Math 241, Fall 2015

Sample Practice Problems for the Final Exam

Remember to revise old Final Exams (available online) and Midterm Exams of both sections.

1. Solve the heat equation $u_t = 5\,u_{xx}$, where $0 < x < 1$, with boundary conditions and initial condition given by $u(0,t) = u(1,t) = 0$ and $u(x,0) = 6\sin(3\pi x)$.

2. Solve the wave equation $u_{tt} = 4\,u_{xx}$ where $0 < x < 1$, with the boundary conditions given by $u(0,t) = u(1,t) = 0$, and initial conditions given by $u(x,0) = \sin(2\pi x)$, $u_t(x,0) = 2\sin(4\pi x)$.

3. Solve the equation $u_t = u_{xx} + e^{-t}\sin(4x)$ where $0 < x < \pi$, with the boundary conditions given by $u(0,t) = 0$, $u(\pi,t) = 3$, and initial condition given by $u(x,0) = \sin(3x)$.

4. Compute the Fourier series of the function $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = x^2 + 1$. For what values of $-1 \leq x \leq 1$ does this series converge to $f(x)$?

5. Compute the Fourier series of the function $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = 2x$. For what values of $-1 \leq x \leq 1$ does this series converge to $f(x)$?

6. Consider the boundary value problem

$$\phi'' + (2 - 4x)\phi' + \lambda\phi = 0, \quad \phi(0) = \phi(1) = 0.$$

   a) Rewrite the above equation in Sturm-Liouville form;
   b) Verify that $\phi(x) = x(1-x)$ is an eigenfunction for this problem, and compute its eigenvalue λ;
   c) Prove that the eigenvalue λ obtained in b) is the first eigenvalue λ_1 of this Sturm-Liouville problem.

7. Find the solution to the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

   on the disk of radius 1, which is bounded at the origin $|u(0,\theta)| < +\infty$, and satisfies $u(1,\theta) = 2 - 5\cos(\theta)$. What are the maximum and minimum values of $u(r,\theta)$?

8. Let T be the triangle $0 \leq x \leq 1$ and $0 \leq y \leq x$. The eigenfunctions of the Laplacian with Dirichlet boundary conditions on T are given by

$$\phi_{nm}(x,y) = \sin(n\pi x)\sin(m\pi y) - \sin(m\pi x)\sin(n\pi y),$$

   where $n = 1, 2, \ldots m-1$ and $m = 2, 3, \ldots$.

a) Compute the eigenvalue λ_{nm} corresponding to the eigenfunction ϕ_{nm};

b) Write the general solution to the wave equation

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & \text{in } T \\ u = 0 & \text{on } \partial T \end{cases}$$

c) Write formulas for the coefficients in the above general solution to match given initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y)$. You may use that $\{\phi_{nm}\}$ are L^2-orthogonal and satisfy $\iint_T \phi_{nm}^2 \mathrm{d}A = \frac{1}{4}$.

9. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $u(x, y, z, t)$ be a solution of

$$\begin{cases} u_{tt} = \Delta u - u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$. Define the *energy* $E(t)$ of this solution by

$$E(t) = \frac{1}{2} \iiint_\Omega u_t^2 + |\nabla u|^2 + u^2 \mathrm{d}V.$$

a) Show that $E(t)$ is constant;

b) Use a) to prove that a solution to the above PDE with given initial position $u(x, y, z, 0)$ and initial velocity $u_t(x, y, z, 0)$ is unique. (Hint: Suppose that u and v are two such solutions, and study the energy of $w = u - v$)

10. Use Fourier transforms to find an explicit formula for the solution $u(x, t)$ of

$$\begin{cases} u_t = 2u_{xx}, \\ u(x, 0) = xe^{-x^2/2} \end{cases}$$

where $-\infty < x < \infty$, $t > 0$.

11. Use Fourier transforms to find the solution $u(x, t)$ of the PDE $2u_t + 5u_x = 0$, where $-\infty < x < \infty$, $t \geq 0$, and $u(x, 0) = f(x)$.

12. Use Fourier transforms to find the solution $u(x, t)$ of the PDE $u_t + 2u_x - u = 0$, where $-\infty < x < \infty$, $t \geq 0$, and $u(x, 0) = f(x)$.