Homework Set 0 (Review problems)

DUE: NEVER

1. Find the most general solution $y(x)$ to each of the following linear ODEs:

$$y' + 4y = 0 \qquad y' - 4y = 0 \qquad xy' + 3x^3 y = 0$$
$$y'' + 4y = 0 \qquad y'' - 4y = 0 \qquad y'' - 6y' + 9y = 0$$

2. Let $u(t)$ be the solution of $u' = 7u$ with initial value $u(0) = A > 0$. Write down an explicit formula for $u(t)$. At what time T is $u(T) = 2A$?

3. Let $u(t)$ be the amount of a radioactive element at time t, and say initially $u(0) = A > 0$. The rate of decay is proportional to the amount present, so

$$\frac{du}{dt} = -cu,$$

where the constant $c > 0$ determines the decay rate. The *half-life* T is the amount of time for half of the element to decay, so $u(T) = \frac{1}{2}u(0)$. Find c in terms of T and obtain a formula for $u(t)$ in terms of T.

4. Write a formula for the solution $y(t)$ of $y'' + y' + \frac{1}{4}\lambda y = 0$ in each of the cases $\lambda < 1$, $\lambda = 1$ and $\lambda > 1$. Show that if $\lambda \geq 1$, then $\lim_{t\to+\infty} y(t) = 0$.

5. What is the dimension n of the space of solutions to the linear ODE $u''' + 9u' = 0$? Find n functions that give a basis of this vector space, and use them to write a formula for the general solution $u(t)$.

6. Say $w(t)$ satisfies the differential equation

$$aw'' + bw' + cw = 0, \tag{1}$$

where $a > 0$, $b \geq 0$, and $c > 0$. Let $E(t) = \frac{1}{2}aw'^2 + \frac{1}{2}cw^2$.

   a) Without solving the differential equation, show that $E'(t) \leq 0$.

   b) Use this to show that if $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all $t \geq 0$.

   c) Suppose that $u(t)$ and $v(t)$ both satisfy equation (1), and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.

7. Let $f(\vec{r}) = \|\vec{r}\|^2$ for all $\vec{r} = (x, y, z) \in \mathbb{R}^3$, and $g(\vec{r}) = \frac{1}{\|\vec{r}\|}$ for all $0 \neq \vec{r} \in \mathbb{R}^3$. Compute ∇f, Δf, ∇g and Δg.

8. Let $D \subset \mathbb{R}^2$ be a bounded (connected) region with smooth boundary ∂D. Given a smooth function $u(x, y)$, write $\Delta u = u_{xx} + u_{yy}$ for its *Laplacian.*

SUGGESTION: First solve this problem for a function of *one* variable, $u(x)$, so $\Delta u = u''$ and, say, D is the interval $\{0 < x < 1\}$.

a) Show that $u\Delta u = \operatorname{div}(u\nabla u) - \|\nabla u\|^2$.

b) If $u(x, y) = 0$ on ∂D, show that

$$\iint_D u\Delta u \, dx \, dy = -\iint_D \|\nabla u\|^2 \, dx \, dy.$$

c) If $\Delta u = 0$ in D and $u = 0$ on ∂D, show that $u(x, y) = 0$ for all $(x, y) \in D$.

9. Say $u(x, t)$ has the property that $\dfrac{\partial u}{\partial t} = 2$ for all points $(x, t) \in \mathbb{R}^2$.

a) Find some such function $u(x, t)$.

b) Find the most general such function $u(x, t)$.

c) If $u(x, 0) = \sin 3x$, find $u(x, t)$.

d) If instead u satisfies $\dfrac{\partial u}{\partial t} = 2xt$, still with $u(x, 0) = \sin 3x$, find $u(x, t)$.

10. Say $u(x, t)$ has the property that $\dfrac{\partial u}{\partial t} = 3u$ for all points $(x, t) \in \mathbb{R}^2$.

a) Find some such function – other than the trivial $u(x, t) \equiv 0$.

b) Find the most general such function.

c) If $u(x, t)$ also satisfies the initial condition $u(x, 0) = \sin 3x$, find $u(x, t)$.

11. A function $u(x, y)$ satisfies $3u_x + u_t = 0$. Find an invertible linear change of variables

$$r = ax + bt$$
$$s = cx + dt,$$

where a, b, c, d are constants, so that in the new (r, s) variables u satisfies $\dfrac{\partial u}{\partial s} = 0$. [REMARK: There are many possible such changes of variable. The point is to reduce $3u_x + u_t = 0$ to the much simpler $u_s = 0$.]