Homework Set 10

Due: Apr 22, 2016 (in class)

1. Bretscher Section 8.3: 4, 15, 20, 21, 24

2. Find an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathbb{R}^2$ such that the basis $\{(-2,3),(1,4)\}$ is orthonormal.

3. Find an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathbb{R}^3$ such that the basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ is orthonormal.

4. Let $A \in M_{n\times n}(\mathbb{R})$ be a nondegenerate symmetric matrix, so that $\langle\!\langle v, w \rangle\!\rangle = \langle Av, w\rangle = w^{\mathrm{t}}Av$ is a (possibly indefinite) inner product. Show that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* of this inner product, that is, $\langle\!\langle T(v), T(w) \rangle\!\rangle = \langle\!\langle v, w \rangle\!\rangle$ if and only if $T^{\mathrm{t}}AT = A$.

   Note: If $A = Id$ is the $n \times n$ identity matrix, then this shows that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an Euclidean isometry if and only if $T^{\mathrm{t}}T = Id$, i.e., $T \in \mathsf{O}(n)$ is an orthogonal matrix.

5. Use the above criterion (Problem 4) to show that for all $-1 < \beta < 1$,

$$T = \frac{1}{\sqrt{1-\beta^2}}\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$$

   is an isometry of Minkowski space $(\mathbb{R}^2, \eta)$, where $\eta(x,y) = -x_1y_1 + x_2y_2$ is the Lorentz inner product corresponding to the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

6. Use the above criterion (Problem 4) to find isometries of $(\mathbb{R}^2, \xi_\theta)$, where the Lorentz inner product $\xi_\theta(x,y) = x^{\mathrm{t}}(A_\theta)y$ corresponds to the matrix $A_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, in terms of the isometries T of the Minkowski space $(\mathbb{R}^2, \eta)$ given in Problem 5.

   Hint: In Problem 5, you were dealing with the case $\theta = \pi$, i.e., $\eta = \xi_\pi$. To reduce this problem to the previous problem, begin by computing $B = R_\alpha A_\theta R_\alpha^{-1}$, where $R_\alpha = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$ is the matrix of counter-clockwise (Euclidean) rotation by α, and then choose a convenient α in terms of θ so that $B = A_\pi$. Then use Problem 4.

   Hint 2: $\cos(a-b) = \cos a \cos b + \sin a \sin b$
   $\sin(a-b) = \sin a \cos b - \sin b \cos a$.