2 Exponents and logarithms

While the section on exponents and logarithms is "math you already know," there is probably a fair amount of new learning for most of you. However, we will start with some algebraic identities that are purely review: you are expected already to know them and will be handling them by MML only. Please begin by reviewing these identities, which you will find in Sections 1.5 and 1.6 of the textbook. Page 37 in Section 1.5 has identities involving exponents, page 44–45 in Section 1.6 has identities involving logarithms, and your MML homework has a bunch of problems on these.

The notation we will use for logarithms in this class is: $\ln x$ for the natural log (base e); $\lg x$ for the base two logarithm, and $\log_b$ for a log to any other base. I may slip up and use log for ln because this is common in my research area. If I'm on the ball, you shouldn't see log without a base in this class.

The one most basic fact of all about logarithms is that $\log_b$ inverts the function $f(x) = b^x$. Therefore $\log_{10} 10^x = x$, meaning that you can tell something about $\log_{10} x$ just by knowing how many digits x has. If x has n digits then $\log_{10} x$ is between $n - 1$ and n.

2.1 Computation

Knowing just a few approximate values concerning logarithms allows you to do most computations without a calculator. On the next page, therefore, please find your very own Logarithm Cheat Sheet. Surprisingly, many important logs are good to within 1% even when you have only the first nontrivial decimal digit.

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \\ \log_{10}(3) &\approx 0.48 \end{aligned}$$

Some other useful quantities to with 1%:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \end{aligned}$$

(ok so technically $\sqrt{2}$ is about 1.015% greater than 1.4 and 0.7 is about 1.015% less than $\sqrt{1/2}$)

Also useful sometimes: $\sqrt{3} = 1.732\ldots$ and $\sqrt{5} = 2.236\ldots$ both to within about 0.003%.

EXAMPLE: What is the probability of getting all sixes when rolling 10 six-sided dice? It's 1 in 6^{10} but how big is that? If we use base-10 logs, we see that $\log_{10}(6^10) = 10\log_{10} 6 = 10(\log_{10}(2) + \log_{10}(3)) \approx 10(.78) = 7.8$. So the number we're looking for is approximatelhy $10^{7.8}$ which is $10^7 \times 10^{0.8}$ or $10,000,000$ times a shade over $10^{.78}$ which is 6. So we're looking at a little over sixty million to one odds against.

This is not just a random example, it is always the best way to get a quick idea of the size of a large power. When the base is 10 we already know how many digits is has, but when the base is something else, we quickly compute $\log_{10}(b^a) = a \cdot \log_{10}(b)$.

EXAMPLE: Why is the 2.3 on your log cheatsheet so important? It converts back and forth between natural and base-10 logs. Remember, $\log_{10} x = \ln x/\ln 10$. Thus the constant $\ln 10$ is an inmportant conversion constant that just happens to be closer than it looks (the actual value is $2.302\ldots$). So for example,

$$e^8 \approx 10^{8/2.3} \approx 10^{3.5} = 100 \times 10^{0.5} \approx 3,000\,.$$

2.2 Exponential and logarithmic relationships

Much of what you learn on this topic will be a kind of insight into the nature of exponential and logarithmic functions:

- What happens to x when e^x doubles?
- Subtracting 5 from x does what to 2^x?
- If $y = Ax^b$ then how are $\ln x$ and $\ln y$ related?
- If we change the units of measurement, how does this affect the logarithm of the measurement?
- Why would we be more likely to compute the difference of the logarithms of measurements of two quantities than the logarithm of a single measure?

I will show you in a minute how to set up equations to answer this kind of question. But I should point out that some portion of what is to be learned is intuition. For example, take doubling: you know what doubling feels like, you have an intuitive feel that doubling $x + y$ is just like doubling x and y separately and adding them, and this is nearly independent of your knowledge of the distributive law. The knowledge I am trying to covey about exponential and logarithmic relationships will be imparted by two in-class activities (SEP 8 and 10) and a hands-on activity (SEP 11).

When answering a question about functional relationships, set up notation that distinguishes between before and after. For example in the first question above, use x_{before} and x_{after} to denote the two values of x, before and after. If that's too cumbersome then try something like x_0 and x_1. Once you choose the right notation, the problem almost solves itself. The information "e^x doubles" becomes the equation

$$e^{x_1} = 2e^{x_0} \,.$$

We are trying to capture the relation between x_0 and x_1 and it looks as if taking (natural) logs of both sides will get us there or nearly:

$$\ln\left(e^{x_1}\right) = \ln\left(2e^{x_0}\right)$$

simplifies to

$$x_1 = \ln 2 + x_0 \,.$$

So there's your answer: it corresponds to an increase by the (additive) amount of $\ln 2$ or roughly 0.7.

2.3 Orders of growth

Conmparisons at infinity

This topic is not covered in the book, though it has some overlap with Section 7.4. The goal is to make some vague statements more precise. We focus on two in particular. Let f and g be positive functions.

- (i) f is asymptotic to g (synonym: f is nearly identical to g).
- (ii) f is much smaller than g.

Usually these comparisons are meant to take place at infinity. The precise meanings and notations are as follows.

(i) The function f is said to be asymptotic to g at infinity, denoted $f \sim g$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1\,.$$

Note that $\lim g(x)/f(x) = 1/\lim f(x)/g(x)$, so one limit is 1 if and only if the other is. This means that the relation is symmetric: $f \sim g$ if and only if $g \sim f$.

Example: Is $e^x + x$ asymptotic to e^x? Yes, as you can easily verify:

$$\lim_{x\to\infty} \frac{e^x + x}{e^x} = \lim_{x\to\infty} 1 + \frac{x}{e^x} = 1$$

because by L'Hôpital's rule, $\lim_{x\to\infty} x/e^x = 0$. Intuitively, $e^x + x$ is nearly identical to e^x because x is so small compared to e^x that it doesn't make a difference.

(ii) The function f is said to be much smaller than g if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0\,.$$

This is denoted $f \ll g$. Another way to write this is $f = o(g)$. This second notation is more common but very misleading. It seems to claim that f is precisely equal to some variant of g, as it were "g made small". It is really not an equality statement at all. In fact the relation $f \ll g$ is anti-symmetric: it is not possible that simultaneously $f \ll g$ and $g \ll f$.

Example: We have just seen that $x = o(e^x)$. In fact, the function $g(x) + h(x)$ will be asymptotic to $g(x)$ exactly when $h(x) \ll g(x)$. Why? Because $(g(x)+h(x))/g(x)$ and $h(x)/g(x)$ differ by precisely 1, so their limits do, meaning that the limit of $h(x)/g(x)$ is zero if and only if the limit of $(g(x) + h(x))/g(x)$ is one.

Philosophical note: Why are we spending our time making a science out of vague statements? Answer: (1) people really think this way and it clarifies your thinking to make these thoughts precise; (2) a lot of theorems can be stated with these as hypotheses; (3) knowing the science of orders of growth helps to fulfill the Number Sense mandate because you can easily fit an unfamiliar function into the right place in the hierarchy of more familiar functions. You can and should read Section 7.4 of the text, but I don't think the treatment there is adequate. Furthermore, they introduce some notions not commonly used, which we may as well avoid.

Comparisons elsewhere

These same notions may be applied elsewhere simply by taking a limit as $x \to a$ instead of as $x \to \infty$. Usually this is done in order to compare how fast f and g go to either zero or infinity as $x \to a$. At a itself, the ratio of f to g might be $0/0$ or ∞/∞, which of course is meaningless, and can be made precise only by taking a limit as x approaches a. Unfortunately, the notation is not built to reflect whether $a = \infty$ or some other number, and relies on this being understood in context!

Example: Compare $\sqrt{x}$ and $\sqrt[3]{x}$. Is one of these functions much smaller than the other as $x \to 0^+$? We try evaluating the ratio: $f(x)/g(x) = x^{1/2}/x^{1/3} = x^{1/2-1/3} = x^{1/6}$. Therefore,

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} x^{1/6} = 0$$

and indeed $x^{1/2} \ll x^{1/3}$. Intuitively, the square root of x and the cube root of x both go to zero as x goes to zero, but the cube root goes to zero a lot slower (that is, it remains bigger for longer).

The most common way to establish $f \sim g$ or $f \ll g$ is via L'Hôpital's rule. There are some identities. Clearly if $f \ll g$ and $g \ll h$ then $f \ll h$. When adding, you can ignore something much smaller: if $f \ll g$ then $f + g \sim g$. Also, if $f \sim g$ then $cf \sim cg$ when c is any constant. You don't need to memorize a bunch of rules, however, because they follow from familiar facts about limits.

Uses of all this

The most direct use of asymptotic equality is that if $f \sim g$ then either function may be used to estimate the other. For example, because $\sqrt{x^2+1} \sim x$, we may estimate $\sqrt{100^1+1}$ as roughly 100. A second use will be when we study improper integrals. A third use will be when we study convergence of infinite series. A fourth will be when we study long term behavior of solutions to differential equations.

For these reasons, it is good to have an understanding of the relative sizes of common functions. Here are some basic facts.

1. Positive powers all go to infinity but at different rates.
2. Exponentials grow at different rates and every exponential grows faster than every power.
3. Logarithms grow so slowly that any power of $\ln x$ is less than any power of x.

These rules are enough to answer questions such as, "which is greater for sufficiently large x, the function $f(x) = e^x/x$ or the function $g(x) = x^7$? The ratio $f(x)/g(x)$ is equal to e^x/x^8. The exponential grows faster than any power, so this goes to infinity. Flipped, this says that $g(x)/f(x) \to 0$, in other words, $x^7 \ll e^x/x$.

"For sufficiently large x"

Often when discussing comparisons at infinity we use the term "for sufficiently large x". That means that something is true for every value of x greater than some number M (you don't necessarily know what M is). For example, is it true that $f \ll g$ implies $f < g$? No, but it implies $f(x) < g(x)$ for sufficiently large x. Any limit at infinity depends only on what happens for sufficiently large x.

Example: We have seen that $\ln x \ll \sqrt{x-5}$. It is not true that $\ln 6 < \sqrt{6-5}$ (the corresponding values are about 1.8 and 1) and it is certainly not true that $\ln 1 < \sqrt{1-5}$ because the latter is not even defined. But we can be certain that $\ln < \sqrt{x}$ for sufficiently large x. The crossover point is a little over 10.