HW 5 # 7

<u>Prop:</u> $\|A\| \leq \|A\|_{HS}$, $\forall A \in M_{n\times n}(\mathbb{R})$.

<u>Pf:</u> Since $\|A\| = \max_{|v|=1} \|Av\|$, it suffices to show that $\|Av\| \leq \|A\|_{HS}$ $\forall v \in \mathbb{R}^n$, $|v| = 1$.

Let $v = \sum_i v_i e_i$, so that $|v|^2 = \sum_i v_i^2 = 1$.

Then $Av = A\left(\sum_i v_i e_i\right) = \sum_i v_i A e_i$, so that

$$\|Av\| = \left\| \sum_i v_i A e_i \right\| \overset{①}{\leq} \sum_i |v_i| \, \|Ae_i\|$$

$$\overset{②}{\leq} \sqrt{\underbrace{\sum_i |v_i|^2}_{1}} \sqrt{\underbrace{\sum_i |Ae_i|^2}_{\|A\|_{HS}^2}} = \|A\|_{HS}.$$

where for ① we use the triangle inequality, and for ② the Cauchy-Schwartz inequality.

Equality $\|A\| = \|A\|_{HS}$ holds if and only if equality in ① and ② hold when v realizes $\|A\| = \|Av\|$. Equality in ① holds if and only if $\{Ae_i\}$ are all colinear, so $\operatorname{rank} A \leq 1$. It is easy to see that all such matrices have $\|A\| = \|A\|_{HS}$. Thus $\|A\| = \|A\|_{HS}$ if and only if $\operatorname{rank} A \leq 1$.

$\square$