constant of integration c must be taken as zero in order that $f(w)$ may vanish when $w = 0$. Hence

$$\phi = \frac{a\alpha^3}{1-\alpha^2}\frac{(r-at)^2}{r}, \tag{8}$$

$$\left.\begin{aligned} u &= \frac{a\alpha^3}{1-\alpha^2}\left(\frac{a^2t^2}{r^2}-1\right), \\ p-p_0 &= 2\rho\frac{a^2\alpha^3}{1-\alpha^2}\left(\frac{at}{r}-1\right). \end{aligned}\right\} \tag{9}$$

If at time $t = 0$, $u = 0$ and $p-p_0 = 0$ everywhere, then at all subsequent times $u = 0$ and $p = p_0$ in the region outside the sphere $r = at$.

It will be seen that both u and $p-p_0$ are constant when r/at is constant, thus points where u and $p-p_0$ have any assigned value are propagated outwards at uniform speeds which are proportional to distance from the centre. Subject to the limitations of the theory of sound therefore* the air wave produced by a uniformly expanding sphere expands at a uniform rate and the velocity and pressure at corresponding points are constant at all stages of the expansion.

This result might have been expected *a priori* but the solution is here given in detail because it forms the starting point of the work which follows.

Analysis when velocity of expansion is not small

It seems likely that a uniformly expanding sphere will be surrounded by a uniformly expanding air wave, accordingly a solution of the complete equations of motion is sought in which u and p are functions of $x = r/t$ only. For such motions

$$\left(\frac{\partial}{\partial t}+\frac{r}{t}\frac{\partial}{\partial r}\right)(u, p, \rho) = 0. \tag{10}$$

The equation of motion is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{11}$$

and in view of equation (10) this may be written

$$(u-x)\frac{du}{dx} = -\frac{1}{\rho}\frac{dp}{dx}. \tag{12}$$

* It is shown later that the sound wave equations themselves are not valid in this case, even when α is small, but this fact does not invalidate the expression (8) regarded as a solution of those equations.

The equation of continuity is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)=0,$$

which in view of equation (10) may be written

$$\frac{u-x}{\rho}\frac{d\rho}{dx}+\frac{du}{dx}+\frac{2u}{x}. \tag{13}$$

The gas equation $p\rho^{-\gamma}=$ constant, together with the expression for the velocity of sound, namely $c^2=\dfrac{dp}{d\rho}=\dfrac{\gamma p}{\rho}$, give

$$\left.\begin{aligned}\frac{1}{\rho}\frac{dp}{dx}&=\frac{1}{\gamma-1}\frac{dc^2}{dx},\\ \text{and}\qquad \frac{1}{\rho}\frac{d\rho}{dx}&=\frac{1}{(\gamma-1)c^2}\frac{dc^2}{dx}.\end{aligned}\right\} \tag{14}$$

Substituting from equations (14) in equations (12) and (13)

$$\frac{dc^2}{dx}=-(\gamma-1)(u-x)\frac{du}{dx}, \tag{15}$$

$$\frac{u-x}{(\gamma-1)c^2}\frac{dc^2}{dx}+\frac{du}{dx}+\frac{2u}{x}=0. \tag{16}$$

For convenience in calculation equation (16) may be replaced by

$$\frac{du}{dx}=-\frac{2u}{x}\left\{1-\left(\frac{u-x}{c}\right)^2\right\}^{-1}. \tag{17}$$

Equations (15) and (17) may be expressed in non-dimensional form by substituting the variables

$$\left.\begin{aligned}\xi&=u/x,\\ \eta&=c^2/x^2,\\ z&=\log_e x.\end{aligned}\right\} \tag{18}$$

The resulting equations are

$$\frac{d\eta}{d\xi}=\frac{2\eta}{\xi}\frac{\eta+(\gamma+1)\xi-\gamma\xi^2-1}{3\eta-(1-\xi)^2}, \tag{19}$$

$$\frac{dz}{d\xi}=-\frac{1}{\xi}\frac{\eta-(1-\xi)^2}{3\eta-(1-\xi)^2}. \tag{20}$$

The solution of equation (19), which contains two variables only, will contain one arbitrary constant. Without attempting to express this solution in mathematical form it is possible to construct by numerical integration a complete set of (ξ, η) relationships each corresponding with a given value of the arbitrary constant and to set them out graphically in a single set of curves on a diagram whose co-ordinates are ξ and η. This diagram is shown in figure 1. The arbitrary constant α is defined so as to correspond with the constant α in equations (5), (8) and (9), and the value of α corresponding with each (ξ, η) curve is shown in figure 1. The single curve which cuts across all the graphical solutions of equation (19) in figure 1 will be explained later.

Boundary condition at the surface of the sphere

At the surface of the expanding sphere $u = r/t = x$ so that

$$\xi = 1. \tag{21}$$

In the sound-wave solution the constant α specifies the velocity of radial expansion of the sphere as a fraction of the velocity of sound. In the complete solution η represents c^2/x^2 at any point so that at the surface of the sphere η represents $\left(\frac{\text{local velocity of sound}}{\text{velocity of expansion}}\right)^2$. Correspondence between the complete solution and the sound-wave solution is therefore attained when the arbitrary constant α is defined by the relation

$$\eta_2 = \alpha^{-2}, \tag{22}$$

where η_2 is the value of η at $\xi = 1$.

The curves in figure 1 were constructed for a series of values of α starting at the point $(\xi = 1, \eta = \alpha^{-2})$ and calculating the change in η step by step for small decrements $\delta\xi$ in ξ through the range $\xi = 1$ to $\xi = 0$. The change δz in z in each interval $\delta\xi$ was also calculated using equation (20). When η has been found as a function of ξ the solution of equation (20) is of the form

$$z - z_2 = (\text{function of } \xi),$$

where z_2 is the constant of integration. If z_2 is chosen so that $z = z_2$ when $\xi = 1$ then

$$\Sigma \delta z = z - z_2 = \log_e r/R. \tag{23}$$

Outer boundary conditions

At the outer boundary of the expanding air it must be possible to connect the still air conditions with those which obtain in the disturbed region. This can be done in two possible ways: either (*a*) it might be found that $u = 0$ at radius $r = at$, so that the (ξ, η) curve passes through the point $(\xi = 0, \eta = 1)$; or (*b*) it might be found that at some radius the pressure temperature and velocity are attained which correspond with the pressure, temperature and velocity immediately behind a shock wave moving into still air with velocity r/t. In case (*b*) the expanding region would be bounded by an expanding

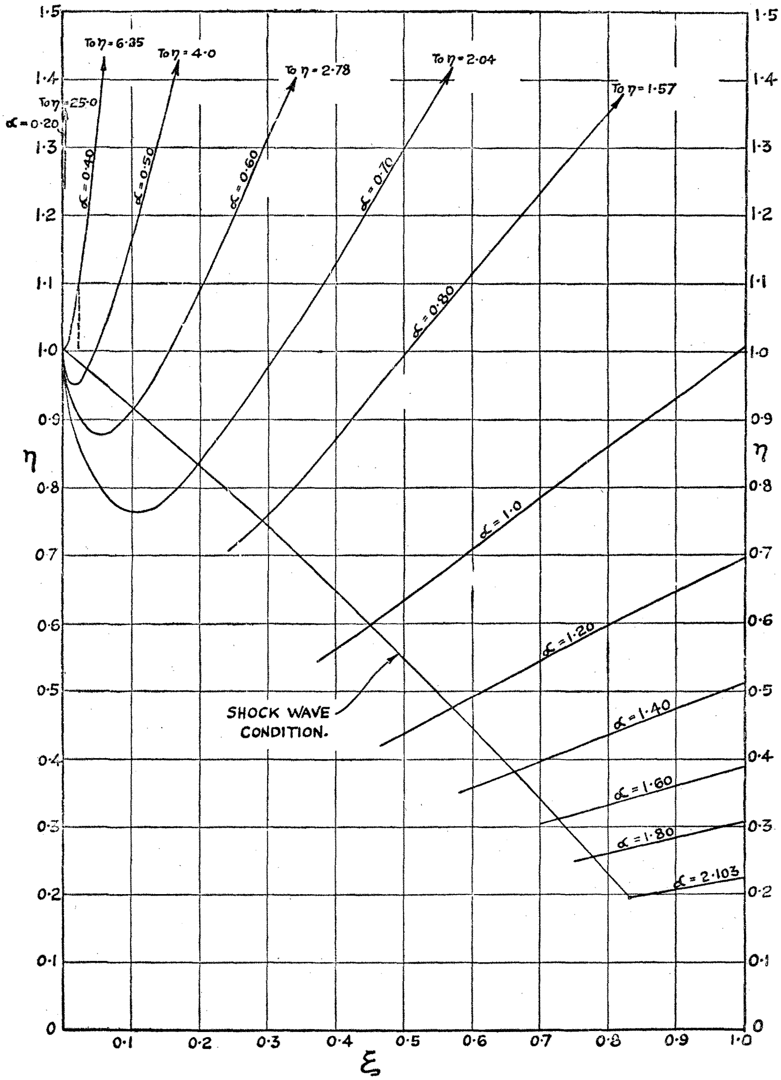

FIGURE 1*

spherical shock wave and the air outside this sphere would be at rest. The sound wave solution (equations (8) and (9)) satisfies condition (*a*), for at $r = at$ both $u = 0$ and $p - p_0 = 0$.

* The area of figure 9 is indicated by a broken line on the right of the area.

For values of α larger than those to which sound wave analysis can be applied it appears that it is not possible to satisfy condition (a). If the solution for $\alpha = 0{\cdot}7$ for instance is followed (see figure 1) for values of ξ decreasing from $\xi = 1$ it is found that z reaches a maximum while ξ is still positive and for smaller values of ξ, z decreases. Thus the same value of z would correspond with two different values of ξ which is physically impossible. It remains to find out whether the alternative condition (b) can be satisfied. For this purpose it is necessary to express the appropriate shock wave conditions in terms of the variables ξ and η.

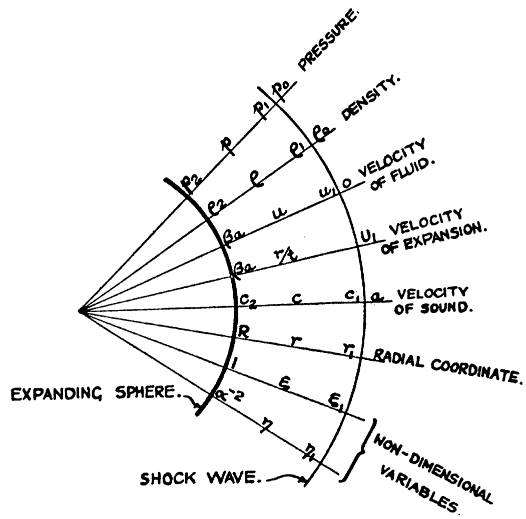

FIGURE 2. Diagram of symbols.

A shock wave is an extremely thin region within which the pressure, density, temperature and velocity change from one set of values to another. The ratios of density and temperature on the two sides of a shock wave depend only on the ratio of the corresponding pressures. If p_1 is the pressure immediately behind a shock wave and p_0 is the atmospheric pressure in front of it, $y = p_1/p_0$ may be regarded as the independent variable in terms of which all other changes occurring at the shock wave may be expressed. Figure 2 shows the positions in the field to which the various symbols apply. The shock wave formulae were first given by Rankine (1870) and later independently by Hugoniot (1889). The ratio of densities on the two sides of the shock wave is

$$\frac{\rho_1}{\rho_0} = \frac{\gamma - 1 + (\gamma + 1)\,y}{\gamma + 1 + (\gamma - 1)\,y}, \tag{24}$$

where γ, the ratio of specific heats, is the same as the "γ" which appears in equation (19). In the present calculations γ is taken as 1·405. Continuity requires that ρ_1/ρ_0 must be equal to the ratio of the velocities of the air on the two sides relative to the shock wave itself. Hence if U_1 is the velocity of the shock wave and u_1 that of the air behind it

$$\frac{U_1 - u_1}{U_1} = \frac{\rho_0}{\rho_1},$$

or, using equation (24),
$$\frac{u_1}{U_1} = \frac{2(y-1)}{\gamma - 1 + (\gamma+1)\,y}. \tag{25}$$

The condition that the shock wave may expand uniformly with the rest of the system is

$$U_1 = r_1/t,$$

where r_1 is the radius of the shock wave.

Hence
$$u_1/U_1 = u_1 t/r = \xi_1,$$

or
$$\xi_1 = \frac{2(y-1)}{\gamma - 1 + (\gamma+1)\,y}. \tag{26}$$

So far as this condition is concerned an appropriate value of y may be chosen and a corresponding possible shock wave found at any point in the field. Equation (26), however, is not the only necessary condition. The velocity of sound in the air behind the shock wave must also satisfy the condition

$$\eta_1 = c_1^2 t^2/r_1^2 = c_1^2/U_1^2, \tag{27}$$

and
$$c_1^2 = \gamma p_1/\rho_1 = a^2 y\left\{\frac{\gamma + 1 + (\gamma - 1)\,y}{\gamma - 1 + (\gamma+1)\,y}\right\}, \tag{28}$$

where a is the velocity of sound in the undisturbed atmosphere. U_1 may be expressed in terms of a making use of the momentum equation

$$\rho_0 U_1 u_1 = p_1 - p_0 = (y-1)\,p_0, \tag{29}$$

substituting a^2/γ for p_0/ρ_0 and u_1/U_1 from (25)

$$\frac{a^2}{U_1^2} = \frac{2\gamma}{\gamma - 1 + (\gamma+1)\,y}, \tag{30}$$

hence from equations (27), (28) and (30)

$$\eta_1 = \frac{2\gamma y\{\gamma + 1 + (\gamma - 1)\,y\}}{\{\gamma - 1 + (\gamma + 1)\,y\}^2}. \tag{31}$$

Values of ξ and η have been calculated from equations (26) and (31) for a sequence of values of y. These are plotted in a curve in figure 1. The intersections of this curve with those which describe the flow in the expanding air determine the values of ξ and η, and hence the value of y, corresponding with possible shock waves. These have been taken from figure 1 and are given in cols. 2 and 3 of table 1 for values of α ranging from 0·5 to 2·1. The corresponding values of r_1/R obtained by numerical integration of equation (20) are also given in col. 4 of table 1. Corresponding values of y are given in col. 5 of table 1.

TABLE 1

1	2	3	4	5	6	7	8
α	ξ_1	η_1	r_1/R	y	β	y'	p_2/p_0
0	—	—	—	—	—	—	—
0·2	—	—	4·93	1·000	0·203	0·928	1·075
0·4	0·0021	0·998	2·44	1·003	0·410	0·775	1·295
0·5	0·033	0·974	1·950	1·050	0·523	0·750	1·400
0·6	0·103	0·916	1·763	1·169	0·638	0·749	1·569
0·7	0·198	0·833	1·503	1·365	0·761	0·755	1·808
0·8	0·291	0·749	1·392	1·629	0·891	0·774	2·105
1·0	0·453	0·597	1·256	2·400	1·180	0·811	2·959
1·2	0·575	0·474	1·182	3·59	1·520	0·847	4·250
1·4	0·662	0·382	1·135	5·60	1·953	0·887	6·32
1·6	0·727	0·313	1·103	9·06	2·560	0·917	9·89
1·8	0·779	0·256	1·083	17·95	3·598	0·92	19·7
2·1	0·832	0·197	1·060	∞	∞	0·93	∞

REDUCTION TO MORE FAMILIAR FORMS

The physical cause of the motion of the expanding air being the motion of the sphere, the results are more comprehensible when expressed in terms of the ratio

$$\beta = \frac{\text{velocity of expanding spherical surface}}{\text{velocity of sound in undisturbed atmosphere}} = \frac{U_2}{a}, \tag{32}$$

rather than in terms of α. Since $U_2 t/U_1 t$ is R/r_1

$$\beta = \frac{U_2}{U_1}\frac{U_1}{a} = \frac{R}{r_1}\sqrt{\left(\frac{\gamma-1+(\gamma+1)\,y}{2\gamma}\right)}, \tag{33}$$

values of β are given in col. 6 of table 1.

The ratio
$$y' = \frac{\text{pressure behind shock wave}}{\text{pressure at surface of sphere}} = \frac{p_1}{p_2}, \tag{34}$$

is related to c_1/c_2 by the equation

$$y'^{(1-\gamma)/\gamma} = c_2^2/c_1^2. \tag{35}$$

Values of y' are given in col. 7 of table 1. By definition of η

$$\frac{c_2^2}{c_1^2} = \frac{\eta_2 R^2}{\eta_1 r_1^2} = \frac{R^2}{r_1^2 \alpha^2 \eta_1},$$

so that
$$\frac{p_2}{p_0} = \frac{y}{y'} = y\left(\frac{R^2}{r_1^2\alpha^2\eta_1}\right)^{\gamma/(\gamma-1)}. \tag{36}$$

Values of p_2/p_0 namely the pressure at the surface of the sphere expressed in atmospheres, are given in col. 8 of table 1. The way in which p_1/p_0 and p_2/p_0 vary with β is shown in figure 3.

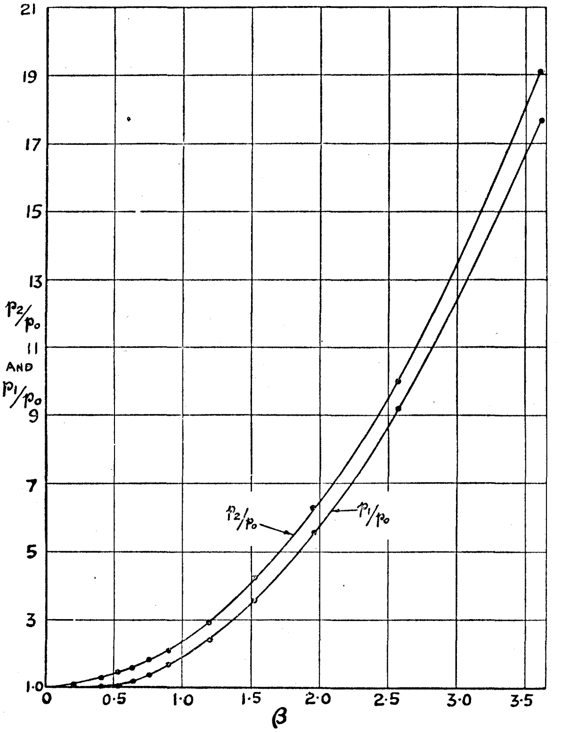

FIGURE 3. Pressure p_2 at sphere and p_1 behind shock wave as multiples of atmospheric pressure p_0.

LIMITING VALUES FOR VERY HIGH RATES OF EXPANSION

The limiting values of ξ_1 and η_1 when $y \to \infty$ are

$$\left.\begin{aligned} \xi &= \frac{2}{\gamma+1} = 0{\cdot}8316, \\ \eta &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}. \end{aligned}\right\} \tag{37}$$

Starting from these values equations (19) and (20) were integrated numerically for increasing values of ξ. At $\xi = 1$ the value of η so found was 0·226 corresponding with $\alpha = (0{\cdot}226)^{-2} = 2{\cdot}103$, and the value of r_1/R was 1·0602. These are given in the last line of table 1. It seems therefore that as the velocity of expansion becomes infinitely great the pressure at the surface becomes infinite, but the density remains finite. The thickness of the layer of expanding air is never less than 6·0 % of the radius of the sphere.

Variation of velocity with radius

The numerical solutions of equations (19) and (20) give η and r/R in terms of ξ. The most convenient variables for describing velocity distribution are u/a and r/at which are connected with ξ, η and r/R by the relations

$$\left.\begin{aligned} u/a &= \beta\xi r/R, \\ r/at &= \beta r/R. \end{aligned}\right\} \tag{38}$$

Some calculated values of u/a and r/at are given in table 2. These velocity distributions are shown in figure 4 for values of β ranging from 0·20 to 1·95.

Table 2

$\alpha = 0{\cdot}2$, $\beta = 0{\cdot}203$

r/at	u/a	p/p_0
0·203	0·203	1·0752
0·214	0·182	1·0745
0·228	0·159	1·0727
0·253	0·127	1·0671
0·300	0·090	1·0571
0·374	0·356	1·0431
0·425	0·042	1·0362
0·594	0·018	1·0196
0·766	0·008	1·0087
1·000	0·000	1·0000

$\alpha = 0{\cdot}4$, $\beta = 0{\cdot}410$

r/at	u/a	p/p_0
0·410	0·410	1·295
0·430	0·369	1·293
0·451	0·334	1·286
0·471	0·303	1·280
0·512	0·211	1·263
0·614	0·162	1·213
0·697	0·113	1·173
0·799	0·069	1·122
0·901	0·035	1·068
0·984		1·015
1·000		1·003

$\alpha = 0{\cdot}5$, $\beta = 0{\cdot}523$

r/at	u/a	p/p_0
0·523	0·523	1·400
0·544	0·481	1·397
0·564	0·444	1·391
0·586	0·411	1·386
0·627	0·353	1·363
0·669	0·304	1·338
0·711	0·262	1·310
0·774	0·209	1·265
0·836	0·162	1·219
0·900	0·120	1·171
0·940	0·093	1·137
0·983	0·065	1·100
1·017	0·031	1·050

$\alpha = 0{\cdot}6$, $\beta = 0{\cdot}638$

r/at	u/a	p/p_0
0·638	0·638	1·569
0·660	0·597	1·566
0·723	0·489	1·539
0·787	0·405	1·494
0·850	0·332	1·437
0·936	0·249	1·349
0·978	0·209	1·300
1·020	0·167	1·245
1·042	0·145	1·186
1·067	0·114	1·169

$\alpha = 0{\cdot}7$, $\beta = 0{\cdot}761$

r/at	u/a	p/p_0
0·761	0·761	1·808
0·782	0·717	
0·826	0·640	1·786
0·890	0·541	1·736
0·934	0·484	1·692
1·000	0·404	1·612
1·043	0·353	1·550
1·087	0·302	1·480
1·109	0·275	1·442
1·130	0·240	1·399
1·145	0·225	1·365

$\alpha = 0{\cdot}8$, $\beta = 0{\cdot}891$

r/at	u/a	p/p_0
0·891	0·891	2·105
0·935	0·805	2·096
0·980	0·729	2·067
1·025	0·662	2·022
1·068	0·598	1·965
1·114	0·537	1·898
1·158	0·478	1·820
1·203	0·417	1·733
1·225	0·384	1·677
1·242	0·357	1·630

TABLE 2 (*continued*)

$\alpha = 1{\cdot}0$, $\beta = 1{\cdot}180$

r/at	u/a	p/p_0
1·180	1·180	2·959
1·227	1·088	2·939
1·274	1·004	2·889
1·321	0·927	2·822
1·368	0·853	2·731
1·415	0·779	2·621
1·463	0·704	2·485
1·482	0·670	2·413

$\alpha = 1{\cdot}2$, $\beta = 1{\cdot}520$

r/at	u/a	p/p_0
1·520	1·520	4·250
1·570	1·421	4·231
1·620	1·330	4·169
1·671	1·233	4·067
1·722	1·157	3·927
1·772	1·071	3·747
1·800	1·029	3·636

$\alpha \doteq 1{\cdot}4$, $\beta = 1{\cdot}953$

r/at	u/a	p/p_0
1·953	1·953	6·317
2·010	1·843	6·286
2·065	1·742	6·191
2·120	1·643	6·033
2·175	1·544	5·811
2·215	1·470	5·607

$\alpha = 1{\cdot}6$, $\beta = 2{\cdot}560$

r/at	u/a	p/p_0
2·560	2·560	9·89
2·603	2·474	9·87
2·649	2·385	9·82
2·696	2·291	9·72
2·750	2·198	9·50
2·824	2·050	9·07

$\alpha = 1{\cdot}8$, $\beta = 3{\cdot}60$

r/at	u/a	p/p_0
3·60	3·60	19·7
3·66	3·47	19·7
3·73	3·35	19·5
3·79	3·22	19·0
3·86	3·09	18·3
3·90	3·03	17·9

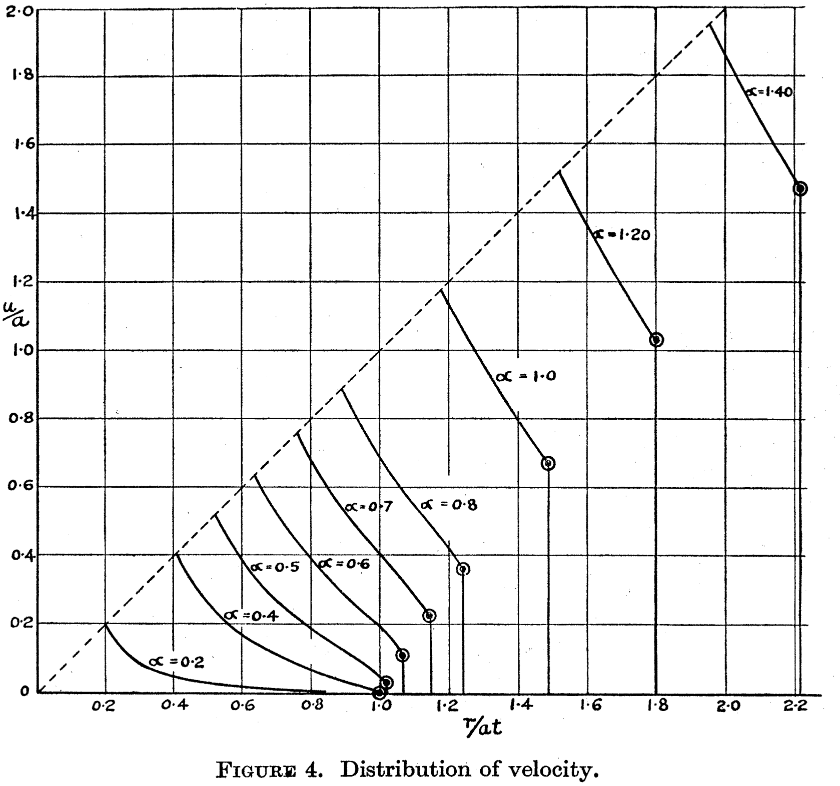

FIGURE 4. Distribution of velocity.

The sudden jump in velocity which occurs at the shock wave is represented in each case by a vertical line and the subsequent increase from the shock wave to the sphere by a sloping curve behind it. The points corresponding with the surface of the expanding sphere lie on the line $u = r/t$ because this is the condition which must be satisfied at the sphere. For high rates of expansion the velocity distribution is practically linear. When the thickness of the layer of expanding air is small compared with the radius of the sphere $\frac{1}{t}\frac{du}{dr}$ does not differ appreciably from its value at the spherical surface which is, according to equation (17), equal to -2. The mean slope of the velocity distribution curve for $\beta = 1{\cdot}95$ is in fact found from figure 4 to be $\tan^{-1}(-1{\cdot}8)$. The mean slopes for $\beta = 2{\cdot}56$ and $3{\cdot}598$ which are outside the range of figure 4 are still closer to the approximate value $\tan^{-1}(-2)$.

The velocity distribution is shown on a larger scale in figure 5 for the case when $\alpha = 0{\cdot}7$, $\beta = 0{\cdot}761$. The calculated points are marked in figure 5. The calculation has been carried beyond the point where the shock wave occurs and the corresponding part of the velocity distribution curve is marked in figure 5 with a broken line. It will be seen that in the virtual part of the curve the velocity for a given radius is no longer single-valued.

Pressure distribution

The pressure p at any point is given by

$$\frac{p}{p_0} = \frac{y}{y'}\left(\frac{\eta r^2 \alpha^2}{R^2}\right)^{\gamma/(\gamma-1)}. \tag{39}$$

Values of p/p_0 for a selection of values of r/at are given in table 2. The pressure distributions for $\alpha = 0{\cdot}2$, $0{\cdot}4$, $0{\cdot}5$, $0{\cdot}6$, $0{\cdot}7$, $0{\cdot}8$, $1{\cdot}0$ and $1{\cdot}2$ are shown in figure 6. It will be noticed that those for $\alpha = 1{\cdot}0$ and $1{\cdot}2$ appear to be nearly parabolic. It can be shown in fact that the distribution is parabolic near the sphere so that when the thickness of the layer of expanding air is small compared with the radius of the sphere the distribution is nearly parabolic through the layer. If

$$s = (r-R)/r \quad \text{so that} \quad x = U_2(1+s), \tag{40}$$

and s is supposed small the approximate linear distribution of velocity is

$$u = U_2(1-2s),$$

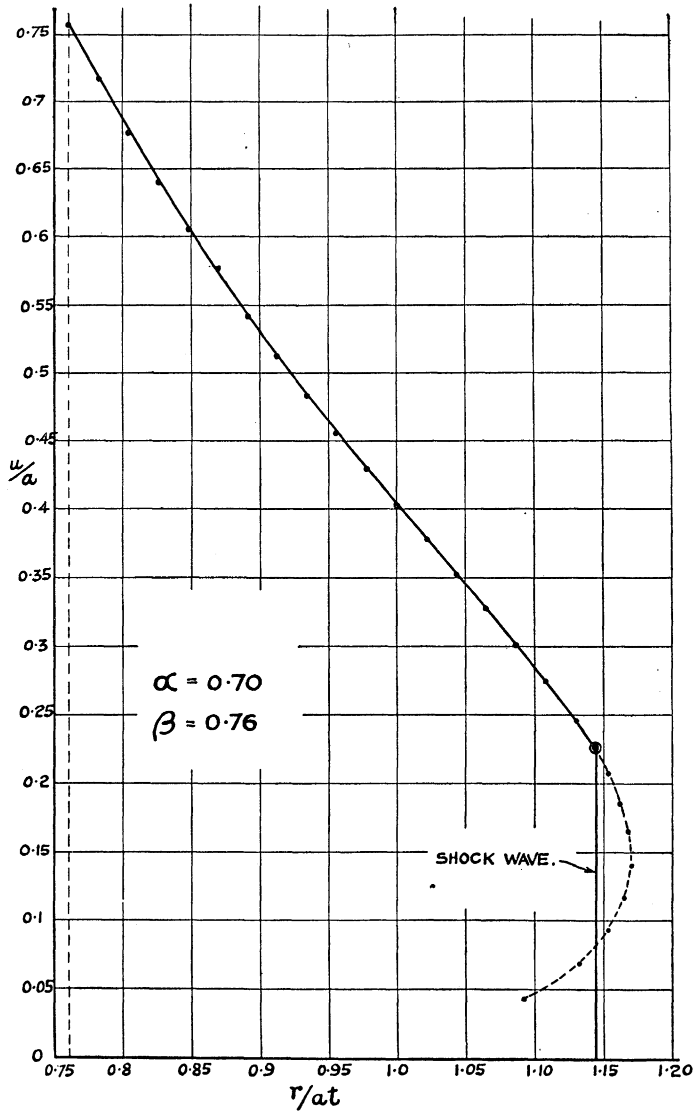

FIGURE 5. Distribution of velocity for $\alpha = 0{\cdot}7$.

equation (15) therefore takes the form

$$\frac{dc^2}{ds} = -6(\gamma-1)\,U_2^2 s,$$

so that $$c_2^2 - c^2 = 3(\gamma-1)\,U_2^2 s^2 \quad \text{or} \quad 1-\frac{c^2}{c_2^2} = 3(\gamma-1)\,\alpha^2 s^2. \tag{41}$$

If s is small
$$\frac{p_2-p}{p_2} = \frac{\gamma}{\gamma-1}\frac{c_2^2-c^2}{c_2^2}$$

so that
$$1-\frac{p}{p_2} = 3\gamma\alpha^2 s^2 = 3\gamma\alpha^2\left(\frac{r-R}{R}\right)^2, \tag{42}$$

which represents a parabolic pressure distribution.

As an example of the application of the approximate expression (42) the values of y' and α corresponding with infinite rate of expansion may be

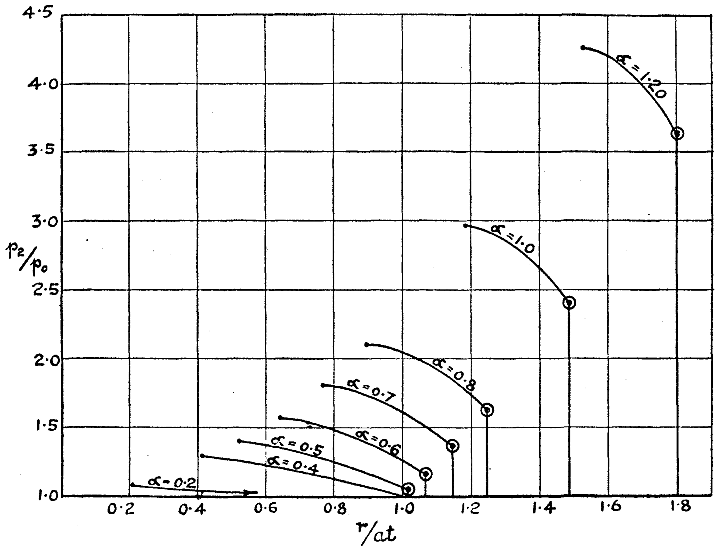

FIGURE 6. Distribution of pressure.

calculated. In this case $\xi_1 = 0{\cdot}8316$, $\eta_1 = 0{\cdot}1968$ (see equation (37)) so that s_1, the value of s at the shock wave is given by

$$\xi_1 = ut/r = 1-3s_1 = 0{\cdot}8316 \quad \text{and} \quad \eta_1 = c^2t^2/r^2 = \alpha^{-2}(1-2s_1).$$

Hence $s_1 = 0{\cdot}056$, so that

$$r_1/R = 1{\cdot}056 \quad \text{and} \quad \eta_1 = 0{\cdot}1968 = \alpha^{-2}\{1-2(0{\cdot}056)\},$$

so that $\alpha = 2{\cdot}12$. The approximate value of y' is $1-3\gamma\alpha^2 s^2$ or $0{\cdot}94$. These values may be compared with those given in table 1 which were found by numerical integration of the full equations, namely, $r_1/R = 1{\cdot}060$, $\alpha = 2{\cdot}103$, $y' = 0{\cdot}93$.

Comparison with sound wave solution for low rates of expansion

For small values of α the sound wave solution of equations (8) and (9) may be expected to afford a good approximation to the true motion. The

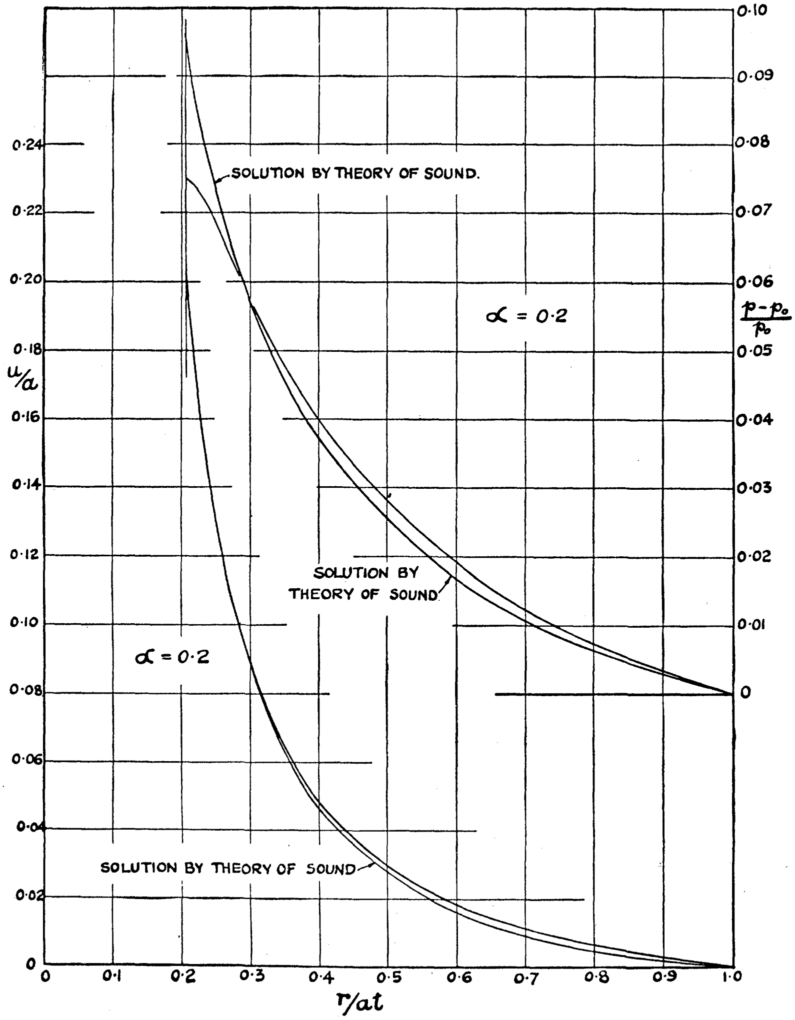

Figure 7. Comparison between velocity and pressure distributions calculated by the theory of sound and by the complete equations.

velocity and pressure distributions calculated from the complete equations and from the approximate equations of the theory of sound are compared in figure 7 for the case $\alpha = 0{\cdot}2$. So far as the velocity distribution is concerned the agreement is good but the pressure distributions are distinctly different near the sphere. The true pressure distribution is in fact parabolic near the sphere and initially $dp/dr = 0$. According to the sound wave equation (9) the value of dp/dr close to the sphere is $-\frac{2\rho\alpha^2}{1-\alpha^2}\left(\frac{a^2}{R}\right)$. The reason for the discrepancy is evidently that it is not justifiable to apply the equations of the theory of sound in the neighbourhood of the sphere owing to the neglect of the term $u\,\partial u/\partial r$ in the equation of motion in comparison with $\partial u/\partial t$. In the correct equation $u\,\partial u/\partial x$ is equal to $-\partial u/\partial t$ at the sphere.

Apart from this difference at the inner boundary the chief contrast between the sound wave solution and the true solution for values of α greater than 0·5 lies in the fact that the former involves no shock wave at the outer boundary. The true solution for $\alpha = 0{\cdot}2$ appears in figure 7 to resemble the sound wave solution in this respect. If this resemblance were true then some limiting value of α would exist below which no shock wave would be produced. Assuming that such a limit exists the author's rough attempts to determine its value placed it between $\alpha = 0{\cdot}4$ and $\alpha = 0{\cdot}5$. The matter was, however, examined later by Dr J. W. Maccoll using more accurate methods of numerical solution, and he came to the conclusion not only that a shock wave is formed when α is less than 0·4 but that no lower limit of the assumed type would be found. Subsequent analysis shows that this prediction is correct.

Form of solution near ($\xi = 0$, $\eta = 1$)

Near the point ($\xi = 0$, $\eta = 1$) equation (19) takes the form

$$\frac{d\zeta}{d\xi} = \frac{\zeta + (\gamma+1)\xi}{\xi}, \tag{43}$$

where $\zeta = \eta - 1$. The solution of equation (43) is

$$\zeta = (\gamma+1)\,\xi \log (A\xi), \tag{44}$$

where A is the constant of integration. As $\xi \to 0$, $\zeta \to 0$ but is negative when $\xi < 1/A$. Multiplying both sides of (44) by A it will be seen that $A\zeta$ is a function of $A\xi$ so that the shape of the (ξ, ζ) curve does not depend on the constant of integration though its scale is proportional to $1/A$.

In the neighbourhood of ($\xi_1 = 0$, $\eta_1 = 1$) the relationship between ξ_1 and η_1 at a shock wave is found by taking $y - 1$ as small in equations (26) and (31).

Hence

$$\operatorname*{Lt}_{\xi_1 \to 0} \frac{\eta_1 - 1}{\xi_1} = -\frac{3-\gamma}{2} = -0{\cdot}797. \tag{45}$$

Comparing equations (45) with (44) the shock wave condition is satisfied if

$$\log_e A\xi = -\frac{3-\gamma}{2(\gamma+1)} = -0{\cdot}3316. \tag{46}$$

Hence

$$\left.\begin{aligned} A\xi &= +0{\cdot}718, \\ A\zeta &= A(\eta-1) = -0{\cdot}572. \end{aligned}\right\} \tag{47}$$

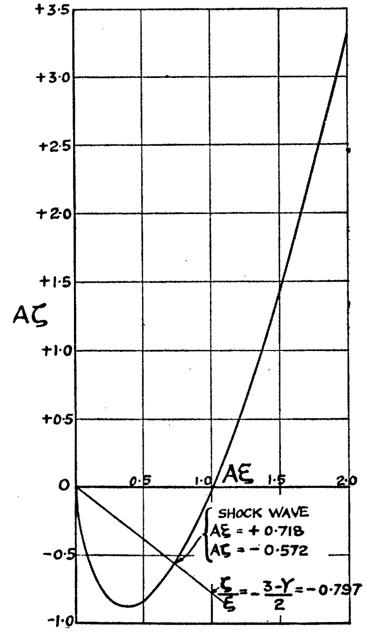

FIGURE 8

Figure 8 shows the form of the solution near ($\xi = 0$, $\eta = 1$) and the shock wave line intersecting the (ξ, η) curve at the point ($+0{\cdot}718$, $-0{\cdot}572$). It may be concluded that provided the numerical solution brings the ($\xi\eta$) curve into the neighbourhood of the point ($\xi = 0$, $\eta = 1$) the expanding air must be bounded by a shock wave. This indeed proves to be the case, but, as will be seen later, the shock wave is of very small intensity when α is less than 0·5.

FITTING THE NUMERICAL SOLUTION TO THE SOLUTION VALID NEAR $\xi = 0$

Writing $\zeta = \eta - 1$ equation (19) becomes

$$\frac{d\zeta}{d\xi} = \frac{2(1+\zeta)\,\zeta + (\gamma+1)\,\xi - \gamma\xi^2}{\xi} \frac{}{2+3\zeta+2\xi-\xi^2}. \tag{48}$$

In the approximate solution (44) ζ/ξ decreases as ξ decreases. In fitting the approximate solution to the numerical solution a pair of values of ξ and ζ may be taken and the value of A found by inserting these in equation (44). To find the error committed in using equation (43) instead of (48) suppose that the solutions are joined where $\zeta/\xi = B$ the true value of $d\zeta/d\xi$ at this point is

$$\frac{d\zeta}{d\xi} = \frac{(1+B\xi)(B+\gamma+1-\gamma\xi)}{2+(3B+2)\xi-\xi^2},$$

expanding this expression in powers of ξ the first two terms are

$$\left[\frac{d\zeta}{d\xi}\right]_B = (B+\gamma+1)\left\{1-\left(1+\tfrac{1}{2}B+\frac{\gamma}{B+\gamma+1}\right)\right\}.$$

The value of $[d\zeta/d\xi]_B$ in the approximate solution is $B+\gamma+1$. The proportional error is therefore

$$\epsilon = \left(1+\tfrac{1}{2}B+\frac{\gamma}{B+\gamma+1}\right)\xi$$

and since in the approximate solution

$$B = (\gamma+1)\log_e(A\xi),$$

the error in $d\zeta/d\xi$ is given by

$$A\epsilon = \left(1+\tfrac{1}{2}B+\frac{\gamma}{B+\gamma+1}\right)e^{(\gamma+1)B}. \tag{49}$$

Values of $A\epsilon$ for a series of values of B are given in table 3.

TABLE 3

B	−1	0	1	2	3	4	5	6	7	10
$A\epsilon$	0·96	1·58	2·9	5·5	9·6	16·9	29·4	50·2	84·4	387·0

$\alpha = 0{\cdot}4$. When $\alpha = 0{\cdot}4$ values of ξ and ζ given in lines 1 and 2 of table 4 were calculated numerically. Values of A calculated from equation (44) are given in line 3, and the proportional error calculated from equation (49) in line 5. The error ϵ is less than 5 % in the first three points.

TABLE 4. $\alpha = 0{\cdot}4$

ξ	0·0081	0·0122	0·160	0·0231	0·0306	0·0384	0·0446	0·0536	0·0651
ζ	0·020	0·044	0·065	0·115	0·170	0·226	0·287	0·351	—
A	342	367	337	342	327	301	325	248	227
B	2·46	3·61	4·0	5·3	5·5	5·9	6·4	6·3	6·5
ϵ	0·021	0·038	0·046	0·105	0·12	0·16	0·19	0·24	0·29

Taking $A = 340$ the (ξ, η) curve shown in figure 9 was calculated. The first three values of η, namely, those corresponding with $\xi = 0{\cdot}0081$, $0{\cdot}0122$ and $0{\cdot}160$ are shown in figure 9. It will be seen that though the curve calculated using equation (44) passes very nearly through these three points the existence of a portion of the $(\xi\eta)$ curve for which ζ is negative would not have been suspected from simple inspection of the apparent trend of the curve calculated step by step through 99 % of the range $\xi = 1$ to 0. This point may perhaps be appreciated more clearly if the area covered by figure 9 is compared with the same area (marked with broken line) on the much smaller scale of figure 1.

It appears from equation (47) that when $\alpha = 0{\cdot}4$ the expanding air is bounded by a shock wave for which

$$\xi_1 = 0{\cdot}718/340 = 0{\cdot}0021,$$
$$\eta_1 - 1 = -0{\cdot}572/340 = -0{\cdot}0017.$$

The corresponding change in pressure at the shock wave is $y - 1 = \gamma\xi_1 = 0{\cdot}003$ of an atmosphere.

$\alpha = 0{\cdot}5$. When $\alpha = 0{\cdot}5$ the four lowest values of η calculated numerically give $A = 24$, 21, 21 and 21. Taking $A = 21$ the shock wave corresponds with

$$\xi_1 = 0{\cdot}718/21 = 0{\cdot}034,$$
$$\eta_1 - 1 = -0{\cdot}572/21 = -0{\cdot}027,$$

so that $$y - 1 = 0{\cdot}034\gamma = 0{\cdot}048,$$

which agrees well with the value $y = 1{\cdot}05$ determined graphically by means of figure 1.

$\alpha = 0{\cdot}2$. When $\alpha = 0{\cdot}2$ the lowest values of ζ calculated numerically are

ξ	0·0002	0·0003	0·0005
ζ	+0·0175	+0·026	+0·043

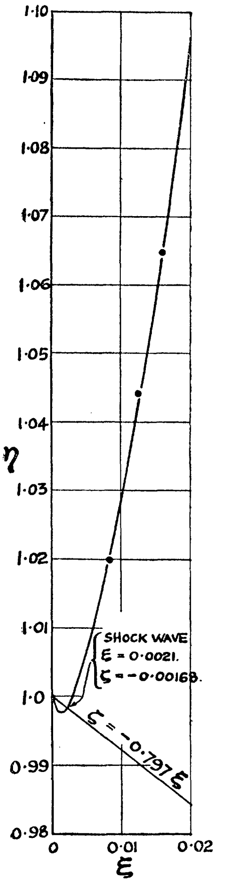

FIGURE 9. ξ, η curve near $\xi = 0$, $\eta = 1$, when $\alpha = 0{\cdot}4$.

When inserted in equation (44) these give values of ξ and η at the shock wave of order 10^{-19}. So far as the equations of a non-viscous fluid are concerned this shock wave seems to be real enough in spite of its extreme smallness but from the physical point of view such a minute shock wave has no meaning. The effect of viscosity and conductivity would in fact become appreciable long before a shock wave with pressure change 10^{-19} atm. could be formed. Nevertheless it is curious that there is this definite mathematical difference between a wave of finite intensity and the equivalent sound wave. It is especially curious that the point where the solution of the complete equation differs from the approximate solution of the theory of sound is in the region of very small velocities and pressure changes, the region in fact where the theory of sound might be expected to be most accurate.

In conclusion I wish to express my thanks to Mrs H. Glauert who carried out some of the calculations and to Dr J. W. MacColl for some valuable suggestions.

Summary

The only case in which the motion of a gas at high speed in three dimensions has so far been discussed mathematically is that of the disturbance produced by a cone moving with velocity greater than that of sound. In the present work another case is analysed, namely, the radial outward flow produced by a uniformly expanding sphere. The region of expanding air is bounded by a shock wave outside which the air is undisturbed. As the radial velocity of the sphere increases the thickness of the layer of disturbed air decreases till at infinite rate of expansion it is only 6 % of the radius of the sphere. The distributions of velocity and pressure are given for a range of rates of expansion. When the radial velocity of the sphere is small an approximate analysis based on the theory of sound yields results which are inaccurate near the sphere and also at the shock wave which forms the outer boundary of the expanding air.

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