HW 2 #7

$$u_{tt} + 2u_t = u_{xx} \quad , \qquad x \in [-1, 1]$$

IC) $u(x,0) = 0$

$u_t(x,0) = \cos \pi x + 3 \sin 3\pi x$

(BC) Periodic boundary conditions

$\left(\text{that is,} \quad \begin{array}{l} u(-1) = u(1) \\ u'(-1) = u'(1) \end{array} \right)$

We use separation of variables: $u(x,t) = X(x)\,T(t)$

$$u_{tt} + 2u_t = u_{xx}$$

$$XT'' + 2XT' = X''T$$

$$\frac{T''}{T} + \frac{2T'}{T} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + 2T' + \lambda T = 0 \end{cases}$$

Using the (BC), we have the following eigenvalue problem for $X(x)$, $x \in [-1, 1]$:

$$\begin{cases} X'' + \lambda X = 0 \\ X(-1) = X(1) \\ X'(-1) = X'(1) \end{cases}$$

By direct computation, or consulting Table 2.4.1 on p. 65 of Haberman, we have that the eigenvalues are

$$\boxed{\lambda_n = (n\pi)^2, \quad n = 0, 1, 2, \ldots}$$

and the associated eigenfunctions are

$$\boxed{X_n(x) = \cos n\pi x \quad \text{and} \quad Y_n(x) = \sin n\pi x .}$$

Since $\lambda_n > 1$ for $n = 1, 2, \ldots$, the ODE for t, $T'' + 2T' + \lambda_n T = 0$ has assoc. polynomial $r^2 + 2r + \lambda_n = 0$, and its roots are:

$$r = \frac{-2 \pm \sqrt{4 - 4\lambda_n}}{2} = -1 \pm \left(\sqrt{\lambda_n - 1}\right) i$$

Thus, the solutions to this ODE are

$$\boxed{T_n(t) = c_1 e^{-t} \cos\left(\sqrt{\lambda_n - 1}\, t\right) + c_2 e^{-t} \sin\left(\sqrt{\lambda_n - 1}\, t\right)}$$

To match the (IC), we use:

$$u_1(x,t) = X_1(x) T_1(t) \qquad (n=1)$$

$$= (\cos \pi x)\left(c_1 e^{-t} \cos\left(\sqrt{\pi^2 - 1}\, t\right) + c_2 e^{-t} \sin\left(\sqrt{\pi^2 - 1}\, t\right)\right)$$

and

$$u_3(x,t) = Y_3(x) T_3(t) = (\sin 3\pi x)\left(c_1 e^{-t} \cos\left(\sqrt{9\pi^2 - 1}\, t\right) + c_2 e^{-t} \sin\left(\sqrt{9\pi^2 - 1}\, t\right)\right) \quad (n=3)$$

$\underline{u(x,0)=0}$

$$u_1(x,0) = (\cos \pi x)(c_1+0) = 0 \quad \Rightarrow \quad \boxed{c_1 = 0 \text{ for } u_1(x,t)}$$

$$u_3(x,0) = (\sin 3\pi x)(c_1+0) = 0 \quad \Rightarrow \quad \boxed{c_1 = 0 \text{ for } u_3(x,t)}$$

$\underline{u_t(x,0) = \cos \pi x + 3\sin 3\pi x}$

$$\frac{\partial u_1}{\partial t}(x,0) = (\cos \pi x)\left(-c_2 e^{-t}\sin\left(\sqrt{\pi^2-1}\,t\right) + c_2\sqrt{\pi^2-1}\,e^{-t}\cos\left(\sqrt{\pi^2-1}\,t\right)\right)\Big|_{t=0}$$
$$= (\cos\pi x)\left(c_2\sqrt{\pi^2-1}\right) \overset{!}{=} \cos \pi x$$

So we choose $\boxed{c_2 = \dfrac{1}{\sqrt{\pi^2-1}} \text{ for } u_1(x,t)}$

$$\frac{\partial u_3}{\partial t}(x,0) = (\sin 3\pi x)\left(-c_2 e^{-t}\sin\left(\sqrt{9\pi^2-1}\,t\right) + c_2\sqrt{9\pi^2-1}\,e^{-t}\cos\left(\sqrt{9\pi^2-1}\,t\right)\right)\Big|_{t=0}$$
$$= (\sin 3\pi x)\left(c_2\sqrt{9\pi^2-1}\right) \overset{!}{=} 3\sin 3\pi x$$

So we choose $\boxed{c_2 = \dfrac{3}{\sqrt{9\pi^2-1}} \text{ for } u_3(x,t)}$

Thus, the solution to our PDE is $u(x,t) = u_1(x,t) + u_3(x,t)$, i.e.

$$\boxed{u(x,t) = \underbrace{\frac{1}{\sqrt{\pi^2-1}}\, e^{-t}\sin\left(\sqrt{\pi^2-1}\,t\right)(\cos \pi x)}_{\substack{u_1(x,t) \\ \begin{pmatrix} n=1 & c_1=0 \\ \lambda=\pi^2 & c_2=\frac{1}{\sqrt{\pi^2-1}} \end{pmatrix}}} + \underbrace{\frac{3}{\sqrt{9\pi^2-1}}\, e^{-t}\sin\left(\sqrt{9\pi^2-1}\,t\right)\sin(3\pi x)}_{\substack{u_3(x,t) \\ \begin{pmatrix} n=3 & c_1=0 \\ \lambda=9\pi^2 & c_2=\frac{3}{\sqrt{9\pi^2-1}} \end{pmatrix}}}}$$

$\square$