SYMPLECTORMOPHISM GROUPS OF NON-COMPACT MANIFOLDS, ORBIFOLD BALLS, AND A SPACE OF LAGRANGIANS

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Abstract. We establish connections between contact isometry groups of certain contact manifolds and compactly supported symplectomorphism groups of their symplectizations. We apply these results to investigate the space of symplectic embeddings of balls with a single conical singularity at the origin. Using similar ideas, we also prove the longstanding expected result that the space of Lagrangian $\mathbb{R}P^2$ in $T^*\mathbb{R}P^2$ is weakly contractible.

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1. Introduction

Since the seminal work of Gromov, [10], the symplectomorphism groups of closed 4-manifolds have been a subject of much research, see for example [1], [20], as have symplectomorphism groups of manifolds with convex ends, see for example [19], [7]. Here we investigate the simplest symplectic manifolds with both convex and concave ends, namely the symplectizations sM of 3-dimensional contact manifolds M. In the case when the contact manifold is a Lens space $L(n,1)$ the compactly supported symplectomorphism group $\mathrm{Symp}_c(sL(n,1))$ has a rich topology. In particular, we obtain the following result:

Theorem 1.1. *The group* $\mathrm{Symp}_c(sL(n,1))$*, endowed with the* C^∞*-topology, has countably many components, each being weakly homotopy equivalent to the based loop space of* $SU(2)$*. There is a natural map from* $\mathscr{L}(\mathcal{C}\,\mathrm{Iso}_n)$*, the based loop group of contact isometry group of* $L(n,1)$*, to* $\mathrm{Symp}_c(sL(n,1))$ *which induces the weak homotopy equivalence.*

Now, if one of our contact manifolds can be embedded in a 4-dimensional symplectic manifold as a hypersurface of contact type then there are natural maps from compact subsets of $\mathrm{Symp}_c(sL(n,1))$ to the symplectomorphism groups of the 4-manifold. But as the symplectomorphism group of the 4-manifold may have much simpler topology, the induced maps on homotopy groups will typically be far from injective. For example, $S^3 \hookrightarrow B^4$ as a contact type hypersurface, but while $\mathrm{Symp}_c(sS^3)$ is weakly homotopy equivalent to the based loop space of $U(2)$, it is a result of [10] that $\mathrm{Symp}_c(B^4)$ is contractible.

Our proof of Theorem 1.1 identifies $\mathrm{Symp}_c(sL(n,1))$ with the based loop space of the Kähler isometry group K_n of the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathscr{O}(n) \oplus \mathbb{C})$. Removing the section at infinity s_∞ from $\mathbb{F}_n$, and blowing down the zero section s_0, one obtains a singular 4-ball with a conical singularity of order n at the origin.

Since the group $\mathrm{Symp}_c(sL(n,1))$ is homotopy equivalent to $\mathrm{Symp}_c(\mathbb{F}_n \setminus \{s_\infty \cup s_0\})$, we can rephrase Theorem 1.1 as a result on the space of symplectic embeddings of a singular ball of size $\epsilon \in (0,1)$ into a singular ball of size 1. In the second part of the paper, we show that Theorem 1.1 is equivalent to the following result:

Theorem 1.2. *The space of symplectic embeddings of a singular ball of size $\epsilon \in (0,1)$ into a singular ball of size 1 is homotopy equivalent to the Kähler isometry group K_n of the Hirzebruch surface $\mathbb{F}_n$. Moreover, the group of reduced, compactly supported symplectomorphisms of a singular ball of size 1 is contractible.*

Note that in the case $n = 1$, the balls are in fact smooth and Theorem 1.2 reduces to the fact that the space of symplectic embeddings $B(\epsilon) \hookrightarrow B(1)$ deformation retracts onto $U(2)$.

In the third part of the paper we apply the techniques used in the proof of Theorem 1.1 in the special case $n = 4$ to obtain the homotopy type of a space of Lagrangian submanifolds:

Theorem 1.3. *The space of Lagrangian $\mathbb{R}P^2$ in the cotangent bunble $T^*\mathbb{R}P^2$, endowed with the C^∞-topology, is weakly contractible.*

It is already known that the space of Lagrangian S^2 in T^*S^2 is contractible, see [11], [12], and Theorem 1.3 may be considered as a $\mathbb{Z}_2$-equivariant version.

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2. Symplectomorphism Groups of $sL(n,1)$

Consider the lens space

$$L(n,1) = \begin{cases} S^3 & n = 1 \\ S^3/\mathbb{Z}_n & n \geq 2 \end{cases}$$

As contact quotients of S^3 with the standard contact form, the lens spaces inherit natural contact one-forms, denoted as λ_n. There is a standard way to associate a non-compact symplectic manifold to a contact manifold, called the *symplectization.* Concretely, we consider $L(n,1) \times \mathbb{R}$ endowed with the symplectic form $d(e^t\lambda_n)$, where t is the coordinate of the second factor $\mathbb{R}$. We denote this symplectic manifold $sL(n,1)$. Compactly supported symplectomorphism groups will be denoted by Symp_c. In this section, we discuss the homotopy type of $\mathrm{Symp}_c(sL(n,1))$, the group of compactly supported symplectomorphisms of $sL(n,1)$.

2.1. Reducing $sL(n,1)$ to compact manifolds.

We first reduce the problem to the symplectomorphism groups of partially compactified symplectic manifolds. Let $\mathscr{O}(n)$ be the complex line bundle over $\mathbb{C}P^1$ with Chern class $c_1 = n$. One can endow the total space of this line bundle with a standard Kähler structure, whose restriction to the zero section is the spherical area form with total area 1. We denote the zero section as C_n.

Proposition 2.1. *The topological group $\mathrm{Symp}_c(sL(n,1))$ is weakly homotopy equivalent to $\mathrm{Symp}_c(\mathscr{O}(n)\backslash C_n)$.*

Proof. Identifying $L(n,1)$ as a circle bundle in $\mathscr{O}(n)$ with contact structure given by the connection 1-form, we get a canonical embedding:

$$\mathscr{O}(n)\backslash C_n \hookrightarrow sL(n,1), \tag{2.1}$$

where the image is $\{(x,t) \in sL(n,1) : t < 1\}$. Let $\mathrm{Symp}_c^r(sL(n,1))$ be the subgroup of $\mathrm{Symp}_c(sL(n,1))$ consisting of symplectomorphisms supported in $\{t < r\}$, then the embedding (2.1) induces an embedding of the corresponding groups of symplectomorphisms, where the image is exactly $\mathrm{Symp}_c^1(sL(n,1))$. On the one hand, for $r > 1$, using the inverse Liouville flow one sees that $\mathrm{Symp}_c^r(sL(n,1))$ deformation retracts to $\mathrm{Symp}_c^1(sL(n,1))$; on the other hand, $\mathrm{Symp}_c(sL(n,1))$ is nothing but the direct limit of $\mathrm{Symp}_c^r(sL(n,1))$ as $r \to \infty$. This concludes the proof. □

2.2. $\mathrm{Symp}_c(\mathscr{O}(n)\backslash C_n)$ **as a loop space.** We will find the weak homotopy type of $\mathrm{Symp}_c(\mathscr{O}(n)\backslash C_n)$ in this section by showing it is weakly homotopy equivalent to a certain loop space. We start with some known results about $\mathrm{Symp}_c(\mathscr{O}(n))$.

Lemma 2.2. $\mathrm{Symp}_c(\mathscr{O}(n))$ *is weakly contractible.*

This result was shown in [5], Proposition 3.2. Coffey proceeded by compactifying $\mathscr{O}(n)$ by adding an infinity divisor to obtain the projectivization of $\mathscr{O}(n)$, which is the Hirzebruch surface $\mathbb{F}_n$. Symplectomorphisms of Hirzebruch surfaces are then studied using holomorphic curves. We note that this can also be deduced from Abreu and McDuff's results in [1].

Now, Coffey also showed that $\mathrm{Symp}_c(\mathscr{O}(n))$ acts transitively on the space $S(C_n)$ of unparametrized embedded symplectic spheres in $\mathscr{O}(n)$ which are homotopic to the zero section. We then have an action fibration

$$\mathrm{Stab}_c(C_n) \to \mathrm{Symp}_c(\mathscr{O}(n)) \to S(C_n)$$

where $\mathrm{Stab}_c(C_n)$ is the subgroup of $\mathrm{Symp}_c(\mathscr{O}(n))$ consisting of symplectomorphisms which preserve the zero section C_n.

Lemma 2.3 (Coffey [5]). *The stabilizer* $\mathrm{Stab}_c(C_n)$*, is contractible.*

Let $\mathcal{G}_\omega(\nu)$ be the symplectic gauge transformations of the normal bundle ν of C_n, that is, sections of $Sp(\nu) \to C_n$, where $Sp(\nu)$ are the fiberwise symplectic linear maps. Notice that $\mathcal{G}_\omega(\nu) \simeq \mathrm{Map}(C_n, Sp(2)) \simeq S^1$ (see [7], [19]).

Let $\mathrm{Fix}_c(C_n)$ be the subgroup of $\mathrm{Stab}_c(C_n)$ consisting of symplectomorphisms which fix the zero section C_n pointwise. We will use the following lemma from time to time.

Lemma 2.4. *The homomorphism* $\mathrm{Fix}_c(C_n) \to \mathcal{G}_\omega(\nu)$ *given by taking derivatives along* C_n *is surjective.*

Proof. Let $g \in \mathcal{G}_\omega(\nu)$. Then each $g(z)$ for $z \in S^2$ is a symplectic transformation of the normal fiber ν_z over z. Any such linear symplectic map is the time 1 Hamiltonian flow ϕ_1 of a unique quadratic form $Q(z)$ on ν_z.

Consider the Hamiltonian function $H(z,v) = \chi(|v|)Q(z)v$ on $\mathscr{O}(n)$, where χ is a bump function equal to 1 near 0 and 0 when $|v| \geq 1$. As $dH = 0$ along C_n the resulting Hamiltonian flow ψ_t lies in $\mathrm{Fix}_c(C_n)$. We will check that the corresponding gauge action at time 1 is precisely g.

For this, let $Y \in \nu_z \cong T_0\nu_z \subset T_z\mathscr{O}(n)$. Then we claim that $d\psi_t(Y) = \phi_t(Y)$, where in the second term Y is considered as a point in ν_z and ϕ_t is the Hamiltonian flow of $Q : \nu_z \to \mathbb{R}$. The vector Y can be extended to a Hamiltonian vector field on

$\mathscr{O}(n)$ generated by a function L which is linear on ν_z. Let X_H be the Hamiltonian vector field generated by H. Then

$$\mathcal{L}_{X_H} Y = [X_H, Y] = X_{\{H,L\}} = X_{dH(Y)}$$

using the same notation throughout for Hamiltonian vector fields. Evaluating at z, our Lie derivative is tangent to the fiber ν_z, and restricting to this fiber the function $dH(Y) = dQ(z)(Y)$ is linear and dual under the symplectic form to $X_Q(Y)$. In other words, $\mathcal{L}_{X_H} Y(z) = X_Q(Y)$, identifying two vectors in ν_z. This is equivalent to our claim and so the proof is complete. $\square$

Let $\mathrm{Fix}_c^{\mathrm{id}}(C_n)$ denote the subgroup of $\mathrm{Fix}_c(C_n)$ consisting of diffeomorphisms whose derivatives act trivially on the normal bundle ν of the zero section. A simple application of Moser's argument shows that $\mathrm{Fix}_c^{\mathrm{id}}(C_n)$ is homotopy equivalent to $\mathrm{Symp}_c(\mathscr{O}(n)\backslash C_n)$, and we will freely switch between these two groups without explicitly mentioning it below.

Let us write $\mathrm{Aut}_\omega(\nu)$ for the group of automorphisms of the normal bundle ν of the zero section C_n which are symplectic linear on the fibers and preserve the symplectic form along the zero section. The group $\mathrm{Stab}_c(C_n)$ acts on $\mathrm{Aut}_\omega(\nu)$ via its derivative along the zero section. Clearly $\mathrm{Stab}_c(C_n)$ acts transitively on C_n and so by Lemma 2.4 the action on $\mathrm{Aut}_\omega(\nu)$ is also transitive. Hence we have the fibration

$$\mathrm{Fix}_c^{\mathrm{id}}(C_n) \hookrightarrow \mathrm{Stab}_c(C_n) \twoheadrightarrow \mathrm{Aut}_\omega(\nu) \tag{2.2}$$

which by Lemma 2.3 yields a weak homotopy equivalence (cf. Proposition 4.66 [9])

$$\mathrm{Fix}_c^{\mathrm{id}}(C_n) \simeq \mathscr{L}\,\mathrm{Aut}_\omega(\nu)$$

where $\mathscr{L}\,\mathrm{Aut}_\omega(\nu)$ is the space of based loops of $\mathrm{Aut}_\omega(\nu)$. Therefore, the following proposition will imply the first part of Theorem 1.1:

Proposition 2.5. *The group $\mathrm{Aut}_\omega(\nu)$ is homotopy equivalent to the Kähler isometry group K_n of the Hirzebruch surface $\mathbb{F}_n$. In particular,*

$$\mathrm{Aut}_\omega(\nu) \simeq K_n \simeq U(2)/\mathbb{Z}_n \simeq \begin{cases} SO(3)\times S^1 & \text{if } n \text{ is even, } n \neq 0 \\ U(2) & \text{if } n \text{ is odd} \end{cases}$$

so that $\mathscr{L}\,\mathrm{Aut}_\omega(\nu)$ has countably many components, where each component is homotopy equivalent to $\mathscr{L}SU(2)$, that is, to the identity component of $\mathscr{L}SO(3)$.

Proof. First notice that $\mathrm{Aut}_\omega(\nu)$ acts transitively on the symplectic reparametrization group of the zero section, or equivalently, the symplectomorphism group of $\mathbb{C}P^1$. We thus have an action fibration

$$\mathcal{G}_\omega(\nu) \hookrightarrow \mathrm{Aut}_\omega(\nu) \twoheadrightarrow \mathrm{Symp}(\mathbb{C}P^1). \tag{2.3}$$

whose fiber is the subgroup which fixes $\mathbb{C}P^1$ pointwise and thus is simply the gauge group $\mathcal{G}_\omega(\nu)$.

Recall that the Hirzebruch surface $\mathbb{F}_n$ is the projectivation $\mathbb{P}(\mathscr{O}(n)\oplus\mathbb{C})$. Under the action of its Kähler isometry group $K_n \simeq U(2)/\mathbb{Z}_n$, the complex surface $\mathbb{F}_n$ is partitionned into three orbits: the zero section C_n, the section at infinity C_n^∞ and their open complement $\mathbb{F}_n\backslash\{C_n \cup C_n^\infty\}$, see Appendix B in [2]. Since the K_n action preserves the ruling $\mathbb{F}_n \to \mathbb{C}P^1$, every element in K_n acts as an isometry of $\mathbb{C}P^1$

and K_n acts faithfully on the normal bundle ν on C_n via derivatives. We thus get a commutative diagram of fibrations

$$\begin{tikzcd} \mathcal{G}_\omega(\nu) \arrow[r] & \mathrm{Aut}_\omega(\nu) \arrow[r, two heads] & \mathrm{Symp}(\mathbb{C}P^1) \\ S^1 \arrow[r] \arrow[u] & K_n \arrow[r, two heads] \arrow[u] & SO(3) \arrow[u] \end{tikzcd}$$

in which the first and third vertical inclusions are homotopy equivalences. It follows that the middle inclusion is a weak homotopy equivalence. Since all spaces involved are homotopy equivalent to CW-complexes, this weak equivalence is a genuine homotopy equivalence. The second part of the statement now follows from substituting $M = SO(3)$ and $N = K_n$ in the following simple lemma:

Lemma 2.6. *Let M be a CW-complex with $\pi_2(M) = 0$ and $\pi_1(M)$ at most countable. Suppose N is an S^1-bundle over M. Then $\mathscr{L}(N)$ has countably many components and we have a weak homotopy equivalence between identity components $\mathscr{L}^0(N) \simeq \mathscr{L}^0(M)$.*

Proof of the lemma. This fact is an elementary consequence of the usual "path-loop" construction. Fix a base point on N and let $P(N) \simeq *$, be the corresponding based path space. The fibration map $\pi : N \to M$ induces the commutative diagram:

$$\begin{tikzcd} \mathscr{L}(N) \arrow[r, hook] \arrow[d, "\tilde{\pi}"] & P(N) \arrow[r, two heads] \arrow[d, "\pi"] & N \arrow[d, "\pi"] \\ \mathscr{L}(M) \arrow[r, hook] & P(M) \arrow[r, two heads] & M \end{tikzcd} \tag{2.4}$$

By assumption, the projections $\pi_* : \pi_k(N) \to \pi_k(M)$, are isomorphisms for $k \geq 2$, and the circle fiber and its multiples are non-zero in $\pi_1(N)$. From the commutative diagram of the long exact sequence of homotopy groups induced by (2.4), we deduce that:

$$\tilde{\pi}_* : \pi_k(\mathscr{L}(N)) \xrightarrow{\cong} \pi_k(\mathscr{L}(M)) \ , \quad \text{when } k \geq 1; \tag{2.5}$$

Moreover, we have noticed that $\pi_1(N)$ is the central extension of $\mathbb{Z}$ and $\pi_1(M)$, hence the lemma follows. □

This conclude the proof of Proposition 2.5 □

2.3. **The loop group of the contact isometries of $L(n,1)$.** In this section, we prove the second part of Theorem 1.1 by showing that a natural inclusion map is a weak homotopy equivalence. Unlike the usual notion of contactomorphism which preserves only the contact structures, we need to consider the automorphisms of $L(n,1)$ called *contact isometries*. These are diffeomorphisms which preserve the contact form λ_n and the round metric induced from the round metric on S^3 under projection. We denote the group of contact isometries of the lens spaces of $L(n,1)$ as $\mathcal{C}\,\mathrm{Iso}_n$. It acts on $L(n,1)$ in such a way that the Reeb orbits are preserved. Therefore, if we think of $L(n,1)$ as a unit circle bundle in $\mathscr{O}(n)$ with the Reeb orbits as the circle fibers, there is an induced isometric action of $\mathcal{C}\,\mathrm{Iso}_n$ on the base $\mathbb{C}P^1$ endowed with the standard round metric. Also, since the action on the fibers is linear, there is a natural inclusion $\mathcal{C}\,\mathrm{Iso}_n \hookrightarrow \mathrm{Aut}_\omega(\nu)$. Therefore, along with (2.3), one obtains the following diagram of fibrations :

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] & \mathcal{C}\,\mathrm{Iso}_n \arrow[r, two heads] \arrow[d, hook] & SO(3) \arrow[d, hook] \\ \mathcal{G}_\omega(\nu) \arrow[r, hook] & \mathrm{Aut}_\omega(\nu) \arrow[r, two heads] & \mathrm{Symp}(\mathbb{C}P^1) \end{tikzcd} \tag{2.6}$$

Notice that we have weak homotopy equivalences in both the base and fiber. Therefore, the natural inclusion of $\mathcal{C}\,\mathrm{Iso}_n$ into $\mathrm{Aut}_\omega(\nu)$ is in fact a (weak) homotopy equivalence.

We now want to describe a natural map from $\mathscr{L}(\mathcal{C}\,\mathrm{Iso}_n)$ to $\mathrm{Symp}_c(\mathscr{O}(n)\backslash C_n)$ (or equivalently $\mathrm{Fix}_c^{\mathrm{id}}(C_n)$, see section 2.2) which induces a weak homotopy equivalence. Given Proposition 2.1 this will imply the remainder of Theorem 1.1. To this end, consider the smooth path space

$$P(\mathcal{C}\,\mathrm{Iso}_n) = \{\phi : (-\infty, +\infty) \to \mathcal{C}\,\mathrm{Iso}_n : \phi(t) = id, t \leq 0, \phi(t) = \phi(1), t \geq 1\}.$$

This is just the usual based path space when restricted to $t \in [0,1]$, thus it is a contractible space. Recall from Section 2.1 that, up to a scaling, we may identify symplectically $\mathscr{O}(n)\backslash C_n$ with $sL(n,1)_{t<2}$. Therefore, for $\phi \in P(\mathcal{C}\,\mathrm{Iso}_n)$, one can define the following diffeomorphism of $sL(n,1)$:

$$\phi' : L(n,1) \times \mathbb{R} \longrightarrow L(n,1) \times \mathbb{R} \tag{2.7}$$

$$(x,t) \longmapsto (\phi(t)x, t) \tag{2.8}$$

By definition, $\phi'|_{t\leq 0} = id$, and $\phi'|_{1\leq t\leq 2}$ is a symplectomorphism induced by a contact isometry multiplied by identity in the $\mathbb{R}$-direction. However, ϕ' fails to be a symplectomorphism in general. Let $\omega_0 = d(e^t\lambda_n)$, the canonical symplectic form on $sL(n,1)$, and $\omega_1 = \phi'^*\omega_0$. Then nevertheless we claim that the exact forms $\omega_u = (1-u)\omega_0 + u\omega_1$ are symplectic for all $0 \leq u \leq 1$.

Proof of claim. To see this, arguing by contradiction, note that if an ω_u fails to be symplectic then it has a kernel of dimension at least 2, which must intersect the tangent space to some level $L(n,1) \times \{t\}$ nontrivially. As our $\phi(t)$ are contact isometries this kernel must be the kernel of $d\lambda_n$, namely the Reeb direction. But as the Reeb direction is preserved and $\phi'_*(\frac{\partial}{\partial t})$ always has a positive $\frac{\partial}{\partial t}$ component, the Reeb vector pairs nontrivially with $\frac{\partial}{\partial t}$ under all ω_t.

Given our claim, we can apply Moser's method, see [18], to isotope ϕ' to a symplectomorphism $\tilde{\phi}$ of $sL(n,1)$ compactly supported in $\{t \geq 0\}$. As $\pi'^*\omega_0|_{t\geq 1} = \omega_0|_{t\geq 1}$, indeed on this region ϕ' preserves our primitive $e^t\lambda_n$, Moser's flow will vanish here, and $\tilde{\phi}|_{t\geq 1} = \phi'|_{t\geq 1}$. Next, as ϕ' is translation invariant on $\{t \geq 1\}$ we can perform a symplectic cut at the level of $\{t = 2\}$ so that $\tilde{\phi}$ descends to a compactly supported symplectomorphism of $\mathscr{O}(n)$ preserving the zero-section, that is, we have a map $P(\mathcal{C}\,\mathrm{Iso}_n) \to \mathrm{Stab}_c(C_n)$, $\phi \mapsto \tilde{\phi}$.

Claim: The following diagram of fibrations is commutative and all maps are continuous. The rightmost vertical arrow is a homotopy equivalence:

$$\begin{tikzcd} \mathscr{L}(\mathcal{C}\,\mathrm{Iso}_n) \arrow[r, hook] \arrow[d] & P(\mathcal{C}\,\mathrm{Iso}_n) \arrow[r, two heads] \arrow[d] & \mathcal{C}\,\mathrm{Iso}_n \arrow[d, hook] \\ \mathrm{Fix}_c^{\mathrm{id}}(C_n) \arrow[r, hook] & \mathrm{Stab}_c(C_n) \arrow[r, two heads] & \mathrm{Aut}_\omega(\nu) \end{tikzcd} \tag{2.9}$$

Proof of claim: The second arrow of the first row is simply the restriction of an element ϕ to $\phi(2)$. The continuity of the vertical maps follows from the continuous dependence of solutions of an ODE on initial conditions when applying Moser's method. The rightmost vertical arrow is the one induced from (2.6) and is thus a homotopy equivalence. The commutativity of the diagram (2.9) is straightforward from definitions. □

Now the middle vertical arrow is a homotopy equivalence due to the contractibility of both spaces, see Lemma 2.3, and the rightmost arrow is also a weak homotopy equivalence from the argument at the start of this subsection. Therefore, the leftmost vertical arrow is a homotopy equivalence as well, and provides the desired mapping. Hence, the second part of Theorem 1.1 follows.

3. Space of Symplectic Embeddings of Orbifold Balls

In this section, we study the space of symplectic embeddings of balls with a single conical singularity at the origin. We first briefly recall the two different notions of maps between orbifolds that we use and the related definition of orbifold embeddings. A comprehensive discussion of orbifold structures and of orbifold maps can be found in [4].

Given an orbifold $\mathcal{A}$, we write $|\mathcal{A}|$ for its underlying topological space. An *unreduced* orbifold map $(f, \{\hat{f}\})$ between two orbifolds $\mathcal{A}$ and $\mathcal{B}$ consists of the following data:

(1) a continuous map $f : |\mathcal{A}| \to |\mathcal{B}|$ of the underlying topological spaces;
(2) for all $x \in |\mathcal{A}|$, the choice of a germ $\hat{f}_x$ of local lift of f to uniformizing charts U and V centered at x and $f(x)$.

A *reduced* orbifold map is a continuous map $f : |\mathcal{A}| \to |\mathcal{B}|$ of the underlying topological spaces such that smooth lifts exist at every point. The set of smooth unreduced orbifold maps between $\mathcal{A}$ and $\mathcal{B}$ will be denoted by $C^\infty_{orb}(\mathcal{A}, \mathcal{B})$, while we will write $C^\infty_{red}(\mathcal{A}, \mathcal{B})$ for the set of smooth reduced orbifold maps. Smooth unreduced or reduced diffeomorphisms are defined accordingly by requiring f to be a homeomorphism and all lifts to be smooth local diffeomorphisms. The sets of all unreduced or reduced diffeomorphisms of an orbifold $\mathcal{A}$ can be naturally endowed with a C^∞ topology that make them Fréchet Lie groups. The short exact sequence

$$1 \to \Gamma_{\mathrm{id}} \to \mathrm{Diff}_{orb}(\mathcal{A}) \to \mathrm{Diff}_{red}(\mathcal{A}) \to 1$$

is then a principal bundle whose fiber Γ_{id} is the (discrete) group of all lifts of the identity map.

A smooth (unreduced, resp. reduced) embedding $f : \mathcal{A} \to \mathcal{B}$ is a smooth (unreduced, resp. reduced) orbifold map which is a diffeomorphism onto its image and which covers a topological embedding $f : |\mathcal{A}| \to |\mathcal{B}|$. We will denote by $\mathrm{Emb}_{orb}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Emb}_{red}(\mathcal{A}, \mathcal{B})$ the corresponding embedding spaces.

If all the uniformizing charts are symplectic, and if all the local group actions preserve the symplectic forms, the orbifold atlas is said to be symplectic. Orbifold symplectic maps are then defined in the obvious way. In particular, since the open set of regular points $\mathcal{A}_{reg}$ becomes an open symplectic manifold, orbifold symplectomorphisms restrict to genuine smooth symplectomorphisms of $\mathcal{A}_{reg}$.

Let us write $B_n(\epsilon)$ for a symplectic ball of size ϵ with a single conical singularity of order $n \geq 1$ at the origin, that is,

$$B_n(\epsilon) := B^4(n\epsilon)/\mathbb{Z}_n$$

where $B^4(r)$ stands for the standard ball of radius $\sqrt{r/\pi}$ in $\mathbb{R}^4$. In this paper, we are only interested in the simplest possible embedding spaces between symplectic orbifolds, namely $\mathrm{Emb}_{red}(B_n(\epsilon), B_n(1))$. In that case, it is easy to see that $\mathrm{Emb}_{orb}(B_n(\epsilon), B_n(1))$ consists of smooth symplectic embeddings $f : B^4(n\epsilon) \to B^4(n)$ of standard smooth balls that are equivariant with respect to the standard $\mathbb{Z}_n$ action, and that

$$\mathrm{Emb}_{red}(B_n(\epsilon), B_n(1)) = \mathrm{Emb}_{orb}(B_n(n\epsilon), B_n(n))/\mathbb{Z}_n$$

This follows from the fact that any local smooth lift at the conical point extends uniquely to the whole ball, see [4]. Since the space of smooth symplectic embeddings retracts onto $U(2)$, and since the $\mathbb{Z}_n$ action belongs to the center of $U(2)$, one can show that the space of $\mathbb{Z}_n$-equivariant embeddings of smooth balls is itself homotopy equivalent to $U(2)$, see [22]. Therefore, we have the following results:

Proposition 3.1. *The space of reduced symplectic embeddings* $\mathrm{Emb}_{red}(B_n(\epsilon), B_n(1))$ *is homotopy equivalent to*

$$K_n := U(2)/\mathbb{Z}_n \simeq \begin{cases} SO(3) \times S^1 & \text{if } n \text{ is even, } n \neq 0 \\ U(2) & \text{if } n \text{ is odd} \end{cases}$$

Just as in the smooth case, one can show that the group of compactly supported and reduced symplectomorphisms of the open orbifold ball $B_n(1)$ acts transitively on $\mathrm{Emb}_{red}(B_n(\epsilon), B_n(1))$, see [22]. We get an action fibration

$$\mathrm{Stab}_{c,red}(\iota) \to \mathrm{Symp}_{c,red}(B_n(1)) \to \mathrm{Emb}_{red}(B_n(\epsilon), B_n(1))$$

where $\iota : B_n(\epsilon) \to B_n(1)$ is the inclusion, and where $\mathrm{Stab}_{c,red}(\iota)$ is the subgroup made of those reduced symplectomorphisms that are the identity on the image $\iota(B_n(\epsilon))$. This subgroup is homotopy equivalent to the group of reduced diffeomorphisms that are the identity near the image $\iota(B_n(\epsilon))$. Performing a symplectic blow-up of the ball $\iota(B_n(\epsilon))$, those symplectomorphisms lift to symplectomorphisms of the Hirzebruch surface $\mathbb{F}_n$ that are the identity near the zero section and near the section at infinity. This last group is itself homotopy equivalent to $\mathrm{Symp}_c(sL(n, 1))$. Hence, we get a homotopy fibration

$$\mathrm{Symp}_c(sL(n, 1)) \to \mathrm{Symp}_{c,red}(B_n(1)) \to \mathrm{Emb}_{red}(B_n(\epsilon), B_n(1))$$

which shows that the homotopy equivalence $\mathrm{Symp}_c(sL(n, 1)) \simeq \mathscr{L}K_n$, together with Proposition 3.1, imply the following mild generalization of a fundamental result due to Gromov:

Proposition 3.2. *The group* $\mathrm{Symp}_{c,red}(B_n(1))$ *of reduced, compactly supported symplectomorphisms of an open ball of size* 1 *with a single conical singularity of order* n *at the origin is contractible.*

This completes the proof of Theorem 1.2.

4. Space of Lagrangian $\mathbb{R}P^2$ in $T^*\mathbb{R}P^2$

We prove Theorem 1.3. Let the space of Lagrangian $\mathbb{R}P^2$ in $T^*\mathbb{R}P^2$ be denoted as $\mathcal{L}$. The group of compactly supported Hamiltonian symplectomorphisms of $T^*\mathbb{R}P^2$ acts transitively on $\mathcal{L}$, see [12], and our point of departure is the corresponding action fibration

$$\mathrm{Stab}_c(\mathbf{0}) \hookrightarrow \mathrm{Symp}_c(T^*\mathbb{R}P^2) \twoheadrightarrow \mathcal{L} \tag{4.1}$$

where $\mathrm{Stab}_c(\mathbf{0})$ is the subgroup of $\mathrm{Symp}_c(T^*\mathbb{R}P^2)$ which preserves the zero section $\mathbf{0}$.

Notice that for any Lie group G, $\pi_0(G)$ inherits a natural group structure from G. It is proved in [7] that:

Theorem 4.1. $\mathrm{Symp}_c(T^*\mathbb{R}P^2)$ *is weakly homotopic to* $\mathbb{Z}$. *Moreover, the generator of* $\pi_0(\mathrm{Symp}_c(T^*\mathbb{R}P^2))$ *as a group is the generalized Dehn twist in* $T^*\mathbb{R}P^2$.

We will also make use of the following fact, which may be well-known but for which the authors unfortunately know of no reference:

Lemma 4.2. *Let* $H \hookrightarrow G \twoheadrightarrow B$ *be a homotopy fibration where* $H \lhd G$ *are groups. Then the following two maps in the induced long exact sequence are both group homomorphisms:*

$$\pi_1(B) \xrightarrow{i} \pi_0(H) \xrightarrow{j} \pi_0(G)$$

Proof. Let $x_0 \in B$ be the image of $id \in G$. Given a loop $\alpha : [0,1] \to B$, $\alpha(0) = \alpha(1) = x_0$, let $\bar{\alpha}$ be the lift of α and $i(\alpha)$ be the connected component of H where $\bar{\alpha}(1)$ lies. Consider another loop $\beta : [0,1] \to B$, $\beta(0) = \beta(1) = x_0$, then the lift of concatenation $\alpha\#\beta$ can be chosen to be

$$\overline{\alpha\#\beta}(t) = \begin{cases} \bar{\alpha}(2t), & t \leq \dfrac{1}{2} \\ \bar{\alpha}(1) \cdot \bar{\beta}(2t-1), & t > \dfrac{1}{2} \end{cases}$$

Therefore, $i(\alpha\#\beta) = \overline{\alpha\#\beta}(1) = \bar{\alpha}(1)\bar{\beta}(1)$, verifying the claim for the map i. The fact that j is a homomorphism is trivial because the inclusion $H \hookrightarrow G$ is a homomorphism. $\square$

To compute the homotopy type of $\mathrm{Stab}_c(\mathbf{0})$ we need to consider the diffeomorphism group of $\mathbb{R}P^2$. We have the following result, of which the proof is postponed to the appendix:

Proposition 4.3. *The diffeomorphism group of* $\mathbb{R}P^2$ *is weakly homotopic to* $SO(3)$. *Moreover, the standard inclusion is a weak homotopy equivalence.*

With this understood, we define $\mathrm{Fix}_c(\mathbf{0})$ to be the subgroup of compactly supported symplectomorphisms of $T^*\mathbb{R}P^2$ which fixes the zero section pointwise. We obtain a further action fibration:

$$\mathrm{Fix}_c(\mathbf{0}) \hookrightarrow \mathrm{Stab}_c(\mathbf{0}) \twoheadrightarrow \mathrm{Diff}(\mathbb{R}P^2).$$

We may also consider the following object: given the standard round metric g_0 on $\mathbb{R}P^2$, let $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$ be the symplectomorphisms which are compactly supported and induce an isometry on the zero section.

Now we have the following commutative diagram of fibrations:

$$\begin{tikzcd} \mathrm{Fix}_c(\mathbf{0}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0}) \arrow[r, two heads] \arrow[d, hook] & SO(3) \arrow[d, hook] \\ \mathrm{Fix}_c(\mathbf{0}) \arrow[r, hook] & \mathrm{Stab}_c(\mathbf{0}) \arrow[r, two heads] & \mathrm{Diff}(\mathbb{R}P^2) \end{tikzcd} \tag{4.2}$$

From Proposition 4.3, we observe that the vertical arrows on the two sides are weak homotopy equivalences, so the middle one is also a weak homotopy equivalence. Note also that the inverse Liouville flow contracts $T^*\mathbb{R}P^2$ to the zero section. Now, taking into consideration the bundle metric on $T^*\mathbb{R}P^2$ induced by g_0, we may talk about the length of cotagent vectors. By the same direct limit and Liouville flow argument as in Proposition 2.1 we may restrict our attention to the symplectomorphisms supported in $T^*_r\mathbb{R}P^2$, which consists of cotagent vectors with length $\leq r$ for some $r > 0$. We will assume that $r = 1$ below.

Lemma 4.4. *(i)* $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$ *is weakly homotopy equivalent to* $\mathbb{Z}$*;*
(ii) $\pi_0(\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0}))$ *is isomorphic to* $\mathbb{Z}$ *as a group.*

Remark 4.5. It is very tempting to conclude (i) directly from the results in the previous sections by setting $n = 4$, see the first paragraph of the proof. However, the connecting map in (4.2) seems then difficult to understand directly. That is why we use a slightly different argument below.

of Lemma 4.4. We first notice the following fact: a symplectomorphism which fixes a smooth Lagrangian pointwise also fixes the framing of the Lagrangian. This follows from the corresponding linear statement that, a symplectomorphism of T^*M which is linear on the fibers is indeed a cotangent map of a diffeomorphism on the base. This is also used in [5], proof of Theorem 1.3. It follows from this that the subgroup of $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$ consisting of maps which act on a neighborhood of $\mathbb{R}P^2$ by the cotangent map of an isometry of $\mathbb{R}P^2$ is weakly homotopy equivalent to $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$. Therefore we are able to consider this subgroup instead of $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$, and use the same notation to denote it throughout the rest of the proof.

Given $\psi \in \mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$, denoting the cotangent map of $\psi|_{\mathbb{R}P^2}$ as c_ψ, we may consider the symplectomorphism $\tilde{\psi} := c_\psi^{-1} \circ \psi$ on $T^*\mathbb{R}P^2$. The map $\tilde{\psi}$ is not compactly supported in $T^*\mathbb{R}P^2$, but it fixes $\mathbb{R}P^2$ pointwise and thus (by our assumption that the maps are cotangents near $\mathbb{R}P^2$) also a neighborhood. Since ψ preserves the round metric on $\mathbb{R}P^2$, c_ψ preserves the Reeb vector field on the level sets of $T^*\mathbb{R}P^2$. Therefore, by a symplectic cut on the level set $r = 1$, one obtains a symplectomorphism ψ' of $T^*\mathbb{R}P^2$ cut along the level $r = 1$. This symplectic manifold is just $\mathbb{C}P^2$ with the standard symplectic form ω_{FS}, see [3] and [14]. From the construction, ψ' preserves the symplectic reduction of the boundary, a symplectic (+4)-sphere which is indeed the quadratic sphere $\{[x, y, z] \in \mathbb{C}P^2 : x^2 + y^2 + z^2 = 0\}$, and it fixes a neighborhood of the standard Lagrangian $\mathbb{R}P^2 = Re(\mathbb{C}P^2)$. Removing the Lagrangian $\mathbb{R}P^2$, one sees that ψ' descends to a compactly supported symplectomorphism of $\mathscr{O}(4)$, which is denoted as $\bar{\psi}$. Define H to be the image of the bar assignment $\psi \mapsto \bar{\psi}$, H is clearly homeomorphic to $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$. In the rest of the proof we investigate the homotopy type of H.

Lemma 4.6. *Let U be a sufficiently small neighborhood of the zero section in $\mathscr{O}(4)$, then $H|_U = SO(3)$, and it acts transitively on the zero section.*

Proof. From the construction, there is a surjective map $f : SO(3) \to H|_U$. But remembering that points of the zero section in $\mathscr{O}(4)$ corresponds to the lifts of a geodesic, the map f is clearly injective. ☐

Remark 4.7. There is an interesting model described to the authors by Yi Liu. Consider $\mathbb{R}^3$ with the standard Euclidean metric g_E. Consider an oriented normal frame (e_1, e_2, e_3) as a point on $\mathbb{R}P^3$, it fibers over S^2 by projection to e_1. Let $\varpi : \mathbb{R}P^3 \to \mathbb{R}P^3$ be the involution sending (e_1, e_2, e_3) to $(-e_1, -e_2, e_3)$ and consider its quotient $L(1, 4)$. This can be identified with the unit cotengent bundle of $\mathbb{R}P^2$ and fibers over $\mathbb{R}P^2$ by the projection

$$[e_1, e_2, e_3] \to [e_1] \tag{4.3}$$

with fiber S^1. On the other hand, one may project $L(1, 4)$ to S^2 by sending

$$[e_1, e_2, e_3] \to e_3 \tag{4.4}$$

Endow all the spaces involved with the metric inherited by g_E and use the obvious $SO(3)$ action on $\mathbb{R}P^2$ as constructed, then the projections interplay correctly with the symplectic structure on $T^*\mathbb{R}P^2$. In other words, given an isometry of $\mathbb{R}P^2$, represented by an element $R \in SO(3)$ in the above model, the corresponding action on the unit cotangent bundle is described by the same element R acting on $L(1, 4)$. In turn R acts on the fibration (4.4). In this way we retrieve the action of $H|_U$.

Returning to the proof of Lemma 4.4, given the round metric g on the zero section C_4 of $\mathscr{O}(4)$, we consider the subgroups

$$\begin{aligned}
\mathrm{Stab}_c(C_4) &= \{\psi \in \mathrm{Symp}_c(\mathscr{O}(4)) \ : \ \psi \text{ preserves the zero section } C_4\} \\
\mathrm{Stab}_c^{\mathrm{Iso}}(C_4) &= \{\psi \in \mathrm{Stab}_c(C_4) \ : \ \psi \text{ restricted to the zero section} \\
&\qquad\qquad \text{is an isometry with respect to the metric } g\} \\
\mathrm{Fix}_c(C_4) &= \{\psi \in \mathrm{Stab}_c^{\mathrm{Iso}}(C_4) \ : \ \psi|_{C_4} = \mathrm{id}\}
\end{aligned}$$

We then again have a commutative diagram of fibrations:

$$\begin{tikzcd}
\mathrm{Fix}_c(C_4) \arrow[r, hook] \arrow[d, equal] & \mathrm{Stab}_c^{\mathrm{Iso}}(C_4) \arrow[r, two heads] \arrow[d, hook] & SO(3) \arrow[d, hook] \\
\mathrm{Fix}_c(C_4) \arrow[r, hook] & \mathrm{Stab}_c(C_4) \arrow[r, two heads] & \mathrm{Symp}(\mathbb{C}P^1)
\end{tikzcd} \tag{4.5}$$

Using the fact that the embedding of $SO(3)$ into $\mathrm{Symp}(\mathbb{C}P^1)$ is a weak homotopy equivalence and Lemma 2.3, we deduce that $\mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$ is also weakly contractible.

Now consider the subgroup $H \subset \mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$. We construct the following group homomorphism from $\mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$ to the gauge group:

$$\phi : \mathrm{Stab}_c^{\mathrm{Iso}}(C_4) \longrightarrow \mathrm{Map}(S^2, Sp(2)) \simeq S^1$$

To define ϕ, let $t \in \mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$. Then $t|_{C_4}$ acts on the zero section C_4 isometrically. By Lemma 4.6 and the remark following it, there exists an element $u \in H$ such that $u|_{C_4} = t|_{C_4}$. Now we define $\phi(t)$ to be the gauge of $t \cdot u^{-1}$.

Notice first that for any $u \in H$, its action on the normal bundle of C_4 is uniquely determined by its action on C_4, hence $\phi(t)$ does not depend on the choice of u and is well-defined.

The homomorphism ϕ is clearly surjective by Lemma 2.4 since $\mathrm{Fix}_c(C_4) \subset \mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$.

It is also not hard to verify that $\ker(\phi) \simeq H$: indeed, for $t \in \ker(\phi)$, by definition there exists $u \in H$, such that $t \cdot u^{-1}$ acts trivially on the normal bundle of C_4. However, up to homotopy, $\ker(\phi)$ consists of t for which there is a $u \in H$ such that $t \cdot u^{-1}$ acts trivially on a neighborhood of C_4. As all elements acting trivially on a neighborhood lie in H, we deduce that all such t lie in H too.

Therefore we have the following fibration:

$$H \hookrightarrow \mathrm{Stab}_c^{\mathrm{Iso}}(C_4) \twoheadrightarrow S^1$$

which implies that H is weakly homotopy equivalent to $\mathbb{Z}$ and, by Lemma 4.2, that $\pi_0(S) \cong \mathbb{Z}$ since $\mathrm{Stab}_c^{\mathrm{Iso}}(C_4)$ is contractible. This concludes our proof of Lemma 4.4. □

Proof of Theorem 1.3: For $\pi_i(\mathscr{L})$, $i \geq 1$ the theorem follows immediately from Lemma 4.4 and the homotopy fibration (4.1). Since the Dehn twists are also contained in the subgroup $\mathrm{Stab}_c^{\mathrm{Iso}}(\mathbf{0})$, one sees that the map $\pi_0(\mathrm{Stab}_c(\mathbf{0})) \to \pi_0(\mathrm{Symp}_c(T^*\mathbb{R}P^2))$ is surjective. However, by Lemma 4.2, since both groups are $\mathbb{Z}$, it can only be an isomorphism. □

Appendix A. The Diffeomorphism Group of $\mathbb{R}P^2$

We give a proof of Proposition 4.3:

Proof. Thinking of $\mathbb{R}P^2 = S^2/\sim$, where the equivalence relation identifies antipodal points, the action of $SO(3)$ on S^2 preserves equivalence classes and thus descends to an action on $\mathbb{R}P^2$. Therefore $\mathrm{Diff}(\mathbb{R}P^2)$ contains $SO(3)$ as a subgroup. We will show that the homogeneous space $\mathrm{Diff}(\mathbb{R}P^2)/SO(3)$ is weakly contractible. Fixing an $x \in \mathbb{R}P^2$, first notice that given $f \in \mathrm{Diff}(\mathbb{R}P^2)$, there exists a unique element $\iota_f \in SO(3)$, such that $\iota_f \circ f$ fixes a framing of x, or rather, up to homotopy we may assume it fixes a neighborhood of x. For the uniqueness, we observe that the antipodal map on S^2 fixes the equivalence class of the north pole, say, but reverses the orientation of a framing. Therefore, as the complement of a ball in $\mathbb{R}P^2$ is a Möbius band, we may identify $\mathrm{Diff}(\mathbb{R}P^2)/SO(3)$ with the compactly supported diffeomorphism group of the Möbius band B with the boundary removed, which we denote as $\mathrm{Diff}_c(B)$. This is homotopic to the diffeomorphisms of the closed Möbius band which fix the boundary. Below, we identify B with the bundle $\pi : B \to S^1$ with fibers the unit interval.

We fix a fiber F_0 over $p_0 \in S^1$ and parameterize F_0 as a map

$$F_0 : (-\infty, +\infty) \to B.$$

Define

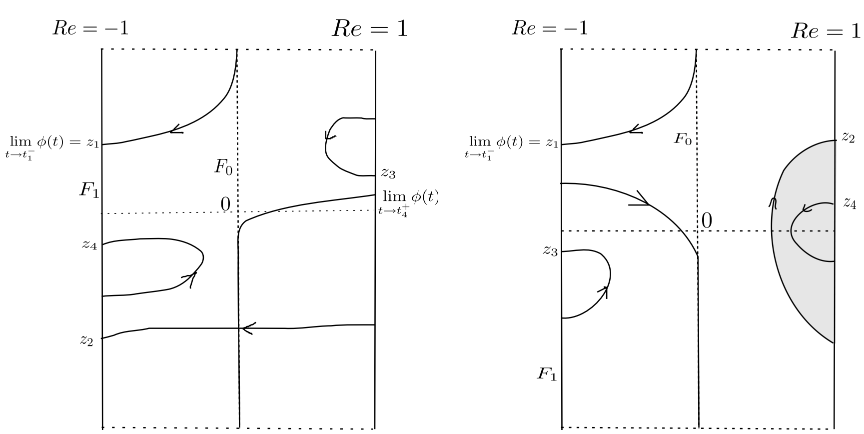
$$\mathcal{F} = \{\phi : (-\infty, \infty) \to B : \phi(t) = F_0(t) \text{ when } |t| > R \text{ for some } R, \phi \text{ is embedded}\}.$$

Then for $\phi \in \mathcal{F}$ we have that $\pi \circ \phi$ is a closed loop in S^1 with a well defined degree. Given this, we partition $\mathcal{F}$ as follows:

$$\mathcal{F}_i = \{\phi \in \mathcal{F} : deg(\pi \circ \phi) = i\}.$$

Lemma A.1. *$\mathcal{F}_i$ is connected when $i = 0$, and empty except when $i = -1, 0, 1$. Curves in $\mathcal{F}_{1,-1}$ divide B into two components.*

Proof. Consider the strip $I = \{|Re(z)| \leq 1\}$ in $\mathbb{C}$, then B is obtained by gluing the two edges of the strip by $z \sim (-z)$ if $|Re(z)| = 1$ (see Figure (a)). We denote the distinguished fiber in B obtained from the glued edges by F_1. For a curve $\phi \in \mathcal{F}$,



(a) "Example" of a curve in $\mathcal{F}_2$ (b) Pushing the curve we may assume that it intersects F_1 transversely. We thus have a finite subset T of $\mathbb{R}$, such that $T = \phi^{-1}(F_1)$. Write $T = \{t_i\}$ where the t_i are in increasing order.

Formally, we now consider ϕ as a map $\phi : \mathbb{R} \setminus T \to \mathring{I}$, to the interior of I such that

$$\lim_{t \to t_i^+} \phi(t) = -\lim_{t \to t_i^-} \phi(t).$$

Let $z_i = \phi(t_i) = \lim_{t \to t_i^-} \phi(t)$. For $\phi \in \mathcal{F}_i$ with $i \neq 0$, T must be non-empty.

Claim. There exists an isotopy of ϕ to a curve $\phi' \in \mathcal{F}$ with corresponding points z'_j such that either $Re(z'_j) = -1$ or $Re(z'_j) = 1$ for all j.

Proof of Claim. If $Re(z_i) = -Re(z_{i+1})$ for some i then the image of $\phi|_{(t_i,t_{i+1})}$ is a curve in $\mathring{I}$ converging at both ends to points on the same edge. It is possible that the region formed by $\phi|_{(t_i,t_{i+1})}$ and F_1 contains other such loops (see the shaded area of Figure (b)). If there are no such loops then the region is empty. Hence we can find a j such that $Re(z_j) = -Re(z_{j+1})$ and the region formed by $\phi|_{(t_i,t_{i+1})}$ and F_1 is empty. Now we can perform an isotopy to remove the intersections z_j and z_{j+1} by pushing ϕ across the region. After such an isotopy the number of intersection points with F_1 will reduce by 2 and so after a finite number we must arrive at a curve satisfying our claim.

Given a curve ϕ we may now assume that $Re(z_i) = Re(z_j)$ for all i, j. Without loss of generality suppose that $Re(z_i) = 1$ for all i. Then if $\phi \in \mathcal{F}_0$ we see that ϕ avoids F_1 completely and thus is isotopic to F_0. This proves the first statement.

For the second statement, assume that $|T| \geq 2$, that is, there are intersections z_1 and z_2 with T_1. Then we observe that all paths $\phi|_{(t_i,t_{i+1})}$ must lie beneath $\phi|_{(t_1,t_2)}$ for all $i \geq 2$, and thus cannot converge towards $+\infty$ in I. This gives a contradiction thus proving the second statement. The final statement is similarly clear. $\square$

Corollary A.2. *The space* $\mathrm{Diff}_c(B)$ *is connected.*

Proof. Indeed, any $f \in \mathrm{Diff}_c(B)$ maps F_0 to a path which, like F_0 cannot divide B. Thus, by Lemma A.1 the image of F_0 lies in $\mathcal{F}_0$ and, moreover, we may assume up to isotopy that f fixes F_0, and by a further isotopy a neighborhood of F_0 and the complement of a compact set in B. But removing a tubular neighborhood of the boundary and F_0 from B leaves a set diffeomorphic to a disk, and as diffeomorphisms of the disk are connected, see [21], our corollary follows. $\square$

Recall that to prove Proposition 4.3 we must show that $\mathrm{Diff}_c(B)$ is contractible. Line fields on B are maps from B to its projectivized unit tangent bundle, where we identify vectors differing up to sign. We will only consider fields which are trivial, that is, coincide with fibers of B, outside of a compact set. The bundle is trivialized by the fibers of B and so line fields are equivalent to maps from B to S^1. Let l_0 be the trivial line field tangent to the fibers. The space of sections $\mathcal{L}_0$ homotopic to l_0 is contractible as all such sections lift to maps to $\mathbb{R}$ with compact support.

Given an $f \in \mathrm{Diff}_c(B)$ the line field $f_* l_0$ is homotopic to l_0 by Corollary A.2. Thus we have a continuous map $\mathrm{Diff}_c(B) \to \mathcal{L}_0$. There is also an inverse map which is well defined at least up to homotopy. For this we need the following claim.

Claim: Line fields in $\mathcal{L}_0$ *have no closed loops.*

Proof of the Claim: We first observe that any closed loops must project from B to S^1 with degree 1 or 2 (up to a sign). This is because line fields lift to line fields on an annulus, and it is easily seen that only the generating homotopy class here can admit a closed orbit. Furthermore, if the loop has degree 2 then it bounds a compact region G in B. Up to an isotopy we may assume that $G \cap F_0$ is an interval and we have a return map from $G \cap F_0$ to itself which reverses the two boundary points. The return map must have a fixed point which corresponds to a loop of degree 1.

Next, we observe that the set of line fields in $\mathcal{L}_0$ which have a closed loop of degree 1 are open. This is because the Poincaré return map defined on a suitable interval transverse to the loop is orientation reversing, so the fixed point is stable. As this subset of $\mathcal{L}_0$ is also closed, and l_0 has no closed loops, we deduce that our subset must be empty, proving the claim. □

Now, starting with a line field in $\mathcal{L}_0$, given the above claim all integral curves correspond to fibers of B outside of a compact set. Thus, following these curves we get a orientation preserving diffeomorphism from S^1 (thought of as the boundary of B) to itself with the following properties: it does not have any fixed points, and squares to identity.

Claim: The space of such diffeomorphisms, denoted as D, is contractible.

Proof of the Claim: Fix a point $x_0 \in S^1$. Given $f \in D$, consider $f(x_0)$ which ranges in a contractible set $S^1 \backslash \{x_0\}$. Such assignment $D \to S^1 \backslash \{x_0\}$ is clearly a fibration.

For any choice of $f(x_0)$, the two points x_0 and $f(x_0)$ divides S^1 into two closed intervals I_1 and I_2 (including these two points themselves). Therefore, f is identified with a diffeomorphism from I_1 to I_2 since it is orientation preserving. Such diffeomorphisms are further identified with $\mathrm{Diff}(I_1)$, which is also contractible. (Thinking of the diffeomorphisms as graphs on the interval, it is a convex set.) The claim then follows. □

Notice that deformations in D can be generated by deformations of the line fields near the boundary of B. Therefore there is a deformation retract from $\mathcal{L}_0$ to line fields whose integral curves coincide with the same fiber outside of a compact set. Up to a choice of parameterizing the curves, such line fields generate elements of $\mathrm{Diff}_c(B)$ by mapping the fibers onto the corresponding integral curves. The resulting map, up to homotopy, is an inverse of the natural map $\mathrm{Diff}_c(B) \to \mathcal{L}_0$ described above. Hence, $\mathrm{Diff}_c(B)$ is homotopic to $\mathcal{L}_0$, which is contractible, and the proof is complete. □

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