Math 465/501 Spring 2018 Homework Set 1

1. Shifrin (P8) Exercise 4

Proof: We parametrize the graph of $y=f(x)$, $a\le x\le b$ by $\alpha(t)=(t,f(t))$, $a\le t\le b$.

To computer the arc length, first compute $\alpha'(t)=(1,f'(t))$,

then $\|\alpha'(t)\|=\sqrt{1^2+(f'(t))^2}$, now use arclength formula:

$$\ell(\alpha)=\int_a^b\|\alpha'(t)\|\,dt=\int_a^b\sqrt{1^2+(f'(t))^2}\,dt. \qquad \square$$

2. Shifrin (P10) Exercise 13.

(a) Proof: $\alpha(t)=r(t)(\cos t,\sin t)$

$$\alpha'(t)=(r'(t)\cos t-r(t)\sin t,\ r'(t)\sin t+r\cos t)$$

$$\|\alpha'(t)\|=\sqrt{r(t)^2+(r'(t))^2}.$$

Then we have arc length of α on $[0,\infty)$ as

$$\ell(\alpha)=\int_0^\infty\|\alpha'(t)\|dt=\int_0^\infty\sqrt{r^2(t)+(r'(t))^2}\,dt,$$ is finite.

Notice that $\int_0^\infty|r(t)|\,dt\le\int_0^\infty\sqrt{r^2(t)+(r'(t))^2}\,dt$, then $\int_0^\infty|r(t)|dt$

is finite. Since $r(t)$ is decreasing, $|r(t)|$ is non-negative,

for $\int_0^\infty|r(t)|dt$ to converge, we have to have $\lim_{t\to\infty}r(t)=0$. $\square$

(b) Proof: When $r(t)=\frac{1}{t+1}$, $\alpha(t)=\left(\frac{\cos t}{t+1},\frac{\sin t}{t+1}\right)$, then

$r'(t)=-\frac{1}{(t+1)^2}$. By computation in part (a), we have

$$\|\alpha'(t)\|=\sqrt{r(t)^2+(r'(t))^2}=\sqrt{\frac{1}{(t+1)^2}+\frac{1}{(t+1)^4}}$$

Compute arc length:

$$\ell(\alpha)=\int_0^\infty\|\alpha'(t)\|dt=\int_0^\infty\sqrt{\frac{1}{(t+1)^2}+\frac{1}{(t+1)^4}}\,dt$$

$$\ge\int_0^\infty\sqrt{\frac{1}{(t+1)^2}}\,dt=\int_0^\infty\frac{1}{|t+1|}\,dt \quad \text{with } t+1>0.$$

We know $\int_0^\infty\frac{1}{t+1}dt$ diverges, hence $\ell(\alpha)$ has infinite length.

$\square$

(c) Claim: $\alpha(t)$ has finite length.

Proof: when $r(t) = \frac{1}{(t+1)^2}$, $\alpha(t) = \left(\frac{\cos t}{(t+1)^2}, \frac{\sin t}{(t+1)^2}\right)$. then

$$r'(t) = -\frac{2}{(t+1)^3}$$ By computation in part (a)

$$\|\alpha'(t)\| = \sqrt{\frac{4}{(t+1)^6} + \frac{1}{(t+1)^4}}$$

Compute arc length:

$$\ell(\alpha) = \int_0^\infty \|\alpha'(t)\| dt = \int_0^\infty \sqrt{\frac{1}{(t+1)^4} + \frac{4}{(t+1)^6}}\, dt.$$

$$= \int_0^\infty \frac{1}{(t+1)^2}\sqrt{1 + \frac{4}{(t+1)^2}}\, dt.$$

$$\leq \int_0^\infty \frac{1}{(t+1)^2}\sqrt{\left(1+\frac{2}{t+1}\right)^2}\, dt$$

$$= \int_0^\infty \frac{1}{(t+1)^2}\left|1+\frac{1}{t+1}\right| dt$$

$$= \int_0^\infty \frac{1}{(t+1)^2} dt + \int_0^\infty \frac{1}{(t+1)^3} dt$$

Since both $\int_0^\infty \frac{1}{(t+1)^2} dt$ and $\int_0^\infty \frac{1}{(t+1)^3} dt$ converge, $l(\alpha)$ converges as well. $\square$

Therefore, $\alpha(t)$ has finite length on $[0,\infty)$

(d) Let $l = \int_0^\infty \sqrt{r^2+(r')^2}\, dt$

Claim: $l < \infty \iff \int_0^\infty |r(t)| dt < \infty$ and $\int_0^\infty |r'(t)| dt < \infty$.

Proof: ($\Rightarrow$) Clearly $\int_0^\infty |r(t)| dt \leq \int_0^\infty \sqrt{r^2+(r')^2}\, dt = l < \infty$

$$\int_0^\infty |r'(t)| dt \leq \int_0^\infty \sqrt{r^2+(r')^2}\, dt = l < \infty$$

($\Leftarrow$) $\sqrt{r^2+(r')^2} \leq |r| + |r'|$ since $a^2+b^2 \leq (a+b)^2$ if $ab>0$

Hence $l < \int_0^\infty |r(t)| dt + \int_0^\infty |r'(t)| dt$

$\square$.

(e) Claim: there exists $r(t)$ s.t $\int_0^\infty |r(t)| < \infty$ and $\int_0^\infty |r'(t)| < \infty$, and $r(t)$ not decreasing, for example: $r(t) = \frac{1}{(t+1)^2} |\sin t|$ satisfies

$$\left| \int_0^\infty \frac{1}{(t+1)^2} |\sin t| \, dt \right| \le \int_0^1 \frac{1}{(t+1)^2} \, dt < \infty$$

$$\left| \int_0^\infty |r'(t)| \, dt \right| = \int_0^\infty + \frac{2}{(t+1)^3} |\sin t| + \frac{1}{(t+1)^2} |\cos t| \, dt \Big|$$

$$\le \left| \int_0^\infty \frac{2}{(t+1)^3} \, dt \right| + \left| \int_0^\infty \frac{1}{(t+1)^2} \, dt \right| < \infty.$$

Hence $r(t)$ satisfies that $\int_0^\infty |r(t)| \, dt < \infty$, $\int_0^\infty |r'(t)| < \infty$, but $r(t)$ is not decreasing.

Note: We can also arrange for $r(t)$ not to vanish at any point with a little more work.

3. Shifrin (P18) Exercise 3(c)

$$\alpha(t) = \left(\sqrt{1+t^2},\ t,\ \ln\left(t+\sqrt{1+t^2}\right)\right).$$

$$\alpha'(t) = \left(\frac{t}{\sqrt{1+t^2}},\ 1,\ \frac{1+\frac{t}{\sqrt{1+t^2}}}{t+\sqrt{1+t^2}}\right) = \left(\frac{t}{\sqrt{1+t^2}},\ 1,\ \frac{1}{\sqrt{1+t^2}}\right).$$

$$\|\alpha'(t)\| = \sqrt{1+\frac{t^2}{1+t^2}+\frac{1}{1+t^2}} = \sqrt{2}.$$

Then $\boxed{T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|} = \frac{1}{\sqrt{2}\sqrt{1+t^2}}\left(t,\ \sqrt{1+t^2},\ 1\right)}$

$$T'(t) = \frac{1}{\sqrt{2}}\left(\frac{1}{(t^2+1)^{3/2}},\ 0,\ -\frac{t}{(t^2+1)^{3/2}}\right)$$

$$\|T'(t)\| = \sqrt{\frac{1}{2}\left(\frac{1}{(t^2+1)^3}+\frac{t^2}{(t^2+1)^3}\right)} = \frac{1}{\sqrt{2}}\sqrt{\frac{1+t^2}{(1+t^2)^3}} = \frac{1}{\sqrt{2}(1+t^2)}$$

$$\boxed{\kappa(t) = \|T'(t)\| = \frac{1}{\sqrt{2}(1+t^2)}.}$$

$$\boxed{N(t) = \frac{T'(t)}{\|T'(t)\|} = \left(\frac{1}{\sqrt{t^2+1}},\ 0,\ -\frac{t}{\sqrt{t^2+1}}\right)}$$

$$\boxed{B(t) = T(t)\times N(t) = \frac{1}{\sqrt{2}\sqrt{1+t^2}}\left(-t,\ \sqrt{1+t^2},\ -1\right).}$$

$$N'(t) = \left(-\frac{t}{(1+t^2)^{3/2}},\ 0,\ -\frac{1}{(1+t^2)^{3/2}}\right)$$

$$\boxed{\tau(t) = N'(t)\cdot B(t) = \frac{t^2}{\sqrt{2}(1+t^2)^2}+\frac{1}{(1+t^2)^2} = \frac{1}{\sqrt{2}(1+t^2)}}$$

4 Shifrin (P18) Exercise 4

Proof: Let $y=f(x)$ be a plane curve, parametrize it by $r(x) = (x, f(x), 0)$

We have the curvature: $k = \dfrac{\|r'(x)\times r''(x)\|}{\|r'(x)\|^3}$

$r'(x) = (1, f'(x), 0)$ $\quad r''(x) = (0, f''(x), 0)$

$r'(x)\times r''(x) = (0, 0, f''(x))$ then $\|r'(x)\times r''(x)\| = \sqrt{(f''(x))^2} = |f''(x)| = |y''|$

$\|r'(x)\| = \sqrt{1+(f'(x))^2} = \sqrt{1+(y')^2}$

Plug these values back into k, we have:

$$k = \frac{\|r'(x)\times r''(x)\|}{\|r'(x)\|^3} = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

$\square$

5. Shifrin (P18) Exercise 7

Proof: Following the hint, let $f(s) = \|\alpha(s)\|^2$.

Then $f(s) = \langle \alpha(s), \alpha(s) \rangle$

$f'(s) = 2\langle \alpha'(s), \alpha(s) \rangle$

$f''(s) = 2\langle \alpha''(s), \alpha(s) \rangle + 2\langle \alpha'(s), \alpha'(s) \rangle$.

Since α is an arc length parametrized curve, $\|\alpha'(s)\| = 1$ and $k(s) = \alpha''(s)$,

then $f''(s) = 2\langle \alpha''(s), \alpha(s) \rangle + 2$

Since $\|\alpha(s)\| \le \|\alpha(s_0)\| = R$ for all s sufficiently close to s_0, s_0 is a local maximum, hence $f''(s_0) \le 0$.

Then $2\langle \alpha''(s_0), \alpha(s_0) \rangle + 2 \le 0$.

$\|\alpha''(s_0)\| \, \|\alpha(s_0)\| \cos\theta \le -1$.

This implies that $\cos\theta < 0$, notice that $\cos\theta \in [-1, 1]$,

then $\|\alpha''(s_0)\| \, \|\alpha(s_0)\| \ge 1$.

Plug-in $\|\alpha(s_0)\| = R$ and $k(s_0) = \alpha''(s_0)$, we have $k(s_0) \ge \frac{1}{R}$. □

6. Proof: Suppose $\alpha: [t_0, t_1] \to \mathbb{R}^n$ is Lipschitz, we need to show $\exists S: [s_0, s_1] \to [t_0, t_1]$ s.t. $\tilde{\alpha} = \alpha \circ S$ satisfies $\|\tilde{\alpha}(s) - \tilde{\alpha}(s')\| = |s - s'| \ \forall s, s' \in [s_0, s_1]$.

Since α is Lipschitz, it is absolutely continuous and hence differentiable almost everywhere, i.e. each coordinate satisfies $\alpha_i(t) - \alpha_i(t') = \int_t^{t'} \alpha_i'(\xi)\, d\xi \ \forall t < t'$

Note: $l(t) = \int_{t_0}^{t} |\alpha'(z)| dz = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} (\alpha_i'(z))^2}\, dz \le \int_{t_0}^{t} \sum_{i=1}^{n} \|\alpha_i'(z)\| dz$

$$= \sum_{i=1}^{n} \int_{t_0}^{t} |\alpha_i'(z)| dz < \infty \quad \forall t \in [t_0, t_1]$$

and $l(t)$ is non-decreasing.

Here we make the simplifying assumption that $l(t)$ is increasing (otherwise, need to use multi-valued inverse). Then let $S = l^{-1}$, and $\tilde{\alpha} = \alpha \circ S$ is the parametrized by arclength by construction. □