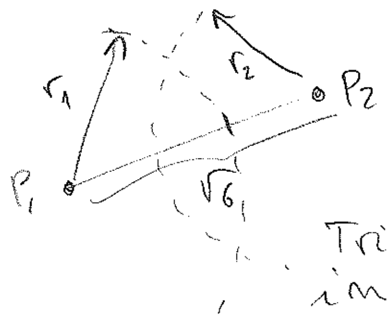
SOLUTIONS TO PRACTICE PROBLEMS FOR MIDTERM 1

#1 $P_1 = (1,0,1)$, $r_1 = 2 \rightsquigarrow \boxed{(x-1)^2 + y^2 + (z-1)^2 = 4}$

$P_2 = (0,2,0)$, $r_2 = 1 \rightsquigarrow \boxed{x^2 + (y-2)^2 + z^2 = 1}$



$$\text{dist}(P_1, P_2) = |\overrightarrow{P_2P_1}| = \sqrt{1+4+1} = \sqrt{6}$$

Triangle ineq.
$$\left(\begin{array}{l} r_1 + r_2 = 3 > \sqrt{6} = |\overrightarrow{P_2P_1}| \\ r_1 + |\overrightarrow{P_2P_1}| = 2 + \sqrt{6} > 1 = r_2 \\ r_2 + |\overrightarrow{P_2P_1}| = 1 + \sqrt{6} > 2 = r_1 \end{array} \right\} \boxed{\text{The spheres intersect!}}$$

#2 • π_1 is the plane through $\vec{0} = (0,0,0)$, with normal

$$\vec{n}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = (0, 0, -2)$$

so it is just the xy-plane.

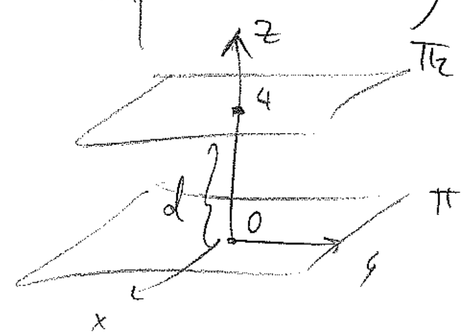
• π_2 contains $P = (0,1,4)$, $Q = (-2,3,4)$, $R = (1,\sqrt{2},4)$, so has normal

$$\vec{n}_2 = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 0 \\ 1 & \sqrt{2}-1 & 0 \end{vmatrix} = \left(0, 0, -2(\sqrt{2}-1) - 2\right)$$
$$= (0, 0, -2\sqrt{2})$$

which means it is *also* parallel to xy-plane
(note: it is the plane with component equation $z=4$.)

The distance between π_1 & π_2 is

$$\boxed{d = 4}$$



#3 $\alpha'(t) = (-\sin t, \cos t, -1)$

$\beta'(t) = (1, 2t, 3t^2)$

$\gamma'(t) = (4t^3 - 2t, e^t\cos t - e^t \sin t, -2e^{2t})$

so $\alpha'(0) = (0,1,-1)$, $\beta'(0) = (1,0,0)$, $\gamma'(0) = (0,1,-2)$.

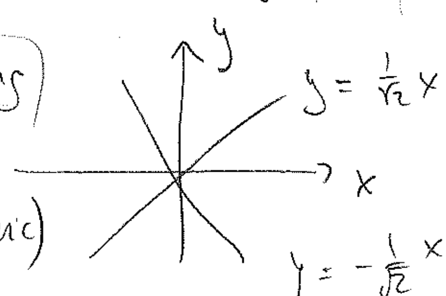
thus, the vol. of the parallelepiped spanned by these vectors is

$$\text{Vol} = \left\| \begin{matrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{matrix} \right\| = |-1+2| = |1| = \underline{\underline{1}}$$

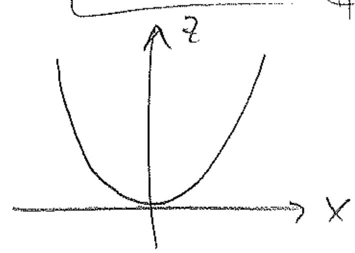
#4 $z = 2x^2 - 4y^2$

<u>$z=0$ section</u> (intersecting w/ xy-plane): $2x^2 = 4y^2 \iff x^2 = 2y^2$

$\iff \boxed{y = \pm\frac{1}{\sqrt{2}}x}$ $\boxed{\text{2 crossing lines}}$ (degenerate conic)

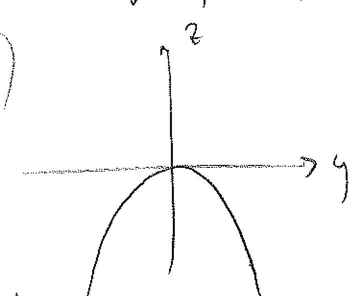


<u>$y=0$ section</u> (intersecting w/ xz-plane): $\boxed{z = 2x^2}$ $\boxed{\text{parabola}}$

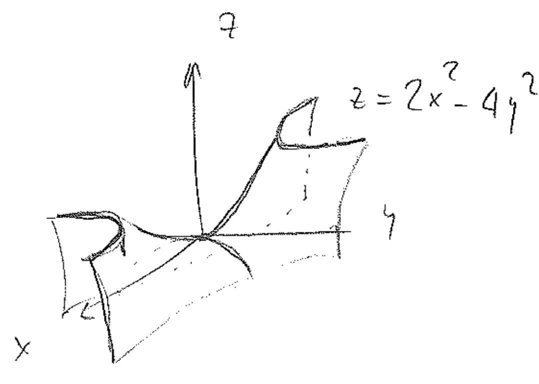


<u>$x=0$ section</u> (intersecting w/ yz-plane):

$\boxed{z = -4y^2}$ $\boxed{\text{parabola}}$



<u>Note</u>: sections $z = \text{const} \neq 0$ are <u>hyperbolas</u>. It is then easy to conclude that the quadric is a <u>saddle</u>:



#5 $x'(t) = \frac{3}{2} t^{1/2}$, $y'(t) = \frac{3}{2}(1-t)^{1/2}(-1) = -\frac{3}{2}\sqrt{1-t}$.

Arc length: $S = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2}\, d\tau = \int_0^t \sqrt{\frac{9}{4}\tau + \frac{9}{4}(1-\tau)}\, d\tau$

$= \int_0^t \frac{3}{2}\, d\tau = \frac{3t}{2}$ //

so the length of the curve between $t=0$ and $t=1$ is $\boxed{s(1) = \frac{3}{2}}$

#6 $\vec{a}(t) = -6t\,\hat{j}$, that is, $\vec{\gamma}''(t) = -6t\,\hat{j}$

so $\vec{\gamma}'(t) = -3t^2\hat{j} + V_0$ and $\boxed{\vec{\gamma}(t) = -t^3\hat{j} + V_0 t + \gamma_0}$

If $V_0 = 10\,\hat{i} + 49\,\hat{j}$, and $\gamma_0 = 0$ is the origin, we have

$$\boxed{\vec{\gamma}(t) = (10t)\,\hat{i} + (49t - t^3)\,\hat{j}}$$

The height of the projectile is $y(t) = 49t - t^3$, which is zero if and only if $y(t) = 0 \iff 49t = t^3 \iff$ $t = 0$ or $49 = t^2 \iff t = 0$ or $t = \pm 7$

So it hits the ground after 7 seconds.

The distance it travelled horizontally is then given by $d = x(7) - x(0)$ where $x(t) = 10t$, so

$\boxed{d = 70 - 0 = 70\text{ m}}$

#7 $\left.\begin{aligned} x'(t) &= \sqrt{2}\cos t \\ y'(t) &= \sin t \\ z'(t) &= -\sin t \end{aligned}\right\} \Rightarrow |\gamma'(t)|^2 = 2\cos^2 t + 2\sin^2 t = 2$

$\Rightarrow |\gamma'(t)| = \sqrt{2}.$

$$\boxed{T = \left(\cos t, \frac{\sin t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}\right)}$$

$$\frac{dT}{dt} = \left(-\sin t, \frac{\cos t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}\right), \quad \left|\frac{dT}{dt}\right|^2 = \sin^2 t + \frac{2}{2}\cos^2 t = 1$$

$$\Rightarrow \left|\frac{dT}{dt}\right| = 1$$

$$\boxed{N = \left(-\sin t, \frac{\cos t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}\right)}$$

$$B = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \frac{\sin t}{\sqrt{2}} & -\frac{\sin t}{\sqrt{2}} \\ -\sin t & \frac{\cos t}{\sqrt{2}} & -\frac{\cos t}{\sqrt{2}} \end{vmatrix} = \left(0, \frac{\cos^2 t + \sin^2 t}{\sqrt{2}}, \frac{\cos^2 t + \sin^2 t}{\sqrt{2}}\right)$$

$$\Rightarrow \boxed{B = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

Thus:

At $t=0$:
$T = (1,0,0)$
$N = (0, 1/\sqrt{2}, -1/\sqrt{2})$
$B = (0, 1/\sqrt{2}, 1/\sqrt{2})$

#8 $\vec{r}''(t) = (e^t, \sin t, t) \Rightarrow \vec{r}'(t) = \left(e^t + \mathring{x}_0, -\cos t + \mathring{y}_0, \frac{t^2}{2} + \mathring{z}_0\right)$

$\vec{r}'(0) = (0,0,1) = (1 + \mathring{x}_0, -1 + \mathring{y}_0, \mathring{z}_0) \Rightarrow \mathring{x}_0 = -1, \mathring{y}_0 = 1, \mathring{z}_0 = 1$

$\Rightarrow \vec{r}'(t) = \left(e^t - 1, 1 - \cos t, \frac{t^2}{2} + 1\right)$

$\Rightarrow \vec{r}(t) = \left(e^t - t + x_0, t - \sin t + y_0, \frac{t^3}{6} + t + z_0\right)$

$\vec{r}(1) = (e, 1, 1) = \left(e - 1 + x_0, 1 - \sin 1 + y_0, \frac{1}{6} + 1 + z_0\right)$

$\Rightarrow x_0 = 1, \; y_0 = \sin 1, \; z_0 = -\frac{1}{6}.$

$$\Rightarrow \boxed{\vec{r}(t) = \left(e^t - t + 1, t - \sin t + \sin 1, \frac{t^3}{6} + t - \frac{1}{6}\right)}$$

#9

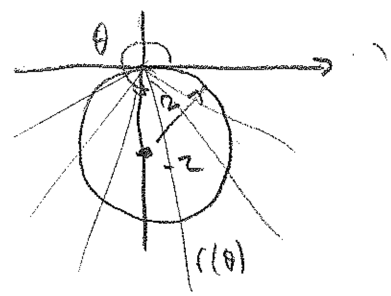
$$r = -4\sin\theta \xrightarrow[\text{by } r]{\text{mult.}} \underbrace{r^2}_{x^2+y^2} = -4\underbrace{r\sin\theta}_{y} \Longrightarrow \boxed{x^2+y^2=-4y}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \rightsquigarrow x^2+y^2=r^2$$

$$\Updownarrow$$

$$\boxed{x^2+(y+2)^2=4}$$

The conic $\boxed{r=-4\sin\theta}$ is therefore a circle of radius 2 centered at the point $(0,-2)$.



#5 continued.

$$\gamma(t) = \left(t^{3/2}, (1-t)^{3/2}\right)$$

$$\gamma'(t) = \left(\frac{3}{2}t^{1/2}, \frac{3}{2}(1-t)^{1/2}\right) = \left(\frac{3}{2}\sqrt{t}, \frac{3}{2}\sqrt{1-t}\right)$$

$$|\gamma'(t)|^2 = \frac{9}{4}t + \frac{9}{4}(1-t) = \frac{9}{4} \Longrightarrow |\gamma'(t)| = \frac{3}{2}$$

$$T = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{3/2}\left(\frac{3}{2}\sqrt{t}, \frac{3}{2}\sqrt{1-t}\right) = \left(\sqrt{t}, \sqrt{1-t}\right)$$

$$\frac{dT}{dt} = \left(\frac{1}{2\sqrt{t}}, \frac{-1}{2\sqrt{1-t}}\right) \Longrightarrow \left|\frac{dT}{dt}\right|^2 = \frac{1}{4t} + \frac{1}{4(1-t)} = \frac{(1-t)+t}{4t(1-t)}$$

$$= \frac{1}{4t(1-t)}$$

$$\Longrightarrow K = \frac{1}{|\gamma'(t)|}\left|\frac{dT}{dt}\right| = \frac{2}{3}\,\frac{1}{2\sqrt{t-t^2}} = \frac{1}{3\sqrt{t-t^2}}.$$

$$K\left(\frac{1}{2}\right) = \frac{1}{3\sqrt{\frac{1}{2}-\frac{1}{4}}} = \frac{1}{3\sqrt{\frac{1}{4}}} = \frac{1}{3\cdot\frac{1}{2}} = \boxed{\frac{2}{3}}$$