HW 6 Solutions.

1. (a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^\infty x^2 e^{-2x}\,dx$

integral by part

$$= \left(-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right)\Big|_0^\infty$$

$$= \frac{1}{4}.$$

So $c = 4$.

(b) $4\int_0^\infty x\cdot f(x)\,dx = 4\int_0^\infty x^3 e^{-2x}\,dx$

integral by part

$$4\left(-\frac{x^3e^{-2x}}{2} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} + \frac{3}{8}e^{-2x}\right)\Big|_0^\infty$$

$$= \frac{3}{2}.$$

2. Note that exponential probability density function with mean μ is

$$f(x) = \begin{cases} 0, & x<0 \\ \frac{1}{\mu}e^{-\frac{1}{\mu}x}, & x \geq 0. \end{cases}$$

thus we need to compute

$$\left(\int_0^\infty (x-\mu)^2 \mu e^{-\frac{1}{\mu}x}\,dx\right)$$

integral by part

$$\left(-(x-\mu)^2 e^{-\frac{x}{\mu}} - 2\mu(x-\mu)e^{-\frac{x}{\mu}} - 2\mu^2 e^{-\frac{x}{\mu}}\right)\Big|_0^\infty$$

$$= -(-\mu^2 + 2\mu^2 - 2\mu^2) = \mu^2$$

Thus $\sigma = \sqrt{\mu^2} = \mu$

3 (a) $a_n = \frac{3}{2+n^2}$. $\lim_{n\to\infty} a_n = 0$.

(b) $\lim_{n\to\infty} \frac{2n}{5n+7} = \lim_{n\to\infty} \frac{2}{5+\frac{7}{n}} = \frac{2}{5}$

(c) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n}+n}{\frac{2}{n}+10} = +\infty$

(d) $\lim_{n\to\infty} 3^{\frac{1}{n}} = 3^{\lim_{n\to\infty}\frac{1}{n}} = 3^0 = 1$.

(e) $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} n = +\infty$

(f) $\cancel{a}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n}$

$= \lim_{n\to\infty} e^{\frac{\ln 4n}{n}}$

$= e^{\lim_{n\to\infty}\frac{\ln 4n}{n}} = e^0 = 1$.

since $\lim_{n\to\infty} \frac{\ln 4n}{n} \overset{\text{L'Hospital}}{=\!=} \frac{\frac{1}{4n}}{1} = 0$

(g) $0 \leq a_n = \frac{1\times 2 \cdots \times n}{n\times n \cdots \times n} \leq \frac{1}{n} \times 1 \times 1 \cdots \times 1 = \frac{1}{n}$.

Thus $\lim a_n = 0$

(h) By definition $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

So $\lim\left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$.

Or $a_n = e^{\ln a_n} = e^{n\cdot\ln\left(\frac{n}{n+1}\right)} = e^{-n\ln\left(\frac{n+1}{n}\right)}$

$= e^{-n\ln\left(1+\frac{1}{n}\right)}$.

$= e$

$$\lim_{n \to +\infty} n \cdot \ln(1+\frac{1}{n}) = \lim_{n \to \infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = 1$$

by L'Hospital rules.

Thus $\lim a_n = \lim e^{\ln a_n} = e^{-1}$