6 Taylor Polynomials

The textbook covers Taylor polynomials as a part of its treatment of infinite series (Chapter 10). We are spending only a short time on infinite series (the next unit, Unit 7) and will therefore learn Taylor polynomials with a more direct, hands-on approach. Accordingly, the readings in the coursepack will be more central, I will be providing a bit more in terms of lecture, the pre-homework will be relatively short, with extra length in the regular homework devoted to problems that would normally be in the pre-homework.

6.1 Taylor polynomials

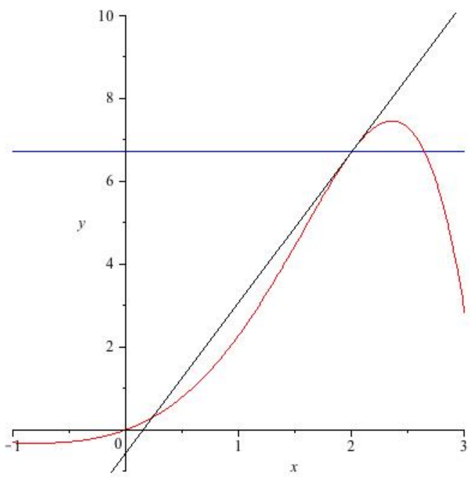
Idea of a Taylor polynomial

Polynomials are simpler than most other functions. This leads to the idea of approximating a complicated function by a polynomial. Taylor realized that this is possible provided there is an "easy" point at which you know how to compute the function and its derivatives. Given a function $f(x)$ and a value a, we will define for each degree n a polynomial $P_n(x)$ which is the "best n^{th} degree polynomial approximation to $f(x)$ near $x = a$."

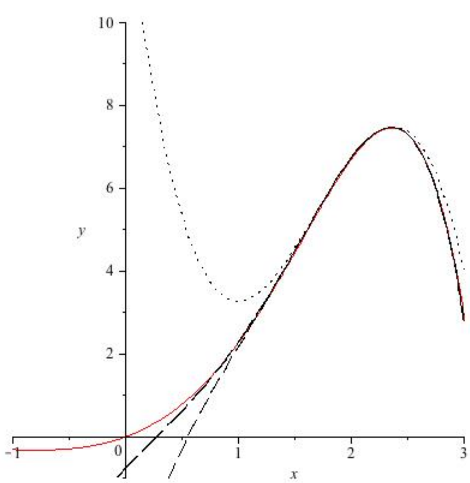
It pays to start very simply. A zero-degree polynomial is a constant. What is the best constant approximation to $f(x)$ near $x = a$? Clearly, the constant $f(a)$. What is the best linear approximation? We already know this, and have given it the notation $L(x)$. It is the tangent line to the graph of $f(x)$ at $x = a$ and its equation is $L(x) = f(a) + f'(a)(x - a)$. So now we know that

$$\begin{aligned} P_0(x) &= f(a) \\ P_1(x) &= f(a) + f'(a)(x - a) \end{aligned}$$

The figure on the next page shows the graph of a function f along with its zeroth and first degree Taylor polynomials at $x = 2$. The zeroth degree polynomial is the flat line and the first degree Taylor polynomial is the tangent line.



Just one more idea is needed to bust this wide open, that is to figure out $P_n(x)$ for all n: the polynomial $P_n(x)$ matches all the derivatives of f at a up to the n^{th} derivative. Check: P_0 matches the zeroth derivative, that is the function value, and P_1 matches the first derivative because both P_1 and f have the same first derivative at a, namely $f'(a)$. The next figure shows $P_3, P_4 and P_5$ at $x = 2$ for the same function, with P_5 shown in long dashes, P_4 in shorter dashes and P_3 in dots. As n grows, notice how P_n beceoms a better approximation and stays close to f for longer.



Taylor's formula

Using what we just said you can solve for what quadratic term is needed to match the second derivative. We used to make students go through this derivation but it took a lot of time and the students did not seem to feel it increased their understanding. Therefore, we will jump straight to the formula.

The definition uses some possibly unfamiliar notation: $f^{(k)}$ refers to the k^{th} derivative of the function f. This is better than f', f'', etc., because we can use it in a formula as k varies. $f^{(0}$ denotes f itself.

> **Definition of Taylor polynomial:** Let a be any real number and let f be a function that can be differentiated at least n times at the point a. The **Taylor polynomial** for f of order n about the point a is the polynomial $P_n(x)$ defined by
>
> $$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Remember to read this sort of thing slowly. Here is roughly the thought process you should go through when seeing this for the first time.

- It looks as if P_n is a polynomial in the variable x with $n+1$ terms.
- When $a = 0$ it's a little simpler:

  $$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k .$$

  The coefficients are the derivatives of f at zero divided by successive factorials.
- Hey, what's zero factorial? Oh, it's defined to be 1. Who knew?
- The degree of $P_n(x)$ will be n unless the coefficient on the highest power $(x-a)^n$ is zero, in which case the degree will be less.

Next you should try a simple example.

EXAMPLE: $f(x) = x$, $n = 3$ and $a = 2$. The value of $f(a)$ is 2 and the first three derivatives of $f(x)$ are constants: 1, 0, 0. Therefore

$$P_3(x) = 2 + 1 \cdot (x-2) + \frac{0}{2!}(x-2)^2 + \frac{0}{3!}(x-2)^3 .$$

In other words, $P_3(x) = x$. Obviously P_4, P_5 and so on will also be x. Maybe this example was too trivial. But it does point out a fact: if f is a polynomial of degree d then the terms of the Taylor polynomial beyond degree d vanish because the derivatives of f vanish. In fact, $P_n(x) = f(x)$ for all $n \geq d$.

EXAMPLE: $f(x) = e^x$, $n = 3$ and $a = 0$. We list the function and its derivatives out to the third one.

k	$f^{(k)}(x)$	$f^{(k)}(a)$	$\frac{f^{(k)}(a)}{k!}(x-a)^k$
0	e^x	1	1
1	e^x	1	x
2	e^x	1	$\frac{x^2}{2}$
3	e^x	1	$\frac{x^3}{6}$

Summing the last column we find that $P_3(x) = 1 + x + x^2/2 + x^3/6$.

EXAMPLE: Let $f(x) = \ln\sqrt{x}$ and expand around $a = 1$. We'll do the first two terms this time.

k	$f^{(k)}(x)$	$f^{(k)}(a)$	$\frac{f^{(k)}(a)}{k!}(x-a)^k$
0	$\ln\sqrt{x}$	0	0
1	$\frac{1}{2x}$	$\frac{1}{2}$	$\frac{1}{2}(x-1)$
2	$\frac{-1}{2x^2}$	$-\frac{1}{2}$	$-\frac{1}{4}(x-1)^2$

Summing the last column we find that $P_2(x) = \frac{x-1}{2} - \frac{(x-1)^2}{4}$.

Tricks for computing Taylor polynomials

You can always compute a Taylor polynomial using the formula. But sometimes the derivatives get messy and you can save time and mistakes by building up from pieces. Taylor polynomials follow the usual rules for addition, multiplication and composition. If f and g have Taylor polynonmials P and Q of order n then $f+g$ has Taylor polynomial $P+Q$. This is easy to see because the derivative is just the sum of the derivatives. Furthermore, the order n Taylor polynomial for fg is $P \cdot Q$ (ignore terms of order higher than n). This is because the product rule for the derivative of fg looks exactly like the rule for multiplying polynomials. I won't present a proof here but you can feel free to use this fact.

EXAMPLE: What is the cubic Taylor polynomial for $e^x \sin x$? The respective cubic Taylor polynomials are $1 + x + x^2/2 + x^3/6$ and $x - x^3/6$. Multiplying these and ignoring terms with a power beyond 3 we get

$$P_3(x) = x\left(1 + x + \frac{x^2}{2}\right) - \frac{x^3}{6} \cdot 1 = x + x^2 + \frac{x^3}{3}.$$

Perhaps the most useful manipulation is composition. I will illustrate this by example. The Taylor polynomial for e^{x^2} is obtained by plugging in x^2 for x in the Taylor polynomial or series for $e^x : 1 + (x^2) + (x^2)^2/2! + \cdots$.

One last trick arises when computing the Taylor series for a function defined as an integral. Suppose $f(x) = \;nt_b^x g(t)\,dt$. Then $f'(x) = g(x)$ so if you know g and its derivatives, you know the derivatives of f. If g has no nice indefinite integral, then you don't know the value of f itself, except at one point, namely $f(b) = 0$. Therefore, a Taylor series at b is the most common choice for a function defined as $\int_b^x$ of another function.

EXAMPLE: Suppose $f(x) = \int_1^x \sqrt{1+t^3}\,dt$. The Taylor series can be computed about the point $a = 1$. From $f'(x) = \sqrt{1+x^3}$, $f''(x) = 3x^2/(2\sqrt{1+x^3})$ we get

$$f(1) = 0, \quad f'(1) = \sqrt{2}, \quad f''(1) = 3/(2\sqrt{2})$$

and therefore $P_2(x) = \sqrt{2}(x-1) + \dfrac{3}{4\sqrt{2}}(x-1)^2$.

Using Taylor polynomials to approximate

In the subsequent sections and in my lectures you will see where Taylor polynomials come from, why they are good approximations to the functions that generate them. You will also see precise statements about how close they are. For now though, we will take this on faith and see how to use them. In case this bothers you I will point out two quick things. (1) The Taylor polynomial of degree 0 is the constant $f(a)$. Surely this is a reasonable, if trivial, approximation to the function $f(x)$ when x is near a. (2) The Taylor polynomial of degree 1 is the linearization $f(a)+f'(a)\cdot(x-a)$. Again, you should already believe that this is a good approximation to $f(x)$ near $x=a$, in fact it is the best possible approximation by a linear function.

Example: What's a good approximation to $e^{0.06}$? A Taylor polynomial at $a=0$ will provide a very accurate estimate with only a few terms. The linear approximation 1.06 is already not bad. The quadratic approximation is

$$1+0.06+(1/2)(0.06)^2 = 1+0.06+0.0018 = 1.0618\,.$$

The true value is 1.0618365... so the quadratic approximation is quite good!

Taylor series are particularly useful in approximating integrals when you can't do the integral. Remember the problem of approximating $\int_0^{1/2}\cos(\pi x^2)\,dx$? It was not se easy to get a good answer with a trapezoidal appoximation. We can do better approximatng cos by a Taylor polynomial around $a=0$. You can directly compute that the first three derivatives are zero, or you can compute P_4 in one easy step like this: for the function $\cos x$, $P_2(x)=1-x^2/2$; now plug in πx^2 for x to get $1-\pi^2x^4/2$. This is P_4. The nice thing about polynomials is that you can always integrate them. In this case,

$$\int_0^{1/2} P_4(x) = \left(x-\frac{\pi^2}{10}x^5\right)\Bigg|_0^{1/2}\,.$$

This comes out to $1/2-\pi^2/320\approx 0.46916$ which is accurate to within 0.001.

6.2 Taylor's theorem with remainder

The central question for today is, how good an approximation to f is P_n? We will give a rough answer and then a more precise one.

Rough answer: $P_n(x) - f(x) \sim c(x-a)^{n+1}$ near $x = a$. For example, the linear approximation P_1 is off from the actual value by a quadratic quantity $c(x-a)^2$. If x differs from a by about 0.1 then $P_1(x)$ will differ from $f(x)$ by something like 0.01. If x agrees with a to four decimal places, then $P_1(x)$ should agree with $f(x)$ to about eight places. Similarly, the quadratic approximation P_2 differs from f by a multiple of $(x-a)^3$, and so on.

You can skip the justification of this answer, but I thought I'd include the derivation for those who want it because it's just an application of L'Hôpital's rule. Once you guess that $P_n(x) - f(x) \sim c(x-a)^n$, you can verify it by starting with the equation

$$\lim_{x \to 0} \frac{f(x) - P_n(x)}{(x-a)^{n+1}} \, ,$$

and repeatedly applying L'Hôpital's rule until the denominator is not zero at $x = a$. Because the derivatives of f and P_n at zero match through order n, it takes at least $n+1$ derivatives to get something nonzero, at which point the denominator has become the nonzero constant $(n+1)!$. The limit is therefore $f^{(n+1)}(a)/(n+1)!$, which may or may not be zero but is surely finite.

We know the Taylor polynomial is an order $(x-a)^{n+1}$ approximation but there is a constant c in the expression which could be huge. What about actual bounds can we obtain on $f(x) - P_n(x)$? These are given by Answer # 2, which is called Taylor's Theorem with Remainder.

> **Taylor's Theorem with Remainder:** Let f be a function with $n+1$ continuous derivatives on and interval $[a, x]$ or $[x, a]$ and let P_n be the order n Taylor polynomial for f about the point a. Then
>
> $$f(x) - P_n(x) = \frac{f^{(n+1)}(u)}{(n+1)!}(x-a)^{n+1}$$
>
> for some u between a and x.

The theorem is telling us that the constant c in the rough answer is equal to $f^{(n+1)}(u)/(n+1)!$ for this unknown u. This is at first a little mysterious and difficult to use, which is why we'll be doing some practice. The exact value of u will depend on a, x, n and f and will not be known. However, it will always be between a and x. This means we can often get bounds. We might know, for example, that $f^{(n+1)}$ is always positive on $[a, x]$ and is greatest at a, which would lead to

$$P_n(x) \leq x \leq P_n(x) + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}.$$

EXAMPLE: Let $f(x) = e^{-x}$, $a = \ln 10$ and $n = 1$. How well does $P_2(x) = \frac{1}{10} - \frac{1}{10}(x - \ln 10)$ approximate e^{-x} for $x = \ln 10 + 0.2 \approx 2.502$? The remainder $R = e^x - P_n(x)$ will equal $f''(u)/2!$ times $(0.2)^2$ for some u between $\ln 10$ and $\ln 10 + 2$. Because $f''(u) = e^{-u}$, we know that $0 < f''(u) < f''(a) = 1/10$. Therefore, with $x = \ln 10 + 0.2$,

$$\frac{1}{10} - \frac{0.2}{10} < e^{-x} < \frac{1}{10} - \frac{0.2}{10} + \frac{1}{20}(0.2)^2.$$

Numerically, $0.08 < e^{-(\ln 10 + 0.2)} < 0.082$. The actual value is $0.081873\ldots$.

Here is another example.

EXAMPLE: Let $f(x) = \cos(x)$, $a = 0$ and $n = 4$. Then $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$. This is also P_5 because $f^{(5)}(0) = 0$. How close is this to the correct value of $\cos x$ at $x = \pi/4$? Because the sixth derivative of cos is $-\cos$, Taylor's theorem says

$$\cos(\pi/4) - P_4(\pi/4) = c(\pi/4)^6$$

where $c = -\cos u/6!$ for some $u \in [0, \pi/4]$. The maximum value of $-\cos$ on $[0, \pi/4]$ is $-\sqrt{1/2}$ and the minimum value is -1, therefore

$$-\frac{1}{720}\left(\frac{\pi}{4}\right)^6 \leq \cos(\pi/4) - P_4(\pi/4) \leq -\frac{1}{720\sqrt{2}}\left(\frac{\pi}{4}\right)^6.$$

For bounds one can compute mentally, we can use the fact that $\pi/4$ is a little less than 1 to get

$$-\frac{1}{720} \leq \cos(\pi/4) - P_4(\pi/4) \leq 0$$

to see that $P_4(\pi/4)$ overestimates $\cos(\pi/4)$ but not by more than $1/720$ which is a little over 0.001.