Math 20580 Name: Solutions

Midterm 1 Instructor:

February 12, 2015 Section:

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + 2x_3 &= 10 \\ x_1 + x_2 &= 3 \end{aligned}$$

Which of the following (x_1, x_2, x_3) are solutions?

(a) $(1,2,1)$, $(-1,4,0)$ and $(-1,-2,3)$
(b) $(1,2,1)$ and $(-1,4,0)$ but not $(-1,-2,3)$
(c) $(1,2,1)$ and $(-1,-2,3)$ but not $(-1,4,0)$
(d) $(-1,4,0)$ and $(-1,-2,3)$ but not $(1,2,1)$
(e) $(1,2,1)$ but not $(-1,4,0)$ or $(-1,-2,3)$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 2 & 3 & 2 & 10 \\ 1 & 1 & 0 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 free. Solutions are $x_2 = 4 - 2x_3$

$x_1 = -1 + 2x_3$

or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$x_3 = 1 \to \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$x_3 = 0 \to \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$

2. For which constants t do the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -4 \\ t \end{bmatrix}$ span all of $\mathbb{R}^3$?

(a) all $t \neq 6$ (b) all $t \neq -30$ (c) all $t \neq 6, -30$ (d) all t
(e) there are no t

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -4 \\ 0 & 6 & t & t \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -5 \\ 0 & 6 & t & t \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & t-6 & t+30 \end{pmatrix}$$

Can never have a row all 0, so vectors always span.

3. Consider the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{v_4} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$.

Which of the following statements are true?

A. $\{\mathbf{v_1}, \mathbf{v_2}\}$ are linearly dependent; ✗

B. $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent; ✓

C. $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$ are linearly dependent; ✓

D. $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ are linearly dependent. ✓

(a) A, B, C and D (b) A, B and C only (c) B, C and D only
(d) B and D only (e) C and D only.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 0 & 3 & -3 \\ 1 & 3 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So v_3, v_4 dependent on previous vectors.

4. Which of the following matrices define a transformation which is one-to-one?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 7 \end{bmatrix}$$

(a) neither A, B nor C (b) A and B only (c) A and C only
(d) B and C only (e) C only.

$T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ so cannot be 1-to-1

B has linearly dependant columns so is not 1-to-1

$$C \longrightarrow \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

So $\det C = -2 \neq 0$ & C is 1-to-1.

5. For which t is the matrix A below invertible?

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 1 & 3 & t \end{bmatrix}$$

(a) $t = 2$ (b) $t \neq 2$ (c) $t = 3$ (d) $t \neq 3$ (e) all t

$$\det A = 1.(2t-6) - 1.\overset{(2t-2)}{\cancel{4}} + 3.4$$

$$= 2t - 6 - 2t + 2 + 12 = 8 \neq 0$$

6. Find the coordinates of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to the basis $\mathcal{B}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

(a) $\begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}$ (c) $\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 4 & 2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{array}\right)$$

7. Find the rank and dimension of the null space (the nullity) of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

$(\text{rank}(A), \text{dimNul}(A)) =$

(a) $(3,1)$ (b) $(2,2)$ (c) $(0,3)$ (d) $(2,1)$ (e) $(1,2)$

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

From this we see 1 free variable, so $\dim \text{Nul}(A) = 1$

& $\text{rank}(A) = 2$

8. Find $\det(AB)$ where

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) 36 (b) 15 (c) 54 (d) 9 (e) 6

$\det A = 6$, $\det B = 1.12 + 0 + 1.(-3) = 9$

$\det(AB) = \det A \det B = 54.$

Part II: Partial credit questions (11 points each). Show your work.

9. Find the solutions to the linear system below in parametric form.

$$x_1 + x_3 + x_4 = 1$$
$$x_1 + 2x_2 + x_3 + 5x_4 = 7$$
$$x_1 + x_2 + x_3 + 4x_4 = 3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 5 & 7 \\ 1 & 1 & 1 & 4 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 4 & 6 \\ 0 & 1 & 0 & 3 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

x_3 free

$x_4 = -1$

$x_2 = 3 - 2x_4 = 5$

$x_1 = 1 - x_3 - x_4 = 2 - x_3$

or
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

10. Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix}, T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}.$$

(a) Find the standard matrix A of T.

$$A = \begin{pmatrix}1 & 2\\-2 & 1\\1 & 0\end{pmatrix}$$

(b) Find a vector $\mathbf{x}$ so that $T\mathbf{x} = \begin{bmatrix}5\\5\\-1\end{bmatrix}$

$$\left(\begin{array}{cc|c}1 & 2 & 5\\-2 & 1 & 5\\1 & 0 & -1\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 2 & 5\\0 & 5 & 15\\0 & -2 & -6\end{array}\right)$$

Soln. $x_2 = 3,\ x_1 = -1$ or $\begin{pmatrix}-1\\3\end{pmatrix}$.

(c) Find a vector $\mathbf{b}$ in $\mathbb{R}^3$ which does not lie in the image of T (or explain why there are none).

Any $\vec{b}$ making $A\vec{x} = \vec{b}$ inconsistent will work.

11. **Find the inverse of A, where**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix}$.

12. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 7 & 4 & -2 \end{bmatrix}.$$

(a) Find a basis for the nullspace Nul(A).

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 7 & 4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

x_3, x_4 free & homog. eqn. has solns. $x_2 = -x_3 + 2x_4$

$x_1 = -2x_2 - x_3 = x_3 - 4x_4$

or $\vec{x} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ & basis is $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

(b) The vector $\mathbf{x} = \begin{bmatrix} 14 \\ -8 \\ 2 \\ -3 \end{bmatrix}$ lies in Nul(A). Find the coordinates of $\mathbf{x}$ relative to the basis found in part (a).

$$\left(\begin{array}{cc|c} 1 & -4 & 14 \\ -1 & 2 & -8 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -4 & 14 \\ 0 & -2 & 6 \\ 0 & 4 & -12 \\ 0 & 1 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

which has unique solution $x_2 = -3, x_1 = 2$

ie. $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$