Math 312, Spring 2016 **Renato Ghini Bettiol**

Practice Problems for the Final Exam

Due: Never (but solutions will be discussed in class on Apr 25 and 27)

1. What are the kernel $\ker T$ and image $\mathrm{Im} T$ of the following linear transformations?

   a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad [T]_{can} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}$

   b) $T\colon \mathbb{R}^2 \to \mathbb{R}^4, \quad [T]_{can} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ -3 & 0 \\ 3 & 2 \end{pmatrix}$

   c) $T\colon \mathbb{R}^3 \to \mathbb{R}, \quad T(x,y,z) = x + y + z$

   d) $T\colon \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \quad T(p(x)) = p'(x)$

   e) $T\colon \mathbb{R}^5 \to \mathbb{R}^5, \quad T(x) = \mathrm{proj}_v(x)$, where $v = (1,0,1,0,1)$

   f) $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}), \quad T(A) = A^{\mathrm{t}}$

   g) $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}), \quad T(A) = A + A^{\mathrm{t}}$

   h) $T\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}, \quad T(A) = \langle A, I\rangle_{\mathrm{HS}}$, where $I \in M_{n\times n}(\mathbb{R})$ is the identity matrix and $\langle \cdot,\cdot\rangle_{\mathrm{HS}}$ is the Hilbert-Schmidt inner product

2. For the linear transformations $T\colon V \to W$ in Exercise 1 c) - h), find bases $\mathcal{B}$ and $\mathcal{C}$ of the source and target vector spaces V and W and then write the matrix $[T]_{\mathcal{B},\mathcal{C}}$ that represents T with respect to these bases.

3. What is the dimension of the following real vector spaces? Use this to decide which are isomorphic to one another.

   $\mathcal{P}_4(\mathbb{R})$, $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^4)$, $\mathbb{R}^5$, $\mathbb{C}$, $\mathcal{P}_{11}(\mathbb{R})$, $M_{3\times 3}(\mathbb{R})$, $\mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^2)$, $M_{2\times 3}(\mathbb{R})$, $\mathbb{R}^{n^2}$, $\{0\}$, $\mathbb{C}^2$, and $\ker T$, $\mathrm{Im} T$ for each of the T's in Exercise 1.

4. Consider the basis $\mathcal{B} = \{(1,1,0),(1,0,1),(0,1,1)\}$ of $\mathbb{R}^3$. Find the coordinates $[v]_{\mathcal{B}}$ of the following vectors with respect to this basis: $v = (1,2,3)$, $v = (0,1,0)$, $v = (-2,5,6)$.

5. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$. Answer the following without too many computations:

   a) What is the dimension of the image of A?

   b) What is the dimension of the kernel of A?

   c) What are the eigenvalues of A?

d) What are the eigenvalues of $B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$? [HINT: $B = A + 3I$].

6. Diagonalize the following matrices A (i.e., find an invertible matrix P such that the matrix $D = PAP^{-1}$ is diagonal):

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},\ A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},\ A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix},\ A = \begin{pmatrix} -1 & 9 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

7. The first three matrices A above in Exercise 5 are symmetric, hence *orthogonally diagonalizable* by the Spectral Theorem. (Check that the basis of eigenvectors you found is orthonormal). Which of these matrices are positive-definite, positive semi-definite, negative-definite and negative semi-definite?

8. Give an example of an $n \times n$ matrix A which is not diagonalizable over $\mathbb{R}$.

9. Suppose $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map represented by the matrix $A \in M_{4\times 6}(\mathbb{R})$.

   a) What are the possible values for the rank of A?

   b) What are the possible values for the dimension of the kernel of A?

   c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^{\perp}$?

10. Let A be an $n \times k$ matrix.

    a) If $\lambda_1 \neq 0$ is an eigenvalue of A^*A, show that it is also an eigenvalue of AA^*. [Note where you use $\lambda_1 \neq 0$].

    b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of A^*A, let $\vec{u}_1 = A\vec{v}_1$, and $\vec{u}_2 = A\vec{v}_2$. Show that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.

11. An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent.)

    a) If λ is an eigenvalue of a nilpotent matrix A, show that $\lambda = 0$. [HINT: start with the equation $A\vec{x} = \lambda\vec{x}$.]

    b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. [HINT: use Part a).]

    c) Let A be the matrix that represents $T\colon \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $T(p(x)) = \frac{\mathrm{d}p}{\mathrm{d}x}$. Without doing any computations, explain why A must be nilpotent.

12. Let A be a real matrix with the property that $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$.
    a) If A is a symmetric matrix, show this implies that $A = 0$.
    b) Give an example of a matrix $A \neq 0$ that satisfies $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$.

13. For certain polynomials $\mathbf{p}(t)$, $\mathbf{q}(t)$, and $\mathbf{r}(t)$, say we are given the following table of inner products:

$\langle\ ,\ \rangle$	$\mathbf{p}$	$\mathbf{q}$	$\mathbf{r}$
$\mathbf{p}$	4	0	8
$\mathbf{q}$	0	1	0
$\mathbf{r}$	8	0	50

    For example, $\langle \mathbf{q}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{q} \rangle = 0$. Let E be the span of $\mathbf{p}$ and $\mathbf{q}$.
    a) Compute $\langle \mathbf{p}, \mathbf{q} + \mathbf{r} \rangle$.
    b) Compute $\|\mathbf{q} + \mathbf{r}\|$.
    c) Find the orthogonal projection $\text{proj}_E \mathbf{r}$. [Express your solution as linear combinations of $\mathbf{p}$ and $\mathbf{q}$.]
    d) Find an orthonormal basis of the span of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. [Express your results as linear combinations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.]

14. Determine the type of the following conics (ellipse. parabola, hyperbola) and compute its principal axes. Give the formula of the curve in the coordinate system defined by the principal axes.
    a) $6x^2 - 7xy + 8y^2 = 1$
    b) $2x^2 + 6xy + 4y^2 = 1$
    c) $6x^2 + 4xy + 3y^2 = 1$

15. Let $A\colon \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that $\dim(\ker A) - \dim(\ker A^*) = k - n$. In particular, for a square matrix, $\dim(\ker A) = \dim(\ker A^*)$.

16. Find a closed formula for the nth element a_n of each of the following recurrent sequences, and compute $\lim\limits_{n\to+\infty} \dfrac{a_{n+1}}{a_n}$.
    a) $a_{n+2} = 4a_{n+1} - a_n$, $a_0 = 0$, $a_1 = 1$;
    b) $a_{n+2} = 4a_{n+1} - a_n$, $a_0 = 2$, $a_1 = 3$;
    c) $a_{n+2} = -a_{n+1} + a_n$, $a_0 = 0$, $a_1 = 1$;
    d) $a_{n+2} = -a_{n+1} + a_n$, $a_0 = 2$, $a_1 = 3$.

17. Determine if the following statements are TRUE or FALSE. If the statement is TRUE, then supply a proof. If the statement is FALSE, then give a counter-example.

a) $\|v\|_1 \leq \|v\|_2$ for all $v \in \mathbb{R}^n$; recall that $\|v\|_1 := \sum_i |v_i|$ and $\|v\|_2 = \sqrt{\sum_i v_i^2}$

b) $\|v\|_2 \leq \|v\|_\infty$ for all $v \in \mathbb{R}^n$; recall that $\|v\|_\infty := \max_{1\leq i\leq n} |v_i|$

c) If $\dim V \neq \dim W$, then V and W are not isomorphic;

d) If $A \in M_{n\times n}(\mathbb{R})$ is orthogonal, then A is symmetric;

e) If $A \in M_{n\times n}(\mathbb{R})$ is orthogonal, then A is invertible;

f) If $A \in M_{n\times n}(\mathbb{R})$ is invertible, then A is diagonalizable;

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i) If $A \in M_{n\times n}(\mathbb{R})$ is symmetric, then A is invertible;

j) If $A \in M_{n\times n}(\mathbb{R})$ is symmetric, then A is diagonalizable;

k) If $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable, then A^k is diagonalizable for all $k \geq 1$;

l) If $A \in M_{n\times n}(\mathbb{R})$ is such that $\ker A = \{0\}$, then $\mathrm{Im}A = \mathbb{R}^n$;

m) If $A \in M_{n\times n}(\mathbb{R})$ is such that $\ker A \neq \{0\}$, then $\mathrm{Im}A \neq \mathbb{R}^n$;

n) If $A \in M_{n\times n}(\mathbb{R})$ is such that $\mathrm{Im}A = \mathbb{R}^n$, then $\ker A = \{0\}$;

o) If $A \in M_{n\times n}(\mathbb{R})$ is such that $A^2 = A$, then $\mathrm{Spec}(A) \subset \{0, 1\}$;

p) If $A \in M_{n\times n}(\mathbb{R})$ is symmetric and $B \in M_{n\times n}(\mathbb{R})$ is skew-symmetric, then AB is skew-symmetric;

q) If $A \in M_{n\times n}(\mathbb{R})$ is skew-symmetric and n is odd, then $\det(A) = 0$;

r) If $A, B \in M_{n\times n}(\mathbb{R})$ are similar, then they have the same trace $\mathrm{tr}(A) = \mathrm{tr}(B)$;

s) If $A, B \in M_{n\times n}(\mathbb{R})$ have the same determinant $\det(A) = \det(B)$, then A and B are similar;

t) If $A \in M_{m\times n}(\mathbb{R})$ and the linear system $Ax = b$ has infinitely many solutions, then $m < \mathrm{rank}(A) = n$.

u) If $A \in M_{m\times n}(\mathbb{R})$ and the linear system $Ax = b$ has a unique solution, then $m = n$;

v) The set $\mathcal{Q}$ of quadratic forms on R^2 has a vector space structure and the subset of positive-definite quadratic forms is a linear subspace of $\mathcal{Q}$;

w) The subset $\{p(x) \in \mathcal{P}_5(\mathbb{R}) : p(3) = 0\}$ is a linear subspace of $\mathcal{P}_5(\mathbb{R})$;

x) The unit sphere $\{x \in \mathbb{R}^2 : \eta(x, x) = 1\}$ in the Minkowski space $(\mathbb{R}^2, \eta)$, $\eta(x, y) = -x_1y_1 + x_2y_2$ is an ellipse;

y) The unit sphere $\{x \in \mathbb{R}^2 : \langle Ax, x\rangle = 1\}$ in $\mathbb{R}^2$ with respect to the inner product $\langle A\cdot, \cdot\rangle$, where A is positive-definite, is an ellipse;

z) Every matrix $M \in M_{n\times n}(\mathbb{Z})$ is invertible mod 13.