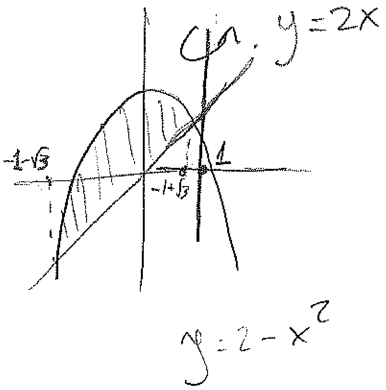
MATH 104 Solution to Practice Problems

#1



$2x = 2 - x^2 \iff x = -1 \pm \sqrt{3} < 1$

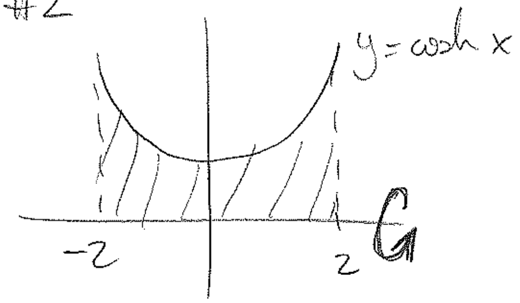
Cyl. Shell Method;

$$V = 2\pi \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} \underbrace{(1-x)}_{\text{[shell rad]}} \cdot \underbrace{(2-x^2-2x)}_{\text{[shell Height]}} \, dx$$

$$= 2\pi \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} x^3 + x^2 - 4x + 2 \, dx$$

$$= 2\pi \left(\frac{x^4}{4} + \frac{x^3}{3} - 2x^2 + 2x \right) \Bigg|_{-1-\sqrt{3}}^{-1+\sqrt{3}} = \cdots = \boxed{16\pi\sqrt{3}}$$

#2



Disk Method;

$$V = \pi \int_{-2}^{2} (\cosh x)^2 dx = \pi \int_{-2}^{2} \frac{(e^x + e^{-x})^2}{4} dx$$

$$= \pi \int_{-2}^{2} \frac{e^{2x} + 2 + e^{-2x}}{4} dx = \frac{\pi}{4} \frac{e^{2x}}{2} \Bigg|_{-2}^{2} + \frac{\pi}{2} \cdot 4 + \frac{\pi}{4} \frac{e^{-2x}}{-2} \Bigg|_{-2}^{2} =$$

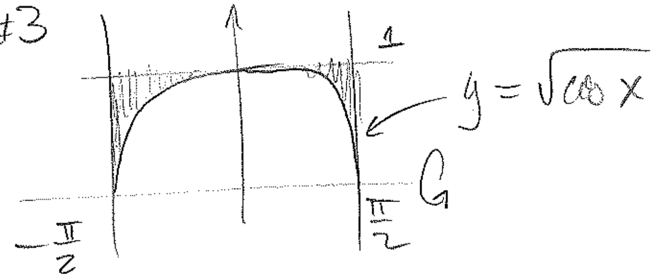
$$= \frac{\pi}{8} \left(e^4 - e^{-4} \right) + 2\pi - \frac{\pi}{8} \left(e^{-4} - e^4 \right) = \boxed{\frac{\pi}{4} \left(e^4 - e^{-4} \right) + 2\pi}$$

Note: Alternatively, note $\cosh^2 x = \frac{1 + \cosh(2x)}{2}$ (prove it!) hence:

$$V = \pi \int_{-2}^{2} \cosh^2 x \, dx = \pi \int_{-2}^{2} \frac{1 + \cosh(2x)}{2} dx = \frac{\pi}{2} \cdot 4 + \pi \frac{\sinh 2x}{4} \Bigg|_{-2}^{2} \underset{\text{sinh odd.}}{=} \boxed{2\pi + \frac{\pi}{2} \sinh(4)}$$

1

#3



Washer Method

$$V = \pi \int_{-\pi/2}^{\pi/2} (1 - \cos x)\, dx$$

$[\text{out}]^2 - [\text{in}]^2$

$$= \pi \left(\pi - \sin x \Big|_{-\pi/2}^{\pi/2} \right) = \pi^2 - \pi\big(1-(-1)\big) = \boxed{\pi^2 - 2\pi}$$

#4 $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}, \quad 1 \le x \le \sqrt{2}$

$$y' = x^3 - \frac{1}{4x^3}$$

$$1 + (y')^2 = 1 + x^6 - \frac{1}{2} + \frac{1}{16x^6}$$

$$= x^6 + \frac{1}{2} + \frac{1}{16x^6}$$

$$= \left(x^3 + \frac{1}{4x^3}\right)^2$$

$$\text{Arclength} = \int_1^{\sqrt{2}} x^3 + \frac{1}{4x^3}\, dx = \left(\frac{x^4}{4} - \frac{1}{8x^2}\right)\Bigg|_1^{\sqrt{2}} = \left(\frac{4}{4} - \frac{1}{16}\right) - \left(\frac{1}{4} - \frac{1}{8}\right)$$

$$\frac{15}{16} - \frac{1}{8} = \boxed{\frac{13}{16}}$$

#5 $y = \dfrac{e^x}{4} + \dfrac{1}{e^x}, \quad 0 \le x \le 1$

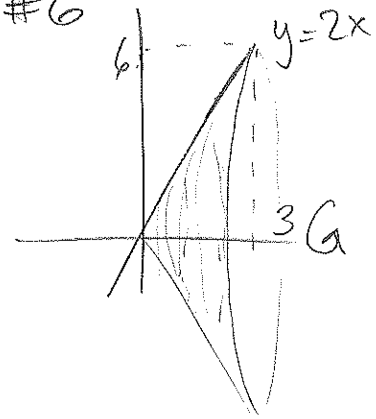
$$y' = \frac{e^x}{4} - \frac{1}{e^x}, \quad 1 + (y')^2 = 1 + \frac{e^{2x}}{16} - \frac{1}{2} + \frac{1}{e^{2x}} = \frac{e^{2x}}{16} + \frac{1}{2} + \frac{1}{e^{2x}}$$

$$= \left(\frac{e^x}{4} + \frac{1}{e^x}\right)^2$$

$$\text{Arclength} = \int_0^1 \frac{e^x}{4} + \frac{1}{e^x}\, dx = \left(\frac{e^x}{4} - \frac{1}{e^x}\right)\Bigg|_0^1 = \left(\frac{e}{4} - \frac{1}{e}\right) - \left(\frac{1}{4} - 1\right)$$

$$= \boxed{\frac{e}{4} - \frac{1}{e} + \frac{3}{4}}$$

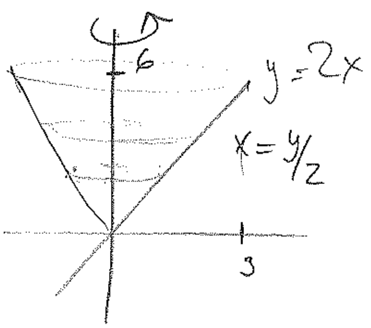
#6



Surface Area

$$S = 2\pi \int_0^3 2x\sqrt{1+4}\, dx = 4\sqrt{5}\,\pi \,\frac{x^2}{2}\Bigg|_0^3 = \boxed{18\sqrt{5}\pi}$$

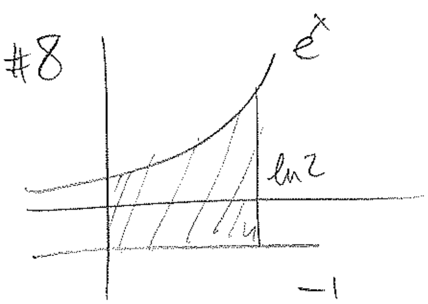
#7



Surface Area

$$S = 2\pi \int_0^6 \frac{y}{2}\sqrt{1+\frac{1}{4}}\, dy = \frac{\pi\sqrt{5}}{2}\,\frac{y^2}{2}\Bigg|_0^6 = \boxed{9\sqrt{5}\pi}$$

#8



$$\bar{x} = \frac{\int_0^{\ln 2} (e^x + 1)x\, dx}{\int_0^{\ln 2} e^x + 1\, dx}, \quad \bar{y} = \frac{\frac{1}{2}\int_0^{\ln 2} e^{2x} - 1\, dx}{\int_0^{\ln 2} e^x + 1\, dx}$$

$$\int_0^{\ln 2} e^x + 1\, dx = (e^x + x)\Big|_0^{\ln 2} = 2 + \ln 2 - 1 = 1 + \ln 2$$

$$\int_0^{\ln 2} xe^x + x\, dx = xe^x\Big|_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx + \frac{x^2}{2}\Bigg|_0^{\ln 2} = 2\ln 2 - 2 + 1 + \frac{(\ln 2)^2}{2}$$

$$= \frac{(\ln 2)^2}{2} + \ln 4 - 1$$

$$\int_0^{\ln 2} e^{2x} - 1 \, dx = \left(\frac{e^{2x}}{2} - x\right)\Big|_0^{\ln 2} = \left(\frac{e^{2\ln 2}}{2} - \ln 2\right) - \frac{1}{2} = 2 - \ln 2 - \frac{1}{2}$$

$$= \frac{3}{2} - \ln 2.$$

So

$$\boxed{\bar{x} = \frac{\frac{(\ln 2)^2}{2} + \ln 4 - 1}{1 + \ln 2}, \quad \bar{y} = \frac{\frac{3}{4} - \frac{\ln 2}{2}}{1 + \ln 2}}.$$

#9

a) $\int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4} + C$

$$\int_0^{\pi/2} \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}\Big|_0^{\pi/2} = 0 + \frac{1}{4} = \boxed{\frac{1}{4}}$$

b) $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$$\left.\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}\right\} = \int u^2 (1 - u^2) \, du = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

$$\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \left(\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x\right)\Big|_0^{\pi/2} = \frac{1}{3} - \frac{1}{5} = \boxed{\frac{2}{15}}$$

c) $\int \sin^3 x \cos^3 x \, dx = \int (1 - \cos^2 x) \sin x \cos^3 x \, dx =$

$$\left.\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}\right\} = -\int (1 - u^2) u^3 \, du = -\int u^3 - u^5 \, du =$$

$$= -\frac{u^4}{4} + \frac{u^6}{6} + C = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$$

$$\int_0^{3\pi/2} \sin^3 x \cos^3 x \, dx = \left(-\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x\right)\Big|_0^{3\pi/2} = -\left(-\frac{1}{4}+\frac{1}{6}\right) = \boxed{\frac{1}{12}}$$

d) $$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \left(1+\tan^2 x\right)\sec^2 x \, dx =$$

$$\left.\begin{array}{l} u = \tan x \\ du = \sec^2 x \end{array}\right\} = \int u^4 (1+u^2)\, du = \int u^4 + u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

$$\int_0^{\pi/6} \tan^4 x \sec^4 x \, dx = \left(\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x\right)\Big|_0^{\pi/6} = \boxed{\frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 + \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7} = \boxed{\frac{26}{945\sqrt{3}}}$$

(this level of simplification is OK!)

#10

a) $$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

b) $$\int e^{-x} \sin x \, dx = -e^{-x}\sin x + \int e^{-x}\cos x \, dx$$

$$= -e^{-x}\sin x - e^{-x}\cos x + \int e^{-x}\sin x \, dx$$

$$\Rightarrow 2\int e^{-x}\sin x \, dx = -e^{-x}(\sin x + \cos x) + C$$

$$\Rightarrow \int e^{-x}\sin x \, dx = -\frac{e^{-x}}{2}(\sin x + \cos x) + C$$

c) $\int x^2 \sin 4x \, dx = x^2\left(-\frac{\cos 4x}{4}\right) + \int \frac{2x \cdot \cos 4x}{4} \, dx$

$$= -\frac{1}{4}x^2 \cos 4x + \frac{1}{2}\int x \cos 4x \, dx$$

$$= -\frac{1}{4}x^2 \cos 4x + \frac{1}{2}\left(x \frac{\sin 4x}{4} - \int \frac{\sin 4x}{4} \, dx\right)$$

$$= -\frac{x^2}{4}\cos 4x + \frac{x}{8}\sin 4x + \frac{1}{8}\frac{\cos 4x}{4} + C$$

$$= \boxed{\left(\frac{1}{32} - \frac{x^2}{4}\right)\cos 4x + \frac{x}{8}\sin 4x + C}$$

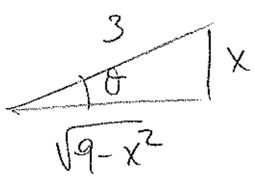
d) $\int \ln(4x) \, dx = x \ln 4x - \int \frac{4x}{4x} \, dx = \boxed{x \ln 4x - x + C}$

#11

a) $\int \frac{x \, dx}{\sqrt{9-x^2}} = \int \frac{3\sin\theta \cdot 3\cos\theta \, d\theta}{3\cos\theta} = 3\int \sin\theta \, d\theta = -3\cos\theta + C$

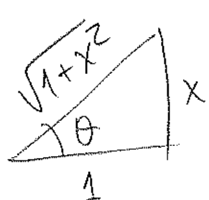
$$\left.\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta \, d\theta \end{aligned}\right\}$$

$$= -3\frac{\sqrt{9-x^2}}{3} + C = \boxed{-\sqrt{9-x^2} + C}$$



b) $\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta \, d\theta}{\sec\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$$x = \tan\theta$$
$$dx = \sec^2\theta \, d\theta$$



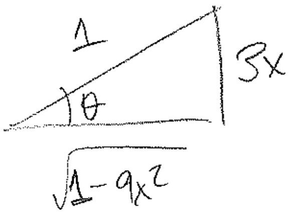
$$= \ln\left|\sqrt{1+x^2} + x\right| + C$$

(Note:
$= \operatorname{arcsinh}(x) + C$)

c) $\int \sqrt{1-9x^2}\,dx = \int \sqrt{1-\sin^2\theta}\,\frac{\cos\theta\,d\theta}{3} = \frac{1}{3}\int \cos^2\theta\,d\theta$

$\left.\begin{aligned} 3x &= \sin\theta \\ 3\,dx &= \cos\theta\,d\theta \end{aligned}\right\}$

$= \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$



$= \frac{1}{6}\left(\arcsin 3x + 3x\cdot\sqrt{1-9x^2}\right) + C$

$= \boxed{\frac{1}{6}\arcsin 3x + \frac{x\sqrt{1-9x^2}}{2} + C}$

d) $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int d\theta = \theta + C$

complete the square

$\left\{\begin{aligned} x-2 &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}\right.$

$= \boxed{\arcsin\left(\frac{x-2}{2}\right) + C}$

12

a) $\frac{1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3)+B(x-1)}{x^2+2x-3}$ $\Rightarrow A(x+3)+B(x-1) = 1$

$\Rightarrow (A+B)x + 3A - B = 1$

$A = -B, \quad 3A - B = 1$

$4A = 1$

$A = 1/4, \quad B = -1/4$

$\int_2^3 \frac{dx}{x^2+2x-3} = \frac{1}{4}\int_2^3 \frac{dx}{x-1} - \frac{1}{4}\int_2^3 \frac{dx}{x+3} = \frac{1}{4}\left(\ln|x-1| - \ln|x+3|\right)\Big|_2^3 =$

$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|\Big|_2^3 = \frac{1}{4}\left(\ln\frac{2}{6} - \ln\frac{1}{5}\right) = \boxed{\frac{1}{4}\ln\frac{5}{3}}$

b) $$\frac{x}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{x^2-2x-3}$$

Set $x=3$; $\quad 3 = 4B \Rightarrow B = 3/4$

Set $x=-1$; $\quad -1 = -4A \Rightarrow A = 1/4$.

$$\int_4^8 \frac{x}{x^2-2x-3}\,dx = \frac{1}{4}\int_4^8 \frac{dx}{x+1} + \frac{3}{4}\int_4^8 \frac{dx}{x-3} = \left(\frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-3|\right)\Bigg|_4^8$$

$$= \left(\frac{1}{4}\ln 9 + \frac{3}{4}\ln 5\right) - \left(\frac{1}{4}\ln 5 + \overset{0}{\cancel{\frac{3}{4}\ln 1}}\right) = \frac{1}{4}\ln\frac{9}{5} + \frac{3}{4}\ln 5 =$$

$$= \boxed{\ln\frac{\sqrt{3}}{5}}$$

c) Long division; $\quad x^3 = (x+1)^2\overbrace{(x-2)}^{\text{quotient}} + \overbrace{3x+2}^{\text{remainder}}$

$$\frac{x^3}{(x+1)^2} = x-2 + \frac{3x+2}{(x+1)^2}$$

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2}$$

$$\Rightarrow A = 3, \quad A+B = 2 \Rightarrow B = -1$$

$$\int_0^1 \frac{3x+2}{(x+1)^2}\,dx = 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left(3\ln|x+1| + \frac{1}{x+1}\right)\Bigg|_0^1 =$$

$$= \ln 8 + \frac{1}{2} - 1 = \ln 8 - \frac{1}{2}$$

$$\int_0^1 \frac{x^3}{(x+1)^2}\,dx = \int_0^1 (x-2)\,dx + \int_0^1 \frac{3x+2}{(x+1)^2}\,dx = -\frac{3}{2} + \ln 8 - \frac{1}{2} = \boxed{-2+\ln 8}$$

d) $$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} = \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$1 = Ax^2+Ax+Bx+B+Cx^2+C = \underbrace{(A+C)}_{=0}x^2+\underbrace{(A+B)}_{=0}x+\underbrace{(B+C)}_{=1}$$

$A=-C$

$A=-B$

$B+C=1 \Rightarrow -2A=1 \Rightarrow \boxed{A=-1/2 \quad B=1/2=C.}$

$$\int_0^1 \frac{dx}{(x^2+1)(x+1)} = -\frac{1}{2}\int_0^1 \frac{x-1}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{dx}{x+1}$$

$$= -\frac{1}{4}\int_0^1 \frac{2x\,dx}{x^2+1} + \frac{1}{2}\int_0^1 \frac{dx}{x^2+1} + \frac{1}{2}\ln|x+1|\Big|_0^1$$

$$= \left(-\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan(x) + \frac{1}{2}\ln|x+1|\right)\Big|_0^1$$

$$= -\frac{1}{4}\ln 2 + \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{2}\ln 2 = \boxed{\frac{\pi}{8} + \frac{\ln 2}{4}}$$