HW6

1. $T(x,y) = x^2+y^2-x-y$

$\nabla T = (2x-1, 2y-1)$

$\nabla T(1,3) = (1,5)$

$\vec{u} = \dfrac{(1,5)}{\sqrt{26}}$ for max temperature change

2. a) $x+y-z = 3$ $(1,1,-1)$

* This is already a plane ; tangent plane is itself

b) $x^2+y^2+z^2 = 4$ $(1,-1,\sqrt{2})$

$f_x = 2x = 2$ $\therefore 2(x-1) - 2(y+1) + 2\sqrt{2}(z-\sqrt{2}) = 0$

$f_y = 2y = -2$

$f_z = 2z = 2\sqrt{2}$

c) $x^2+2xy-y^2+z^2 = 5$ $(1,0,2)$

$f_x = 2x+2y = 2$ $\therefore 2(x-1) + 2y + 4(x-2) = 0$

$f_y = 2x-2y = 2$

$f_z = 2z = 4$

d) $e^{xz}+yz-x-z-1 = 0$ $(0,1,2)$

$f_x = ze^{xz}-1 = 1$ $\therefore x + 2(y-1) = 0$

$f_y = z = 2$

$f_z = xe^{xz}+y-1 = 0$

3. $f(x,y) = x^2-y^2$

a) $\nabla f = (2x,-2y)$

b) $\nabla f = (2x,-2y) = \vec{0}$

CP: $(x,y) = (0,0)$

c) $f_{xx} = 2$ $f_{yy} = -2$

$f_{xy} = 0$

$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow$ det $= -4$ * Saddle Point

$g(x,y) = x^4+y^4+4xy$

a) $\nabla g = (4x^3+4y, 4y^3+4x)$

b) $4x^3+4y = 0 \Rightarrow y = -x^3$

$4y^3+4x = 0$

$-4x^9+4x = 0$

$x(-x^8+1) = 0$

$x = 0, \pm 1$

$y = 0, \mp 1$

c) $f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = 4$

$\begin{bmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{bmatrix}$

$(0,0)$: det < 0 Saddle

$(1,-1)$: det > 0 trace > 0 local min

$(-1,1)$: det > 0 trace > 0 local min

$h(x,y,z) = \frac{1}{x} + xy + \frac{1}{y} + z^2$

a) $\nabla h = \left(-\frac{1}{x^2}+y \,,\, x+\frac{-1}{y^2} \,,\, 2z\right)$

CP: $(x,y,z) = (1,1,0)$

b) $\left.\begin{array}{r} -\frac{1}{x^2}+y = 0 \\ x-\frac{1}{y^2} = 0 \\ 2z = 0 \end{array}\right] \Rightarrow \begin{array}{l} z = 0 \\ x = \frac{1}{y^2} \\ \frac{-1}{(\frac{1}{y^2})^2} + y = 0 \\ y - y^4 = 0 \\ y(1-y^3) = 0 \\ y = \cancel{0}, 1 \end{array}$

$\uparrow$ y domain ; $y \neq 0$

c) $f_{xx} = \frac{2}{x^3}$ $\quad f_{xy} = 1$ $\quad f_{xz} = 0$

$f_{yy} = \frac{2}{y^3}$ $\quad f_{yz} = 0$

$f_{zz} = 2$

$$\begin{bmatrix} \frac{2}{x^3} & 1 & 0 \\ 1 & \frac{2}{y^3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} \det = 2\cdot2\cdot2 - 2 = 6 \\ \text{trace} = 2+2+2 = 6 \end{array}$$

Local min

4. $f(x,y) = x^2 - xy + y^2$, $0 \le x \le 1$ $\quad 0 \le y \le 1$

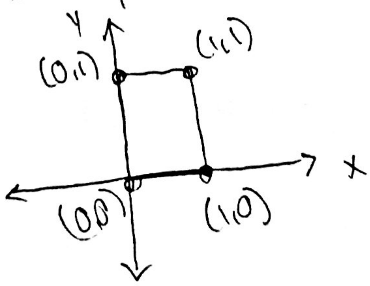
* $\nabla f = (2x-y \,,\, 2y-x) = \vec{0}$

$2x-y = 0$

$2y-x = 0$

CP: $(0,0)$; also belongs to domain

* Boundary



$x=0$: $f(0,y) = y^2$

$f'(y) = 2y = 0$

$(0,0)$

$x=1$: $f(1,y) = 1 - y + y^2$

$f'(y) = 2y-1 = 0$

$(1,\frac{1}{2})$

$y=0$: $f(x,0) = x^2$

$f'(x) = 2x = 0$

$(0,0)$

$y=1$: $f(x,1) = x^2 - x + 1$

$f'(x) = 2x-1 = 0$

$(\frac{1}{2},1)$

Candidates :

(x,y)	$f(x,y)$	
$(0,0)$	0	-> Min
$(0,1)$	1	max
$(1,0)$	1	max
$(1,1)$	1	max
$(1,\frac{1}{2})$	3/4	
$(\frac{1}{2},1)$	3/4	

5. $f(x,y) = \sqrt{x^2+y^2} \xrightarrow{D^2} f(x,y) = x^2+y^2$

$$\sqrt{x^2+y^2} = D = \sqrt{x^2+y^2}$$

Constraint: $g(x,y) = x^2+xy+y^2-1$

$\nabla f = \lambda \nabla g$

$(2x, 2y) = \lambda(2x+y,\ 2y+x)$

$2x = \lambda(2x+y)$ $\quad$ * Assume $2x+y \neq 0$,

$2y = \lambda(2y+x)$ $\quad$ $2y+x \neq 0$

Test later

$$\frac{2x}{2x+y} = \frac{2y}{2y+x}$$

$x^2+xy+y^2 = 1$

$2x(2y+x) = 2y(2x+y)$

$\cancel{4xy} + 2x^2 = \cancel{4xy} + 2y^2$

$2x^2 = 2y^2$

$x = \pm y$

For $x=y$:

$x^2+xy+y^2=1$

$x^2+x^2+x^2=1$

$x = \pm\sqrt{\frac{1}{3}}$

$(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$

$(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$

For $x=-y$:

$x^2+xy+y^2=1$

$x^2-x^2+x^2=1$

$x = \pm 1$

$(1,-1)$

$(-1,1)$

* $2x+y=0$ case

$2x = \lambda(2x+y) = 0$

$x = 0$

$2x+y=0$

$y=0$

But $(0,0)$ doesn't belong to ellipse

* $2y+x=0$ case

$2y = \lambda(2y+x) = 0$

$y=0$

$2y+x=0$

$x=0$

$\longrightarrow$ same reason

(x,y)	$f(x,y) = x^2+y^2 = D^2$	
$(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$	$\frac{2}{3}$	closest
$(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}})$	$\frac{2}{3}$	
$(1,-1)$	2	farthest
$(-1,1)$	2	