Homework # 6
Solutions

1. Decide if the following series converge or diverge. If they converge, find their limit.

a) $\sum_{n=1}^{\infty} \frac{\pi}{5^n}$

Geometric series with $a = \pi$, $r = \frac{1}{5}$. Hence $\boxed{\text{converges}}$ to $ar/(1-r) = (\pi/5)/(4/5) = \boxed{\pi/4}$

b) $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$

Geometric series, $a = 1$, $r = \left(\frac{4}{5}\right)$, Hence converges to $ar/(1-r) = (4/5)/(1-4/5) = \boxed{4}$

c) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}$

Telescoping series:

$$S_m = \sum_{n=1}^{m} \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} = \left[\cancel{\frac{1}{\ln(3)}} - \frac{1}{\ln(2)}\right] + \left[\cancel{\frac{1}{\ln(4)}} - \cancel{\frac{1}{\ln(3)}}\right]$$

$$+ \dots + \left[\frac{1}{\ln(m+2)} - \cancel{\frac{1}{\ln(m+1)}}\right] = -\frac{1}{\ln(2)} + \frac{1}{\ln(m+2)}$$

Then $\lim_{m \to \infty} S_m = \boxed{-\frac{1}{\ln(2)}}$, and the series $\boxed{\text{converges}}$ to this value.

d) $\sum_{n=1}^{\infty} \frac{4}{n^2} - \frac{4}{(n+1)^2}$

Telescoping Series:

$S_m = \left[\frac{4}{1^2} - \cancel{\frac{4}{2^2}}\right] + \left[\cancel{\frac{4}{2^2}} - \cancel{\frac{4}{3^2}}\right] + \left[\cancel{\frac{4}{3^2}} - \frac{4}{4^2}\right] + \ldots + \left[\frac{4}{m^2} - \frac{4}{(m+1)^2}\right]$

$= 4 - \frac{4}{(m+1)^2}$

So $\lim_{m\to\infty} S_m = \boxed{4}$, and the series $\boxed{\text{converges}}$ to this value.

e) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

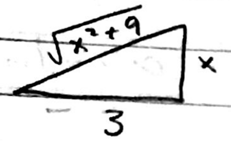
By the n^{th}-term test, as $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)}$

$= \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} = 1$ is not zero, the series $\boxed{\text{diverges}}$.

2. Use the integral test to decide if the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+9}$

We have $\int_1^{\infty} \frac{1}{x^2+9}\,dx$. Doing trig-sub:



$\cos(\theta) = \frac{3}{\sqrt{x^2+9}} \Rightarrow \frac{1}{3}\cos(\theta) = \frac{1}{\sqrt{x^2+9}} \Rightarrow \frac{1}{9}\cos^2\theta = \frac{1}{x^2+9}$

$\tan\theta = \frac{x}{3} \Rightarrow 3\tan\theta = x \Rightarrow 3\sec^2\theta\,d\theta = dx$

So $\rightsquigarrow \int \frac{1}{3}\,d\theta = \frac{1}{3}\theta = \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_1^{\infty}$

$= \pi/6 - \frac{1}{3}\arctan(1/3)$, so the integral converges.

Therefore, by the integral test, the series $\boxed{\text{converges}}$ as well.

~~b)~~ c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

We have $\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx$. Using integration by parts:

$u = \ln(x) \qquad v = -\frac{1}{x}$

$du = \frac{1}{x}\,dx \qquad dv = \frac{1}{x^2}\,dx$

$\rightsquigarrow -\frac{1}{x}\ln(x) + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln(x) - \frac{1}{x}\Big|_1^{\infty}$

$\lim_{x\to\infty} -\frac{\ln(x)}{x} - \cancel{\frac{1}{x}}^{\,0} \overset{LH}{=} \lim_{x\to\infty} -\frac{(1/x)}{1} = 0,$

so the integral $= 0 - (-1) = 1$.

Therefore, as the integral converges, by the integral test, the series $\boxed{\text{converges}}$ as well.

b) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

We have $\int_2^{\infty} \frac{1}{x\ln(x)} dx$ $\quad u = \ln(x)$, $du = \frac{1}{x} dx$

$\rightsquigarrow \int \frac{1}{u} du = \ln(u) = \ln(\ln(x)) \Big|_2^{\infty}$

$= \left[\lim_{x \to \infty} \ln(\ln(x))\right] - \ln(\ln(2)) = \infty - \ln(\ln(2))$

$= \infty$.

Therefore, by the integral test, the series $\boxed{\text{diverges}}$ as well.

3. a) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{\ln(n)}$

$\boxed{\text{diverges}}$

$\lim_{n\to\infty} \frac{2^{n+1}}{\ln(n)} \overset{LH}{=} \lim_{n\to\infty} \frac{2^{n+1}\cdot \ln(2)}{1/n} = \lim_{n\to\infty} n\cdot 2^{n+1}\cdot \ln(2)$

$=\infty$. Therefore, by the n^{th} term test, as $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

b) $\sum_{n=1}^{\infty} \frac{2^n}{1+4^n}$

$\boxed{\text{Converges}}$

By the direct comparison test: $\sum_{n=1}^{\infty} \frac{2^n}{4^n}$ has $b_n \geq a_n$, and is a geometric series with $r = 2/4 < 1$, and hence is convergent. Therefore the original series is convergent.

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+n}}$

$\boxed{\text{converges}}$

For $n \geq 2$, $\frac{(-1)^n}{n^{1+n}} < \frac{1}{n^2}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series,

by the direct comparison test, the original series converges.

d) $\sum_{n=1}^{\infty} \frac{n+1}{n^3-3n^2+4}$

$\boxed{\text{Converges}}$

Near infinity, the terms look like $\frac{n}{n^3} = \frac{1}{n^2}$, so we use the limit comparison test with $b_n = 1/n^2$:

$$\lim_{n\to\infty} \left(\frac{n+1}{n^3-3n^2+4}\right) / (1/n^2) = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-3n^2+4} = 1.$$

As this limit is finite and nonzero, the limit comparison test applies, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the original series converges as well.