8 Introduction to differential equations

8.1 Modeling with differential equations

A differential equation is an equation, involving derivatives, in which the quantity that you must solve for in order to make the equation true is an unknown *function.* A classical example is Malthusian population growth. This simple population growth model (due to Malthus) postulates that the (instantaneous) rate of growth of a population is proportional to its present size. If we let $A(t)$ denote the size of the population at time t, then the equation representing this is

$$\frac{dA}{dt} = kA(t)\,. \tag{8.1}$$

Here k is a constant of proportionality. It is important that you understand its units! To make the equation work out, k must have units of inverse time. The value $k = 0.03\text{years}^{-1}$ for example, would mean that if you measure time in years, and the present population is one million, then the instantaneous growth rate would be 30,000 people per year.

The *solution* to a differential equation such as (8.1) is any function $A(t)$ that makes the equation true. Typically there will be more than one such equation. For example, the general solution to $A'(t) = 0.03A(t)$ is $Ce^{0.03t}$, where C is any real number. We are going to postpone until Unit 9 the business of to find nice solutions such as this one. In Unit 8 we will concentrate instead on understanding pretty much everything else: how to set up a differential equation, what it means, and what the solution will look like qualitatively.

Verifying that you have found a solution is a lot easier than finding a solution. To check that Ce^{kt} solves (8.1), just evaluate both sides when $A(t) = Ce^{kt}$. The left side is the derivative of $A(t)$ which is Cke^{kt}. The right side is k times $A(t)$ which is Cke^{kt}. They match – whoopee!

The reason you might expect there to be many solutions to an equation such as (8.1) is that it is an equation of evolution. Once you know where you start, everything else should be deterimined, but there is nothing in the equation that tells you where you start. A differential equation together with a value at a certain time is called an *initial value problem.* For example, $A'(t) = 0.03t;\ A(0) = 1,000,000$ is an initial value problem.

Standard form for first-order differential equations

A differential equation could be abitrarily complicated. The equation

$$f(x)\sqrt[3]{1+\left(\frac{df}{dx}\right)^2} - \ln f(x+a) + \exp\left(\frac{d^2 f}{dx^2}\right) = \arctan(x + f(x))$$

is a differential equation but way too complicated for us to have any hope of figuring out what functions f satisfy it. Note the appearance of a second derivative, the square of the first derivative, a big messy cube root and the appearance of the unknown function f as the argument of the arctangent. We will stick to a much simpler class of differential equations, called **first order differential equations in standard form**. This is the form

$$\frac{dy}{dx} = F(x, y)\,. \tag{8.2}$$

Be sure you understand what this means. The unknown function in this case is the function $y(x)$. We call y the "dependent variable" and x the "independent variable". The function F is an abstraction representing that the right-hand side is some function of x and y. Here are some examples:

$$\begin{aligned}
\frac{dy}{dx} &= x^{-2} \\
\frac{dy}{dx} &= ky \\
\frac{dy}{dx} &= x - y \\
\frac{dy}{dx} &= \sqrt[3]{y + e^x}
\end{aligned}$$

Even though you don't yet know much about differential equations, there is a lot you can say looking at these examples. (i) It is possible that $F(x, y)$ will be a function of just x, as in the first equation. This means that $y(x)$ is just the integral of this function. So you can already solve this one: it is $y(x) = -1/x + C$. (ii) It is possible that $F(x, y)$ will be a function of just y, as is the case in the second equation. In that case it's not so obvious how to solve it, but you actually already know the solution to this particular equation because it is just (8.1). (iii) In general, a first order equation in standard form can be simple, like the third one, or complicated, like the fourth. The simple ones are usually exactly solvable (the third one will be solved in Unit 9.4) and the more complicated ones are not. The fourth equation, while not exactly solvable will still yield plenty of information; this is what Unit 8 is mostly about.

A point of notation: should we use y' or dy/dx? Both mean the same thing, but dy/dx is clearer because it tells you which is the dependent variable. If you wrote $y' = -ce^{tx}$ it would be unclear whether t or x was the independent variable (or c could be too, but we never choose c for a variable name because it sounds too much like it should be a constant). We will use both notations, as both are common in real life. One more point: when we want to emphsize that the unknown fariable is a function, we sometimes use a name like f or g instead of y. For example, $f' = -xf$ is a differential equation (it is understood that the indpendent variable is x). The most common independent variable names are x and t, with t usually chosen when it represents time.

Integral equations

Certain equations with integrals in them can be made into differential equations by differentiating both sides (this uses the Fundamental Theorem of Caculus). For example the integral equation

$$f(t) = 12 - \int_5^t 3f(s)\,ds$$

can be differentiated with respect to t to obtain

$$f'(t) = -3f(t)\,.$$

The integral equation has only one solution but this differential equation has many. This means that there was initial value data in the integral equation that we forgot to include in the differential equation. Can you spot it? Really we should have translated the integral equation into the initial value problem:

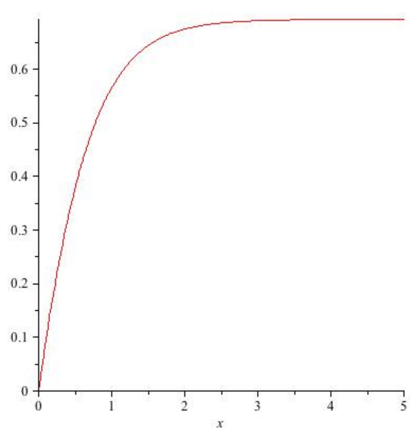
$$f'(t) = -3f(t) \ \ ; \ \ f(5) = 12\,.$$

8.2 Slope fields

Slope field drawings are a way to enable you to sketch solutions to differential equations. The equation $dy/dx = f(x, y)$ tells you what the slope of the graph of y should be at any point (x, y), if indeed that point is on the graph. We make a grid of points and through every (x, y) in the grid we put a little line segment of slope $f(x, y)$. We then try to sketch solutions that are always tangent to the line segments, following them as they change direction. Pages 538–539 of the text do a good job explaining this. PLEASE READ THESE! We will then spend a day practicing.

Slope fields are a *qualitative* approach to understanding the solution to a differential equation, meaning that you get information about the nature of the solution even when you can't find the exact solutions. Here's an example.

Why can we tell that the solution to $y' = 2 - e^y$ should approach a limit of $\ln 2$? It's because when $y(x) < \ln 2$ then $2 - e^y$ is positive and the function threrefore increases while when $y(x) > \ln 2$ then $2 - e^y$ is negative and the function therefore decreases. It seems clear from this that if the function begins below $\ln 2$ it will steadily increase but at a lesser and lesser rate and never get above $\ln 2$, while if the function begins above $\ln 2$ then it will steadily decrease but at a lesser and lesser rate and never get below $\ln 2$. We therefore have a very good idea what this function looks like without ever solving the equation:



8.3 Euler iteration

Euler's method, or "Euler iteration" is a way of finding a numerical approximation for the solution of an initial value problem at some later time. In other words, for the equation $y' = f(y,t); y(t_0) = y_0$, you can compute an approximation to $y(t_1)$ when t_1 is any time greater than t_0.

The idea behind Euler iteration is that you follow the slope field for a small amount of time Δt, which is fixed at some value such as 0.5 or 0.1. Let $t_1 = t_0 + \Delta t$ be the new time and let y_1 be the approximation you get by followingthe slope field fo time Δt. In other words, $y_1 = y_0 + (\Delta t) f(y_0, t_0)$. The slope at the point (t_1, y_1) will in general be different. Follow that slope for time Δt, and repeat.

I don't have a lot to add to what's in the textbook on Page 539–541. Euler's method is important because it gives you an in-principle understanding of what a solution should be like, whether or not you can produce an analytic solution. This is important for your understanding even if you rarely use Euler's method in practice.

Different notions of solution

Our last order of business in this section is some philosophy. You need to understand what is meant by a solution to a differential equation. The simplest differential equation is of the form $y' = f(x)$, in other words, the right-hand side does not depend on y. You already know how to solve this: $y(x) = \int f(x)\,dx$. But wait, what if it's something you can't integrate? An example of this would be $dy/dx = e^{x^2}$. We could write a solution like $y(x) = y(0) + \int_0^x e^{t^2}\,dt$, but is this really a solution? The answer is yes. Here's why.

Euler's method allows you to approximate values of the independent variable. For example, given $y' = f(x,y)$ and $y(0) = 5$ we could use Euler's method to evaluate $y(2)$. What you need to understand is that yes we can do it but it's tedious and not all that accurate unless you use a miniscule step size. By contrast, using Riemann sums to estimate $\int_0^2 e^{t^2}\,dt$ is a piece of cake. Keep in mind the relative difference in difficulty between Riemann sums and Euler's method as we discuss three levels of possible solution to a differential equation.

1. If you can find a solution $y = f(x)$ where f has an explicit formula then that is obviously the best. Your calculator or computer (or maybe even your phone) can evaluate this, and typically you have other information associated with f such as how fast it grows, whether it has asymptotes, and so forth.

2. Next best is if you can write a formula for y that involves functions without nice names, defined as integrals of other functions. You already realize that many simple looking functions such as e^{x^2} and $\ln(x)/(1+x)$ have no simple anti-derivative. The differential equation $y' = e^{x^2}$ is trivial from a differential equations point of view (it is in the form $y' = f(x)$ which we discussed above) but still we can do no better than to write the solution as $y = \int e^{x^2}\, dx$. This is perfectly acceptable and counts as solving the equation.

3. Lastly, for the majority of equations, we can't write a solution even if allowed to use integrals of functions. In this case the best we can do is to numerically approximate particular values and to give limiting information or orders of growth for y. For example, if $y' = 2 - e^y$ then $\lim_{x\to\infty} y(x) = \ln 2$.

One last thing that Euler iteration does for us is to convince us that an initial value problem should have a solution. After all, if you look at an equation with functions and derivatives, there is no reason to believe that there is a function satisfying the equation. But Euler iteration shows you that there has to be. Just do Euler iteration and make the steps smaller and smaller; in the limit it will produce a function satisfying the differential equation. This is the basis for a theorem. The theorem is not officially part of this course but you might be interested to know what it says.

> **Theorem:** Let $f(x, y)$ be a continuous function. Then the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$ has a unique solution, at least for a small amount of time (after that it might become discontinuous). This solution can be obtained by taking the limit of what you get from Euler iteration as the step sizes go to zero.