HW 5 # 3 (Haberman 5.3.9)

$$x^2\phi''(x) + x\phi'(x) + \lambda\phi(x) = 0, \qquad \phi(1) = 0 = \phi(b)$$

a) $\mu = \frac{1}{x} \rightsquigarrow \underbrace{x\phi'' + \phi'}_{(x\phi')'} + \frac{\lambda}{x}\phi = 0$

So the equation above is a Sturm-Liouville equation:

$$(x\phi')' + \frac{\lambda}{x}\phi = 0 \qquad \begin{pmatrix} p = x \\ q = 0 \\ \sigma = \frac{1}{x} \end{pmatrix}$$

b) Multiplying the equation by ϕ and integrating, we have:

$$\int_1^b (x\phi')'\phi\,dx + \lambda\int_1^b \frac{\phi^2}{x}\,dx = 0$$

$$\Rightarrow \underbrace{x\phi'\phi\Big|_1^b}_{\overset{\text{(BC)}}{=}0} - \int_1^b x\phi'\phi'\,dx + \lambda\int_1^b \frac{\phi^2}{x}\,dx = 0$$

$$\Rightarrow \lambda = \frac{\int_1^b x(\phi')^2\,dx}{\int_1^b \frac{\phi^2}{x}\,dx} \geq 0$$

is a quotient of 2 nonnegative integrals ($x \in [1,b]$.)

c) Since this ODE is equidimensional, we can solve it explicitly: we know solutions are of the form

$$\phi(x) = x^p$$

Substituting,

$$\phi(x) = x^p$$
$$\phi'(x) = p x^{p-1}$$
$$\phi''(x) = p(p-1) x^{p-2}$$

Thus

$$p(p-1) x^p + p x^p + \lambda x^p = 0$$
$$(p^2 - p + p + \lambda) x^p = 0$$
$$p^2 + \lambda = 0 \implies p = \pm i\sqrt{\lambda}$$

So the solutions to the ODE are:

$$\phi(x) = x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln x} = \cos(\sqrt{\lambda}\ln x) \pm i\sin(\sqrt{\lambda}\ln x)$$

Rewriting in real terms, the 2 linearly independent solutions are

$$\phi(x) = \sin(\sqrt{\lambda}\ln x) \quad \text{and} \quad \psi(x) = \cos(\sqrt{\lambda}\ln x)$$

We were also given boundary conditions:

(BC) $\phi(1) = 0 = \phi(b)$

Clearly $\psi(x) = \cos(\sqrt{\lambda} \ln x)$ cannot satisfy these (BC) because

$$\psi(1) = \cos(\sqrt{\lambda} \ln 1) = \cos(0) = 1 \neq 0,$$

Regarding the other solution $\phi(x) = \sin(\sqrt{\lambda} \ln x)$, we have

$$\phi(1) = \sin(\sqrt{\lambda} \ln 1) = \sin(0) = 0 \checkmark$$

$$\phi(b) = \sin(\sqrt{\lambda} \ln b) = 0 \iff \underline{\sqrt{\lambda} \ln b = n\pi} \text{ for some } n \in \mathbb{N}$$

Thus, we find that the <u>eigenvalues</u> are

$$\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2, \quad n = 1, 2, \ldots$$

In particular $\lambda = 0$ is <u>not</u> an eigenvalue.
From the above explicit computation of $\{\lambda_n\}_{n=1}^{\infty}$,
this is an infinite sequence with $\lambda_1 > 0$ and $\lambda_n \xrightarrow{n\to\infty} \infty$.

d) According to Sturm-Liouville theory, the eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ of the above problem are orthogonal with weight $\sigma(x) = \frac{1}{x}$, that is, in the space $L^2([1,b], \sigma dx)$ with inner product

$$\langle \phi, \psi \rangle_\sigma = \int_1^b \phi(x) \psi(x) \underbrace{\frac{1}{x}\, dx}_{\sigma dx}.$$

Since we explicitly computed the eigenfunctions (in c)):

$$\phi_n(x) = \sin\left(\sqrt{\lambda}\ln x\right) = \sin\left(\frac{n\pi \ln x}{\ln b}\right)$$

we can verify their orthogonality:

$$\langle \phi_n, \phi_m \rangle_\sigma = \int_1^b \sin\left(\frac{n\pi \ln x}{\ln b}\right)\sin\left(\frac{m\pi \ln x}{\ln b}\right)\frac{1}{x}\,dx$$

substitute
$$\begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases} \quad = \int_0^{\ln b} \sin\left(\frac{n\pi u}{\ln b}\right)\sin\left(\frac{m\pi u}{\ln b}\right)du$$

By the usual orthogonality of sines

$$= \begin{cases} 0 & \text{if } n \neq m \\ \frac{(\ln b)}{2} & \text{if } n = m \end{cases}$$

e) Let us count how many zeroes the eigenfunction

$$\phi_n(x) = \sin\left(\frac{n\pi \ln x}{\ln b}\right)$$

has in the open interval $(1,b)$. That corresponds to

$$\sin\left(\frac{n\pi \ln x}{\ln b}\right) = 0 \iff \frac{n\pi \ln x}{\ln b} = m\pi \quad \text{for some } m \in \mathbb{Z}.$$

$$\iff \ln x = \frac{m}{n}\ln b$$

$$\iff x = e^{\frac{m}{n}\ln b} = b^{\frac{m}{n}}.$$

How many solutions does $x = b^{\frac{m}{n}}$ has if $x \in (1, b)$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}$ is fixed?

Clearly at most n, since if $m < 0$ or $m > n$, then $b^{\frac{m}{n}} \notin (1,b)$. Actually, at most $n-1$, since if $m = n$, we get $b^{\frac{m}{n}} = b \notin (1,b)$. Finally, for each $m = 1, 2, \dots, n-1$, we have $x = b^{\frac{m}{n}} = \sqrt[n]{b^m} \in (1,b)$ is a real solution.

Therefore, $\phi_n(x)$ has <u>exactly</u> $n-1$ zeroes, as predicted by Sturm-Liouville theory. ✓