Homework 3 solutions.

Evaluate the following indefinite integrals:

a) $A = \int (x^3 - x^2) e^x \, dx = \int x^3 e^x dx - \int x^2 e^x dx$

Integration by parts for the first part:

$u = x^3 \qquad v = e^x$

$du = 3x^2 dx \qquad dv = e^x dx$

$A = \left(x^3 e^x - 3 \int x^2 e^x dx \right) - \int x^2 e^x dx$

$= x^3 e^x - 4 \int x^2 e^x dx.$

Int. by parts: $\left\{ \begin{array}{ll} u = x^2 & v = e^x \\ du = 2x\,dx & dv = e^x dx \end{array} \right.$

$A = x^3 e^x - 4\left(x^2 e^x - 2 \int x e^x dx \right)$

Int. by parts: $\left\{ \begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x dx \end{array} \right.$

$A = x^3 e^x - 4\left(x^2 e^x - 2\left(x e^x - \int e^x dx \right) \right)$

$= x^3 e^x - 4x^2 e^x + 8x e^x - 8 e^x + c.$

b) $\int \cos 3z \cos 7z \, dz = \frac{1}{2} \int \left(\cos(7+3)z + \cos(7-3)z \right) dz$

$= \frac{1}{2} \left(\frac{\sin 10z}{10} + \frac{\sin 4z}{4} \right) + c.$

c) $C = \int x \sin^2 x \, dx = \int x \left(\frac{1 - \cos 2x}{2} \right) dx = \int \frac{x}{2} dx - \int \frac{x \cos 2x}{2} dx$

$= \frac{x^2}{4} - \frac{1}{2} \int \frac{x \cos 2x}{\cancel{2x}} dx.$

Int. by parts: $u = x \qquad v = \frac{\sin 2x}{2}$

$du = dx \qquad dv = \cos 2x \, dx$

$$C = \frac{x^2}{4} - \frac{1}{2}\left(\frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right)$$

$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{1}{8}\cos 2x + \text{constant.}$$

d) $D = \int x e^{4x} dx$

Int. by parts: $u = x$ $\quad v = \frac{1}{4}e^{4x}$

$du = dx$ $\quad dv = e^{4x} dx$

$$D = \cancel{\int x} \ \frac{1}{4} x e^{4x} - \frac{1}{4}\int e^{4x} dx = \frac{1}{4} x e^{4x} - \frac{e^{4x}}{16} + c$$

e) $E = \int \sec^3 x \, dx$

Int. by parts: $u = \sec x$ $\quad v = \tan x$

$du = \sec x \tan x \, dx$ $\quad dv = \sec^2 x \, dx$

$$\int \sec^3 x dx = E = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x \, dx \right)$$

$$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x| \right) + c.$$

f) $F = \int e^{-y} \sin 2y \, dy$

Int. by parts: $u = e^{-y}$ $\quad v = \frac{-\cos 2y}{2}$

$du = -e^{-y} dy$ $\quad dv = \sin 2y \, dy$

$$F = \left(-\frac{e^{-y}\cos 2y}{2} - \frac{1}{2}\int e^{-y} \cos 2y \, dy \right)$$

Int. by parts: $u = e^{-y}$ $\quad v = \frac{\sin 2y}{2}$

$du = -e^{-y}dy \quad dv = \cos 2y\, dy$

$$F = -\frac{e^{-y}\cos 2y}{2} - \frac{1}{2}\left(\frac{1}{2}e^{-y}\sin 2y + \int \frac{e^{-y}\sin 2y}{2}dy + c\right)$$

$$\int e^{-y}\sin 2y\, dy$$

$$\Rightarrow \frac{5}{4}\int e^{-y}\sin 2y\, dy = -\frac{e^{-y}\cos 2y}{2} - \frac{1}{4}e^{-y}\sin 2y + c_1$$

$$\rightarrow \int e^{-y}\sin 2y\, dy = -\frac{2}{5}e^{-y}\cos 2y - \frac{1}{5}e^{-y}\sin 2y + c_2$$

g) $G = \int \cos^3 x \sin^3 x\, dx = \int (1-\sin^2 x)\cos x \sin^3 x\, dx$

Substitution: $u = \sin x$, $du = \cos x\, dx$

$$G = \int (1-u^2)u^3 du = \int (u^3 - u^5)du = \frac{u^4}{4} - \frac{u^6}{6} + c$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$$

h) $H = \int \sqrt{x}\, e^{\sqrt{x}} dx$

Substitution: $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$$H = \int 2x\, e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx = \int 2u^2 e^u du$$

Int. by parts: $v = u^2$ $\quad w = e^u$

$dv = 2u\,du \quad dw = e^u du$

$$H = \cancel{\int} 2\left(u^2 e^u - 2\int u e^u du\right)$$

Int. by parts: $v = u$ $\quad w = e^u$

$dv = du \quad dw = e^u du$

$H = 2\left(u^2e^u - 2\left(ue^u - \int e^u du\right)\right)$

$= 2u^2e^u - 4ue^u + 4e^u + C$

$= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C.$

i) $I = \int \sin^2\theta \cos\theta \, d\theta$

Subs. $u = \sin\theta$, $du = \cos\theta \, d\theta$

$I = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3\theta}{3} + C.$

j) $\int \sqrt{1-\cos 2w}\, dw = \int \sqrt{2\sin^2 w}\, dw = \int \sqrt{2}\, |\sin w|\, dw$

$= \sqrt{2}\,\mathrm{sgn}(\sin w) \cdot (-\cos w) + C$ where $\mathrm{sgn}(\sin w)$ denotes the sign of $\sin w$.

<u>Note</u>: When you evaluate this integral w/ bounds, be careful w/ the constant, and these formulas are only valid on intervals where $\sin w$ doesn't switch signs.

k) $K = \int \frac{1}{x \ln x} dx.$

Subs. $u = \ln x$, $du = \frac{1}{x} dx.$

$K = \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C.$

l) $L = \int \frac{1}{x(\ln x)^2} dx.$

Subs: $u = \ln x$, $du = \frac{1}{x} dx.$

$L = \int \frac{1}{u^2} du = \frac{-1}{u} + C = \frac{-1}{\ln x} + C.$