Homework 7 solutions

1. a) $\sum_{n=1}^{\infty} \frac{x^n}{2n}$.

Ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/2(n+1)}{x^n/(2n)}\right| = \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot x\right| = |x|<1$$

So $-1 < x < 1$.

Endpoints.

$x=1$: $\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges (p-series)

$x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}$ converges (alt. series test)

So the interval of convergence is $-1 \leq x < 1$.

b) $\sum_{n=0}^{\infty}\frac{(-1)^n n^2 x^n}{2^n}$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^2 x^{n+1}}{2^{n+1}}}{\frac{(-1)^n n^2 x^n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{2n^2}\,x\right| = \left|\frac{x}{2}\right| < 1$$

So $-2 < x < 2$.

End points:

$X=-2$: $\sum_{n=0}^{\infty} \frac{(-1)^n n^2 (-2)^n}{2^n} = \sum_{n=0}^{\infty} n^2$ diverges

(nth term test)

$X=2$: $\sum_{n=0}^{\infty} \frac{(-1)^n n^2 2^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n n^2$ diverges

(nth term test)

Interval: $-2 < x < 2$.

c) $\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}}$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}}{\frac{2^n (x+3)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{2\sqrt{n}}{\sqrt{n+1}} (x+3) \right|$$

$$= \lim_{n\to\infty} \left| \frac{2}{\sqrt{1+\frac{1}{n}}} (x+3) \right| = |2(x+3)| < 1$$

So $-1 < 2(x+3) < 1$

$-\frac{1}{2} < x+3 < \frac{1}{2}$

$-\frac{7}{2} < x < \frac{-5}{2}$

End points:

$x = -\frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

(alt. series test)

$x = \frac{-5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series)

Interval: $-\frac{7}{2} \leq x < \frac{-5}{2}$.

d) $\sum_{n=0}^{\infty} \frac{n!\,(x-7)^n}{2^n}$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(x-7)^{n+1}}{2^{n+1}}}{\frac{n!\,(x-7)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{n}{2}(x-7) \right|$$

$$= \begin{cases} 0 & \text{if } x = 7 \\ \infty & \text{if } x \neq 7 \end{cases}$$

So ~~xxxx~~ the series ~~only~~ converges only when $x = 7$.

2. a) $f(x) = x^3 - x^2 + x - 1$ $\qquad f(0) = -1$

$f'(x) = 3x^2 - 2x + 1$ $\qquad f'(0) = 1$

$f''(x) = 6x - 2$ $\qquad f''(0) = -2$

$f'''(x) = 6$

$f^{(4)}(x) = f^{(5)}(x) = \dots = 0$

$$f(x) = -1 + x - \frac{2}{2!}x^2 + \frac{6}{3!}x^3 = -1 + x - x^2 + x^3$$

b) $f(1) = 0$, $f'(1) = 2$, $f''(1) = 4$, $f'''(1) = 6$.

$$f(x) = 2(x-1) + \frac{4}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$= 2(x-1) + 2(x-1)^2 + (x-1)^3$$

c) $f(x) = \cos(2x^2)$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Replace x by $2x^2$:

$$\cos(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{4n}}{(2n)!}$$

d) $f(x) = \ln(3x+1)$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Replace x by $3x$:

$$\ln(1+3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{n+1}}{n+1}$$

3. a) $f'(x) = 1 - 2x + 3x^2$

b) $f'(x) = 2 + 4(x-1) + 3(x-1)^2$

c) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n (4n) x^{4n-1}}{(2n)!}$

d) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} (n+1) x^n}{n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n 3^{n+1} x^n.$$

4. a) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Replace x by x^2:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

b) Integration term-by-term: (note that the integral of $\frac{1}{1+x^2}$ is $\arctan x$)

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

c) Plug in $x = 1$:

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

So $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$