HW 8 #1

Using the hint, we let $v(x) = (x^2-1)^n$, and note that clearly

$$(x^2-1)\,v' = 2nxv$$

Differentiating both sides of the above equation $(n+1)$ times; using product rule; we have:

$$(x^2-1)\,v^{(n+2)} + \binom{n+1}{1} 2x\, v^{(n+1)} + \binom{n+1}{2} 2\, v^{(n)} = 2nx\, v^{(n+1)} + \binom{n+1}{1} 2n\, v^{(n)}$$

[0 derivatives on (x^2-1), $n+1$ derivatives on v (occurs once)]

[1 der. on (x^2-1), n der. on v (occurs $\binom{n+1}{1}$ times)]

[2 der. on x^2-1, $n-1$ der. on v (occurs $\binom{n+1}{2}$ time)]

[0 der. on (x^2-1), $n+1$ der on v (occurs once)]

[1 der. on (x^2-1), n der. on v. (occurs $\binom{n+1}{1}$ times)]

So, simplifying,

$$(x^2-1)\,v^{(n+2)} + (n+1)2x\,v^{(n+1)} + n(n+1)\,v^{(n)} = 2nx\,v^{(n+1)} + 2n(n+1)\,v^{(n)}$$

$$(x^2-1)v^{(n+2)} + 2x\,v^{(n+1)} - n(n+1)\,v^{(n)} = 0$$

$$\left((x^2-1)\,v^{(n+1)}\right)' - n(n+1)\,v^{(n)} = 0$$

So $P(x) = v^{(n)} = \dfrac{d^n}{dx^n}(x^2-1)^n$ satisfies the Legendre equation

$$\frac{d}{dx}\left((x^2-1)\frac{dP}{dx}\right) - \mu P = 0$$

w/ eigenvalue $\mu = n(n+1)$. $\square$