Symplectic isotopy classes of ellipsoids and polydisks in dimension greater than four

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Abstract

[1)] In any dimension $2n \geq 6$ we show that certain spaces of ellipsoid and polydisk embeddings into a product $B^4 \times \mathbb{R}^{2(n-2)}$ of a 4-ball and Euclidean space, are not path connected. Thus a theorem of McDuff, saying that the space of symplectic embeddings of one 4-dimensional ellipsoid into another is always path connected, fails to be true in higher dimensions.

1 Introduction

We study symplectic embeddings into Euclidean space $\mathbb{R}^{2n}$, with coordinates x_j, y_j, $1 \leq j \leq n$, equipped with its standard symplectic form $\omega = \sum_{j=1}^n dx_j \wedge dy_j$. Often it is convenient to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by setting $z_j = x_j + iy_j$. The basic domains for symplectic embedding problems are ellipsoids E and polydisks P which we define as follows.

$$E(a_1, \ldots, a_n) = \{\sum_j \frac{\pi |z_j|^2}{a_j} \leq 1\};$$

$$P(a_1, \ldots, a_n) = \{\pi |z_j|^2 \leq a_j \,\text{for all}\, j\}.$$

These are subsets of $\mathbb{C}^n$ and so inherit the symplectic structure. A ball of capacity R is simply an ellipsoid $B^{2n}(R) = E(R, \ldots, R)$. We will also frequently use the notation $Z(R) = B^4(R) \times \mathbb{R}^{2(n-2)}$ to denote the product of a 4-ball of capacity R and Euclidean space.

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Definition 1.1. Two embeddings $f_0, f_1 : E(a_1, \ldots, a_n) \to W$ are symplectically isotopic in a symplectic manifold W if there exists a smooth family of symplectic embeddings $f_t : E(a_1, \ldots, a_n) \to W$ for $t \in [0, 1]$, interpolating between the original maps.

We emphasize this definition since in dimension $2n > 4$ there is also a weaker equivalence relation on embeddings, asking just that there is a symplectic isotopy of W mapping the image $f_0(E)$ onto $f_1(E)$. In dimension 4 the two notions are the same; this follows from Gromov's theorem [7] that the compactly supported symplectomorphism group of a ball is contractible. We do not know if the notions coincide in higher dimension, and our examples will be nonisotopic in the sense of Definition 1.1.

In dimension 4, that is, when $n = 2$, a theorem of McDuff says that the space of symplectic embeddings of one ellipsoid into another is always path connected.

Theorem 1.2. *(McDuff [20] Corollary* 1.6*, see also [21]) For any a, b, a', b', the space of symplectic embeddings $E(a, b) \to \mathring{E}(a', b')$ is path connected whenever it is nonempty.*

In this paper we show that McDuff's theorem is not true in higher dimensions, that is, when $n \geq 3$ some spaces of ellipsoid embeddings are not path connected. For example we will show the following.

Theorem 1.3. *There exist nonisotopic symplectic embeddings*

$$E(1.4, 5.59, 5.65, \ldots, 5.65) \to Z(2.83) = B^4(2.83) \times \mathbb{R}^{2(n-2)}.$$

Since $B^4(2.83) \times \mathbb{R}^{2(n-2)}$ is exhausted by ellipsoids $E(2.83, 2.83, S, \ldots, S)$ with $S \to \infty$, we may replace the product $B^4(2.83) \times \mathbb{R}^{2(n-2)}$ in Theorem 1.3 by an $E(2.83, 2.83, S, \ldots, S)$ for any S sufficiently large. Thus we obtain non isotopic ellipsoid embeddings into an ellipsoid. The theorem extends to ellipsoids with parameters satisfying certain inequalities which we state in Theorem 3.1. However, we cannot claim that *all* spaces of ellipsoid embeddings into an ellipsoid are disconnected in dimension greater than 4. In particular, we do not know if the space of embeddings of a ball into an ellipsoid is ever disconnected.

In dimension 4 there are results showing that spaces of embeddings of polydisks are not path connected. The first was due to Floer, Hofer and Wysocki, showing that spaces of embeddings of a polydisk into a polydisk may be disconnected.

Theorem 1.4. *(Floer-Hofer-Wysocki [6] Theorem 4) Let* $\max(a,b) < R < a+b$*. Then* $g_1 : (z_1, z_2) \mapsto (z_1, z_2)$ *and* $g_2 : (z_1, z_2) \mapsto (z_2, z_1)$ *give nonisotopic embeddings* $P(a,b) \to P(R,R)$*.*

Note that if $R > a + b$ then the embeddings are isotopic in $P(R,R)$ through unitary maps. The condition $\max(a,b) < R$ ensures that the g_i have images in $P(R,R)$.

The starting point for our approach to Theorem 1.3 is the following theorem about polydisks embedded in a ball.

Theorem 1.5. *(Hind, [8] Theorem 1.1) There does not exist a Hamiltonian diffeomorphism* ϕ *with support contained in* $B^4(2a+b)$ *such that* $\phi(P(a,b)) \subset \mathring{B}^4(a+b)$*.*

This leads immediately to examples of nonisotopic polydisks symplectomorphic to $P(a,b)$ with $b > 2a$. Indeed, by a symplectic fold, for any $\epsilon > 0$ there exists a symplectic embedding $P(a,b) \to B^4(2a + \frac{b}{2} + \epsilon)$, see [24], Proposition 4.3.9.

It turns out that Theorem 1.5 does have a generalization to higher dimensions, not only for polydisks but also for polylike domains (products of a disk and an ellipsoid) Q which we define as follows.

$$Q(b, a_2, a_3, \dots, a_n) = \{\pi|z_1|^2 \le b, \sum_{j=2}^{n} \frac{\pi|z_j|^2}{a_j} \le 1\}.$$

Then a generalization of Theorem 1.5 is as follows. Note that by inclusion the polylike domain $Q(b, a_2, \dots, a_n)$ sits inside $B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

Theorem 1.6. *Suppose that* $a_2 < b$ *and* $a_j > 2a_2$ *for all* $j \ge 3$*. There does not exist a Hamiltonian diffeomorphism* ϕ *with support contained in* $B^4(2a_2 + b) \times \mathbb{R}^{2(n-2)}$ *such that* $\phi(Q(b, a_2, \dots, a_n)) \subset \mathring{B}^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$*.*

Theorem 1.6 can be easily applied to give examples of nonisotopic polydisks inside products $B^4(R) \times \mathbb{R}^{2(n-2)}$. The folding mentioned above applied to the first two complex coordinates gives a symplectic embedding $P(b, a_2, \dots, a_n) \to B^4(2a_2 + \frac{b}{2} + \epsilon) \times \mathbb{R}^{2(n-2)}$ for any $\epsilon > 0$, and if $2a_2 < b$ then ϵ can be chosen such that $2a_2 + \frac{b}{2} + \epsilon < a_2 + b$. Hence, as $Q(b, a_2, \dots, a_n) \subset P(b, a_2, \dots, a_n) \subset B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$, Theorem 1.6 implies that this embedding cannot be symplectically isotopic to the inclusion. Similarly, if

$a_3 < b$ then by switching the z_1 and z_3 coordinates we get another embedding $P(b, a_2, \dots, a_n) \to \mathring{B}^4(a_2+b) \times \mathbb{R}^{2(n-2)}$. Therefore we have the following corollary about polydisk embeddings.

Corollary 1.7. *Let $a_3, \dots, a_n > 2a_2$ and choose R with $a_2+b < R < 2a_2+b$. Suppose either $2a_2 < b$ or $a_3 < b$. Then there exists a symplectic embedding $P(b, a_2, \dots, a_n) \to B^4(R) \times \mathbb{R}^{2(n-2)}$ which is not symplectically isotopic to the inclusion inside $B^4(R) \times \mathbb{R}^{2(n-2)}$. Furthermore, even the images are not symplectically isotopic.*

It is also possible to work directly with higher dimensional polydisks and obtain a similar result.

Theorem 1.8. *If $a_1 \le a_2 < a_3 \le \dots \le a_n$ and $a_1 + a_3 < R < 2a_1 + a_3$ then the the space of embeddings $P(a_1, \dots, a_n) \to B^4(R) \times \mathbb{R}^{2(n-2)}$ is not path connected.*

More precisely, the embedding $f : (z_1, z_2, z_3, \dots, z_n) \mapsto (z_1, z_3, z_2, z_4, \dots, z_n)$ is not isotopic to any map with image contained in $\mathring{B}^4(a_1 + a_3) \times \mathbb{R}^{2(n-2)}$. In particular, the inclusion is not isotopic to f.

We remark that each of Theorems 1.6 and 1.8 generalize both of Theorems 1.4 and 1.5. For example, in the case of Theorem 1.8, the map f and the inclusion would be isotopic in $B^4(2a_1+a_3) \times \mathbb{R}^{2(n-2)}$ if we could find an isotopy of the (z_2, z_3) plane rotating the polydisk $P(a_2, a_3)$ inside $B^2(a_1+a_3) \times \mathbb{R}^2$ to switch the two coordinates (since then, fixing the remaining coordinates, we would have an isotopy of $P(a_1, \dots, a_n)$ inside $B^4(2a_1+a_3) \times \mathbb{R}^{2(n-2)}$ switching the z_2 and z_3 coordinates). Setting $a_1 = a_2 = a$ and $a_3 = b$ we therefore have a corollary to Theorem 1.8 generalizing Theorem 1.4.

Corollary 1.9. *Let $a < b$ and $b < R < a + b$. Then the two embeddings $g_1 : (z_1, z_2) \mapsto (z_1, z_2)$ and $g_2 : (z_1, z_2) \mapsto (z_2, z_1)$ from $P(a, b)$ into $B^2(R) \times \mathbb{R}^2$ are not symplectically isotopic.*

The proof of Theorem 1.8 is very similar to that of Theorem 1.6, although there are more closed orbits to analyze on the boundary of a polydisk itself. In this paper we focus on the proof of Theorem 1.6 because polylike domains are in some sense closer to ellipsoids, and indeed we will use some analysis from the proof of Theorem 1.6 to deduce the existence of nonisotopic ellipsoids.

Outline of the paper.

The proof of Theorem 1.6 is contained in section 2. The techniques borrow heavily from the proof of Theorem 1.5, but with some additional technicalities due to working in higher dimension. The rough outline is as follows.

In section 2.1 we describe the basic arrangement. The product $B^4(R) \times \mathbb{R}^{2(n-2)}$ for some $R < 2a_2 + b$ is partially compactified to $\mathbb{C}P^2 \times \mathbb{R}^{2(n-2)}$ and the polylike domain Q is approximated by a smooth domain W. We argue by contradiction and assume that there exists a symplectic isotopy W_t, $0 \le t \le 1$, with $W_0 = W$ and $W_1 \subset \mathring{B}^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

Next, in section 2.2 the symplectic manifolds $X_t = (\mathbb{C}P^2 \times \mathbb{R}^{2(n-2)}) \setminus W_t$ are given almost-complex structures with cylindrical ends and we compute index and area formulas for finite energy holomorphic curves. We refer to the series of papers of Hofer, Wysocki and Zehnder, [12], [13], [14], [15], for the definitions and theory of finite energy curves.

In section 2.3 we study moduli spaces $\mathcal{M}_t$ of holomorphic curves corresponding to the W_t. The constituent curves have area bounded above by $R - (a_2 + b)$ and a monotonicity theorem as in [8] implies that $\mathcal{M}_1$ is empty. On the other hand we show that $\mathcal{M}_0$ has a single element. To complete the proof of Theorem 1.6 we prove a compactness theorem, following [3], showing that $\mathcal{M}_0$ and $\mathcal{M}_1$ must be cobordant. This gives the required contradiction.

Finally then we address the case of ellipsoids. The strategy is to show that two nonisotopic embeddings of polylike domains extend to embeddings of ellipsoids. We give the ellipsoid embeddings in section 3.1, the second one is an extension of an embedding of a polylike domain $Q(b, 1, c, \dots, c)$ into $B^4(S) \times \mathbb{R}^{2(n-2)}$ with $S < 1 + b$. However we do not know whether the first embedding restricts to an embedding of the polylike domain which is isotopic the inclusion and therefore Theorem 1.6 does not apply directly to give Theorem 1.3. More precisely the construction of a nonempty moduli space of holomorphic curves in section 2.3 does not apply. The remainder of the proof consists in showing that a relevant moduli space is nevertheless nonempty and this is covered in section 3.2, where the nontriviality of the moduli space is reduced to a Proposition 3.6. This proposition is proven in section 3.3. Throughout the proofs we rely on the specific parameters of the ellipsoids involved to exclude the existence of curves not in our moduli spaces, and hence to establish compactness of these spaces.

2 Finite energy curves.

This section gives a proof by contradiction of Theorem 1.6. Some preliminary analysis is carried out in subsections 2.1 and 2.2, then we complete the proof in subsection 2.3.

2.1 Approximation of Q.

Here we describe our smooth approximation W of $Q = Q(b, a_2, \ldots, a_n)$, together with the closed characteristics on the boundary ∂W. The analysis is similar to that in [8], section 2.1.

We start by fixing δ and ϵ with $0 < \delta << \epsilon$. Recall that our argument will be by contradiction and so we are assuming that there exists a symplectic isotopy $Q_t \subset B^4(R) \times \mathbb{R}^{2(n-2)}$ with $Q_0 = Q$ and $Q_1 \subset B^4(S) \times \mathbb{R}^{2(n-2)}$, where $R < 2a_2 + b$ and $S < a_2 + b$. We will need to assume that ϵ is small relative to both $a_2 + b - S$ and $2a_2 + b - R$. Also, by perturbing the a_j if necessary, we may assume that ϵ, $1/\epsilon$ and the $1/a_j$ are linearly independent over the rationals.

Now we choose a function $f : [0, b] \to [0, 1]$ with $f(0) = 0$, $f(b) = 1$, $f'(x), f''(x) \geq 0$ and with the property that there exists an x_0 such that $f'(x) = \epsilon$ for $x < x_0 - \delta$ and $f'(x) = \frac{1}{\epsilon}$ for $x > x_0 + \delta$.

Given this, we define W as follows. It will be convenient to use symplectic polar coordinates on $\mathbb{R}^{2n} = \mathbb{C}^n$, so we set $R_j = \pi |z_j|^2$ and $\theta_j = \arg z_j \in S^1$.

$$W = \{f(R_1) + \sum_{j=2}^{n} \frac{R_j}{a_j} \leq 1\}.$$

The boundary ∂W is foliated by the Lagrangian tori $L_c = \{R_j = c_j\}$ which degenerate precisely when some of the $R_j = 0$. However, using the coordinates θ_j we can identify the nondegenerate L_c with a fixed torus T^n and the integer homology with $H_1(T^n, \mathbb{Z}) = \mathbb{Z}^n$.

The characteristic foliation $\ker \omega|_{\partial W}$ is generated by the (Reeb) vectorfield

$$R_W = f'(R_1) \frac{\partial}{\partial \theta_1} + \sum_{j=2}^{n} \frac{1}{a_j} \frac{\partial}{\partial \theta_j}.$$

In particular the Reeb vectorfield is tangent to the Lagrangian toric fibers L_c.

The Reeb vectorfield has two kinds of periodic orbits. The first are the elliptic orbits $\gamma^k = \{z_j = 0, j \neq k\} \cap \partial W$, $k = 1, \dots, n$. We use the notation $r\gamma^k$ to denote the r-fold cover of γ^k.

Since the $1/a_j$ are linearly independent all other periodic orbits lie in one of the complex 2-planes $P_k = \{z_j = 0, j \neq 1, k\}$ for $2 \leq k \leq n$. As ϵ, $\frac{1}{\epsilon}$ and $\frac{1}{a_k}$ are linearly independent orbits in these planes are either elliptic or are called hyperbolic and lie in the region where $x_0 - \delta < R_1 < x_0 + \delta$.

Suppose there exists such an R_1 and a rational number written in lowest terms as $\frac{m}{n}$ such that $f'(R_1) = \frac{m}{na_k}$. Then the corresponding torus fiber over $c = (R_1, 0 \dots, 0, a_k(1 - f(R_1)), 0, \dots, 0)$ (the nonzero entries are in positions 1 and k) is foliated by a 1-parameter family of periodic Reeb orbits in the homology class $(m, 0 \dots, 0, n, 0, \dots, 0)$. We denote these orbits by $\gamma^k_{m,n}$. The r-fold cover of $\gamma^k_{m,n}$ is written as $\gamma^k_{rm,rn}$.

Now, if we fix a symplectic trivialization of $T\mathbb{R}^{2n}|_\gamma$, the tangent bundle of $\mathbb{R}^{2n}$ restricted to a closed orbit γ of R of period T, then the derivative of the Reeb flow (extended to act trivially normal to ∂W) gives a map $\psi : [0, T] \to \mathrm{Symp}(2n, \mathbb{R})$, where $\mathrm{Symp}(2n, \mathbb{R})$ is the group of $2n \times 2n$ symplectic matrices. Associated to such a path is a Conley-Zehnder index $\mu(\gamma)$ defined in this case by Robbin and Salamon in [23]. The analogue of Lemma 2.2 in [8] is the following.

Lemma 2.1. *With respect to the standard basis of $\mathbb{R}^{2n}$ the Conley-Zehnder indices are as follows.*

$$\begin{aligned} \mu(r\gamma^k) &= 2r + n - 1 + 2\lfloor \epsilon r a_k \rfloor + 2\sum_{j \neq k} \lfloor \frac{r a_k}{a_j} \rfloor, \text{if} k \neq 1 \\ \mu(r\gamma^1) &= 2r + n - 1 + 2\sum_j \lfloor \frac{\epsilon r}{a_j} \rfloor \\ \mu(\gamma^k_{m,n}) &= 2(m+n) + \frac{1}{2} + (n-2) + 2\sum_{j \neq k} \lfloor \frac{n a_k}{a_j} \rfloor. \end{aligned} \tag{1}$$

2.2 Index and area formulas.

We compactify the open ball $\mathring{B}^4(R)$ by identifying it with the affine part of $\mathbb{C}P^2(R)$, the complex projective plane with its Fubini-Study form scaled so that lines have area R. We are considering a symplectic isotopy

$$Q_t \subset \mathring{B}^4(R) \times \mathbb{R}^{2(n-2)} \subset \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$$

which restricts to an isotopy $W_t \subset \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$ of W.

Let $X_t = \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)} \setminus W_t$ equipped with the restricted symplectic form. We can choose tame almost-complex structures with cylindrical ends J_t on X_t as in [5] and then study finite energy curves asymptotic to closed Reeb orbits as in [12], [13], [14], [15]. It is convenient to define the degree d of a finite energy curve to be its intersection number with $\mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)}$, where $\mathbb{C}P^1(\infty)$ is the line at infinity in $\mathbb{C}P^2(R)$. The basic arrangement here has been described in [8], section 2.2.1, but here we work in $\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$ rather than $\mathbb{C}P^2$.

In this subsection we give an approximate formula for the area and the virtual index formula for finite energy curves, the analogues of Lemmas 2.3 and 2.7 in [8].

Let C be a genus 0 finite energy plane with e^k punctures asymptotic to multiples of γ^k, $1 \le k \le n$, the ith one asymptotic to $r_i^k \gamma^k$, $1 \le i \le e^k$. (Here r_i^k is a natural number depending upon i and k, hopefully this is not too confusing.) Also, let C have h^k punctures asymptotic to hyperbolic orbits in P_k, $2 \le k \le n$, with the ith one asymptotic to $\gamma^k_{m_i^k, n_i^k}$, $1 \le i \le h^k$.

Proposition 2.2. *Up to an error of order ϵ, the symplectic area of C is given by*

$$\mathrm{area}(C) = \int_C \omega = dR - \sum_{i=1}^{e^1} r_i^1 b - \sum_{k=2}^{n} \sum_{i=1}^{e^k} r_i^k a_k - \sum_{k=2}^{n} \sum_{i=1}^{h^k} (m_i^k b + n_i^k a_k).$$

Note that the formula immediately implies that any nonconstant curves (which have positive area) must have degree $d \ge 1$.

Proof. To see this we can glue a disk in ∂W_t to each asymptotic end to produce a closed cycle of degree d in $\mathbb{C}P^2$, which has area dR. The areas of these disks are roughly the negative terms in our formula (the error term comes because our hyperbolic orbits lie on ∂W_t rather than the singular part of ∂Q_t). □

Proposition 2.3. *The virtual index of C in the space of curves with asymp-*

totic limits allowed to vary is given by

$$
\begin{aligned}
\text{index(C)} = (n-3)(2 - \sum_{k=1}^{n} e^k - \sum_{k=2}^{n} h^k) + 6d \\
- \sum_{i=1}^{e^1} (2r_i^1 + n - 1 + 2 \sum_{j} \lfloor \frac{\epsilon r_i^1}{a_j} \rfloor) \\
- \sum_{k=2}^{n} \sum_{i=1}^{e^k} (2r_i^k + n - 1 + 2\lfloor \epsilon r_i^k a_k \rfloor + 2 \sum_{j \neq k} \lfloor \frac{r_i^k a_k}{a_j} \rfloor) \\
- \sum_{k=2}^{n} \sum_{i=1}^{h^k} (2(m_i^k + n_i^k) + (n-2) + 2 \sum_{j \neq k} \lfloor \frac{n_i^k a_k}{a_j} \rfloor).
\end{aligned}
\tag{2}
$$

Note here that each elliptic limit not a cover of γ^1 contributes a negative term on the third line of the index formula, the limits asymptotic to γ^1 contribute negative terms on the second line, and the hyperbolic limits each contribute a negative term on the last line.

Proof. The general index formula for genus 0 curves with s negative ends is

$$\text{index}(C) = (n-3)(2-s) + 2c_1(C) - \sum_{i=1}^{s} (\mu(\gamma_i) - \frac{1}{2}\text{dim}V_i).$$

For this formula, see [2]. Here $c_1(C)$ is the Chern class which we have normalized to be $3d$, where d is the degree, $\mu(\gamma_i)$ is the Conley-Zehnder index of the limiting Reeb orbit γ_i corresponding to the ith end, and $\text{dim}V_i$ is the dimension of the family of orbits containing γ_i. In our case this dimension is 0 for an elliptic orbit and 1 for a hyperbolic orbit. Substituting the Conley-Zehnder indices from Lemma 2.1 we get the formula as required. □

In the remainder of this subsection we record a few algebraic consequences of the area and index formulas.

Lemma 2.4. *Suppose that a finite energy curve* C *has degree* 1 *and* $\text{area}(C) \leq a_2$ *(up to an error of order* ϵ*). Then* C *either has a single hyperbolic asymptotic limit* $\gamma^2_{1,1}$*, or all asymptotic limits are elliptic and satisfy*

$$b < \sum_{i=1}^{e^1} r_i^1 b + \sum_{k=2}^{n} \sum_{i=1}^{e^k} r_i^k a_k < 2a_2 + b.$$

Proof. As nonconstant curves must have positive area, the area inequality is equivalent to

$$R - a_2 \leq \sum_{i=1}^{e^1} r_i^1 b + \sum_{k=2}^{n} \sum_{i=1}^{e^k} r_i^k a_k + \sum_{k=1}^{n} \sum_{i=1}^{h^k} (m_i^k a_k + n_i^k b) \leq R.$$

As $a_2 + b < R < 2a_2 + b$ (and ϵ is small relative to the differences) this gives

$$b < \sum_{i=1}^{e^1} r_i^1 b + \sum_{k} \sum_{i=1}^{e^k} r_i^k a_k + \sum_{k=1}^{n} \sum_{i=1}^{h^k} (m_i^k a_k + n_i^k b) < 2a_2 + b.$$

Since $a_k > 2a_2$ for all $k \geq 3$ we see that if there exists a hyperbolic orbit it must be of type $\gamma_{1,1}^2$ and be the only asymptotic limit. On the other hand, if all limits are elliptic then they satisfy the inequality of the lemma. □

Lemma 2.5. *Suppose that a finite energy curve C has degree 1, virtual index at least -1, and only elliptic asymptotic limits. Then it has only a single asymptotic limit, that is, C is a finite energy plane.*

Proof. Let E be the total number of elliptic asymptotic limits. Since all terms in the sums on the second and third lines of the index formula of Proposition 2.3 are at least $n+1$, we have the formula

$$-1 \leq \text{index(C)} \leq (n-3)(2-E) + 6 - (n+1)E = 2(n-(n-1)E).$$

Hence $(n-1)E \leq n$ and so as $n \geq 3$ we have $E \leq 1$ as required. □

Putting the previous two lemmas together we have the following, which describes the curves we will be interested in.

Lemma 2.6. *Suppose that a finite energy curve C has degree 1 and $\text{area}(C) \leq a_2$ and $\text{index}(C) \geq -1$. Then C is a finite energy plane asymptotic to either $\gamma_{1,1}^2$, $2\gamma^1$ or $2\gamma^2$.*

Proof. By Lemmas 2.4 and 2.5, if the curve C is not asymptotic to $\gamma_{1,1}^2$ then it is a finite energy plane asymptotic to a cover of one of the γ^k, say asymptotic to $r\gamma^k$.

Suppose first that $k = 1$. Then by Lemma 2.4 we have $b < rb < 2a_2 + b$ and Proposition 2.3 again implies that $r \leq 2$. Putting the two together we have $r = 2$.

Next suppose that $k = 2$. Again by Lemma 2.4 we have $b < ra_2 < 2a_2 + b$ and by Proposition 2.3 we have

$$\text{index}(C) \leq (n-3) + 6 - (2r + (n-1)).$$

As $\text{index}(C) \geq -1$ this implies that $r \leq 2$. Since by our original hypothesis $a_2 < b$, combining the two inequalities gives $r = 2$.

Finally suppose that $k \geq 3$. By hypothesis $a_k > 2a_2$ and so the term $2\lfloor \frac{ra_k}{a_2} \rfloor$ in the index formula contributes at least 4. Hence

$$\text{index}(C) \leq (n-3) + 6 - (2r + (n-1) + 4) \leq -2r$$

a contradiction as required. □

2.3 Moduli spaces of finite energy planes.

Let us fix an orbit η_t of type $\gamma^2_{1,1}$ in each ∂W_t. Consider the corresponding moduli space

$$\mathcal{M}_t = \{u : \mathbb{C} \to X | \text{degree}(u) = 1, \overline{\partial}_{J_t} u = 0, u \sim \eta_t\}/G$$

where $u \sim \eta$ means that u is asymptotic at infinity to η, and G is the reparameterization group of $\mathbb{C}$. The area formula of Proposition 2.2 gives says that curves in $\mathcal{M}_t$ have area roughly $R - (a_2 + b)$.

We will need to choose the almost-complex structure J_0 such that the line at infinity $\mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)}$ is complex and such that it is invariant with respect to the T^{n-2} action rotating the $(z_3, \ldots, z_n)$ planes. This is possible since $W = W_0$ is invariant under the same action. A genericity assumption will also be made as explained in Lemma 2.8. The almost-complex structure J_1 can be assumed to be the standard product integrable structure on $(\mathbb{C}P^2(R) \setminus B(S)) \times \mathbb{R}^{2(n-2)}$ for some $S < a_2 + b$, as $W_1 \subset \mathring{B}(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

Lemma 2.7. *The virtual dimension of $\mathcal{M}_t$ is 0.*

Proof. Proposition 2.3 gives virtual dimension 1 for finite energy planes of degree 1 asymptotic to an orbit of type $\gamma^2_{1,1}$. However a curve lies in $\mathcal{M}_t$ only if it is asymptotic to the specific orbit η_t, and this imposes a 1-dimensional constraint. □

The moduli spaces when $t = 0, 1$ are easily described.

Lemma 2.8. *There exists an almost-complex structure J_0 such that the moduli space $\mathcal{M}_0$ consists of a single, regular curve.*

As this is a direct generalization of Lemma 2.8 in [8], utilizing the analysis in [10] to extend the results to higher dimension, we restrict here to an outline.

Outline of proof. As J_0 is invariant under rotations of the $(z_3, \ldots, z_n)$ planes, the (z_1, z_2)-plane $P_1 = \{z_3 = \cdots = z_n = 0\}$ is J_0-invariant. Hence J_0 can be restricted to $Y = X_0 \cap P_1$ to give an almost-complex manifold with a cylindrical end over $\partial Y := \partial W_0 \cap P_1$. The almost-complex manifold Y is exactly the one studied in [8], and elements of $\mathcal{M}_0$ lying in Y form a moduli space $\tilde{\mathcal{M}}_0$ in their own right. In particular Lemma 2.8 from [8] implies that $\tilde{\mathcal{M}}_0$ is nonempty, that is, there exists an element of $\mathcal{M}_0$ lying in Y. To complete the proof we will show first that there can be no more than one element of $\tilde{\mathcal{M}}_0$ and second that, for a generic choice of invariant J_0, all elements of $\mathcal{M}_0$ must lie in Y. Lemma 3.17 in [10] shows that, for invariant almost-complex structures, curves in $\tilde{\mathcal{M}}_0$ are regular in $\mathcal{M}_0$.

For the first part, we argue by contradiction and suppose that two distinct curves u_0 and u_1 represent equivalence classes in $\tilde{\mathcal{M}}_0$. Automatic regularity in dimension 4 (see [26], Theorem 1, or the discussion after Theorem 2.9 in [8]) implies that u_0, say, can be included in a local 1-parameter family of curves u_t, $-\epsilon < t < \epsilon$, with a single curve in the family asymptotic to each $\gamma^2_{1,1}$ orbit close to η_0. Meanwhile, as u_0 and u_1 are both asymptotic to η_0, on a suitable subset of the cylindrical end $(-\infty, S_0] \times \partial Y$ we can represent u_1 as a section ξ of the normal bundle to the image of u_0. Furthermore, if S_0 is sufficiently negative, the section ξ has no zeros and so defines a winding of u_1 about u_0. For this see [14]. This winding is the same as the winding of an eigenvector of an asymptotic operator associated to the orbit η_0, and as we are dealing with a negative puncture the associated eigenvalue must be positive.

Now, the asymptotic operator acts on sections of the normal bundle to η_0 in ∂Y, which has an induced complex structure still called J_0. With respect to a basis of the normal bundle extending a tangent vector to the space of $\gamma^2_{1,1}$ orbits, the asymptotic operator takes the form

$$-J_0 \frac{d}{dt} - T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where T is the period of η_0. We see that the only eigenvectors with winding number 0 have eigenvalues 0 or $-T$ and so can conclude that in this basis u_1

must wind around u_0. Hence u_1 must intersect the u_t, which have winding 0 because they are asymptotic to different orbits. However, by gluing planes inside W_0 the images of u_1 and the u_t can be included in cycles of degree 1 in $\mathbb{C}P^2$, which therefore have intersection number 1. The added planes can be assumed to have a unique (positive) intersection point at the origin and so the intersections of u_1 and u_t contribute 0. This contradicts positivity of intersection.

For the second part of the proof we must exclude curves in $\mathcal{M}_0$ not lying in Y. As J_0 is T^{n-2} invariant, any such curves appear in $(n-2)$-dimensional families and so are certainly not regular. Hence if we are able to assume that J_0 is regular for $\mathcal{M}_0$ and at the same time T^{n-2} invariant then no such curves exist. The proof of the existence of regular invariant almost-complex structures follows the usual regularity argument working with invariant rather than general almost-complex structures. For this to work, instead of assuming that our holomorphic curves are somewhere injective we need the stronger assumption that corresponding to each curve in $\mathcal{M}_0$ there exists an orbit of the T^{n-2} action which intersects the curve in a single point, see the proof of Proposition 3.16 in [10]. This is automatic in our case since by positivity of intersection a degree 1 curve must intersect $\mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)}$ exactly once transversally, and hence intersect exactly one T^{n-2} orbit in $\mathbb{C}P^1(\infty)\times\mathbb{R}^{2(n-2)}$, in a single point.

□

Lemma 2.9. *For J_1 chosen as above, the moduli space $\mathcal{M}_1$ is empty.*

Proof. This is identical to the proof of Lemma 2.11 in [8]. Indeed, the image of any curve in $\mathcal{M}_1$ can be restricted to a proper curve in $(\mathbb{C}P^2(R) \setminus B(S)) \times \mathbb{R}^{2(n-2)}$. As the complex structure is assumed to be a product the curve projects to a holomorphic curve in $\mathbb{C}P^2(R) \setminus B(S)$, and then by a monotonicity theorem, see [8], Lemma 2.12, we see that it has area at least $R-S$. This is a contradiction as curves in any $\mathcal{M}_t$ have area $R-(a_2+b)$. □

Next we consider the universal moduli space

$$\mathcal{M} = \{(u,t) | u : \mathbb{C} \to X_t, \mathrm{degree}(u) = 1, \overline{\partial}_{J_t} u = 0, u \sim \eta_t, t \in [0,1]\}/G.$$

This has virtual dimension 1, but to show that it is a compact 1-dimensional manifold (the source of our contradiction) we will need some assumptions on the family of almost-complex structures J_t.

First of all, since the curves in $\mathcal{M}$ have degree 1 they are not multiply covered and so we may choose a family J_t so that $\mathcal{M}$ is a 1-dimensional manifold giving a cobordism between $\mathcal{M}_0$ and $\mathcal{M}_1$. The J_t can be chosen to coincide with those we already have when $t = 0, 1$. Indeed, J_0 is regular by Lemma 2.8, and since no curves lie entirely in $(\mathbb{C}P^2(R) \setminus B(S)) \times \mathbb{R}^{2(n-2)}$ we are free to take J_1 standard here and perturb elsewhere to obtain regularity if necessary.

Second, a collection of families $\{J_t\}$ of the second category is regular in the sense that any somewhere injective J_{t_0}-holomorphic finite energy curve, for $t_0 \in [0, 1]$, has deformation index at least -1 (amongst J_{t_0} curves). We will also assume then that our J_t are regular in this sense.

Finally, the cylindrical ends of the X_t are all symplectomorphic, and after identifying them by a symplectomorphism we may assume that all J_t are identical outside of a compact set. This implies that they induce identical translation invariant almost-complex structures on the symplectization $S(\partial W) = \mathbb{R} \times \partial W$. Holomorphic curves in $S(\partial W)$ are either translation invariant, which means they are covers of cylinders over Reeb orbits, or come in families of dimension at least 1. Therefore, if an almost-complex structure is regular, somewhere injective finite energy curves are either trivial cylinders or have deformation index at least 1. As above such almost-complex structures form a subset of the second category and we will assume our J_t induce a structure in this class.

The final lemma is the following, which contradicts Lemmas 2.8 and 2.9.

Lemma 2.10. *The universal moduli space $\mathcal{M}$ is sequentially compact.*

Proof. The general compactness theorem for finite energy curves can be found in [3]. In our situation, it implies that a sequence of finite energy curves u_i representing classes in $\mathcal{M}_{t_i}$ with $t_i \to t_\infty$, after taking a subsequence, converge in the sense of [3] to a holomorphic building in X_{t_∞}. For components in X_{t_∞} to be nonconstant they must have positive degree (see the comment after Proposition 2.2), and so since degree is preserved in the limit and the u_i have degree 1 our limit must consist of a single curve u in X_{t_∞} of degree 1. Therefore the curve is also somewhere injective. By regularity of the family of J_t we have $\text{index}(u) \geq -1$, and as the u_i have area roughly $R - (a_2 + b)$ the area of u is bounded above by $R - (a_2 + b)$. This excludes planes asymptotic to $2\gamma^1$ and hence by Lemma 2.6, the curve u is a finite energy plane asymptotic to either an orbit $\gamma^1_{1,1}$ or to $2\gamma^2$. In the first case, as the limit preserves area, it must be asymptotic to η_{t_∞} itself (as otherwise we would see symplectization components of positive area). Hence (u, t_∞) represents a class in $\mathcal{M}$ and we have compactness as required.

It remains to exclude limiting curves asymptotic to $2\gamma^2$ and for this we look at components of the limit mapping to the symplectization layers $S(\partial W)$. There is a single curve in the highest level with positive end asymptotic to $2\gamma^2$. If the curve is a multiple cover then it has two negative ends, both asymptotic to γ^2. But this leads to a contradiction since our curve has genus 0 and so only one of the ends can connect to the lowest negative end η_{t_∞}. On the other hand, each of these ends have action less than the action of η_{t_∞}. Hence the curve in the highest level is a somewhere injective curve. The negative ends have total action less than $2b$, but one of them must have action at least $a_2 + b$ (if it is to be connected through lower level curves to the negative end at $\gamma_{1,1}^2$). As $2b < R < 2a_2 + b$ (as the component in X_{t_∞} has positive area) any other negative end must have area bounded above by a_2, a contradiction. Thus we see that the only possibility is a cylinder with negative end asymptotic to $\gamma_{1,1}$. A variation of Proposition 2.3 shows that such cylinders have deformation index 0. This is a contradiction to our choice of regular almost-complex structure on $S(\partial W)$ above. □

3 Isotopies of ellipsoids.

The main result of this section is the following theorem, from which Theorem 1.3 follows directly by setting $b = 3/2$ and checking the inequalities.

Theorem 3.1. *Let $A, B, C_3, \ldots, C_n, R$ be parameters satisfying the following inequalities for some $1 < b < 2$.*

i. $1 < A < \frac{3}{2}$;

ii. $1 + \frac{b}{4} < A < \frac{1}{2} + \frac{3b}{4}$;

iii. $\frac{Ab}{A-1} < B < 4A < C_3 < 2R < A + b + 3$;

iv. $\frac{4A-b}{C_3} + \frac{b}{B} < 1$;

v. $R - A < b$;

vi. $R < \min(\frac{b+7}{3}, 2b, \frac{3+2b}{2})$;

vii. $C_3 < C_i$ *for all* $i > 3$;

viii. $C_3 < A(4A - b)$.

Then there exist nonisotopic embeddings $E(A, B, C_3, \ldots C_n) \to B^4(R) \times \mathbb{C}^{n-2}$.

An immediate consequence of these inequalities is that $1 + b < R < 3 < 2 + b$. To see that $R > 1 + b$, by condition (iii) we have $R > 2A$, and by condition (ii) we have $2A > 2 + b/2$. Then as $b < 2$ we have $2 + b/2 > 1 + b$. Also $R < \frac{b+7}{3}$ by condition (vi) and $\frac{b+7}{3} < 3 < 2 + b$ since $1 < b < 2$. Therefore, in dimension 4, the inclusion gives an embedding $P(1, b) \to B^4(R)$ and Theorem 1.5 implies that the inclusion is not isotopic within $B^4(R)$ to an embedding into $\mathring{B}^4(1 + b)$.

In subsection 3.1 we describe our ellipsoid embeddings and state a result on isotopies of polylike domains which implies that the ellipsoids are non-isotopic in the sense of Definition 1.1. In section 3.2 we prove the result on polylike isotopies modulo an existence result for certain holomorphic curves which is reserved for section 3.3.

3.1 Construction of ellipsoid embeddings.

Suppose we are given an ellipsoid $E(B, A, C_3, \ldots C_n)$ with parameters satisfying the inequalities of Theorem 3.1. Note that for notational convenience we have reversed the order of the first two factors. The lower bounds on B and C_3 in condition (iii) imply that $P(1, b) \subset E(A, B)$ and furthermore that $Q(b, 1, 4, \ldots, 4) \subset E(B, A, C_3, \ldots C_n)$. However we need to work with a slightly larger polylike domain.

By condition (iv) there exists a c with $c > 4A - b$ and $\frac{c}{C_3} + \frac{b}{B} < 1$. We claim that $Q(b, 1, c, \ldots, c) \subset E(B, A, C_3, \ldots C_n)$. Indeed, suppose $(z_1, \ldots, z_n) \in Q(b, 1, c, \ldots, c)$, that is, $\pi|z_1|^2 \leq b$ and $(z_2, \ldots, z_n) \in E(1, c, \ldots, c)$. Then we have

$$\begin{aligned} &\frac{\pi|z_1|^2}{B} + \frac{\pi|z_2|^2}{A} + \frac{\pi|z_3|^2}{C_3} + \ldots \frac{\pi|z_n|^2}{C_n} \\ \leq\ & \frac{b}{B} + \frac{c}{C_3}\left(\frac{C_3\pi|z_2|^2}{Ac} + \frac{\pi|z_3|^2}{c} + \cdots + \frac{\pi|z_n|^2}{c}\right) \\ \leq\ & \frac{b}{B} + \frac{c}{C_3}\left(\pi|z_2|^2 + \frac{\pi|z_3|^2}{c} + \cdots + \frac{\pi|z_n|^2}{c}\right) \leq 1 \end{aligned}$$

by our bounds on c. For the inequality between lines two and three note that by condition $(viii)$ we have $C_3 < A(4A - b) < Ac$ and so $\frac{C_3}{Ac} < 1$.

By condition (ii) we have $4A - b < 2 + 2b$ and so we may assume that $4A - b < c < 2 + 2b$. Note that as $4A > 4 + b$ we automatically have $c > 4$ but $c < 2 + 2b < 4 + b < 4A$.

We will produce two embeddings of the ellipsoid which will be shown to restrict to nonisotopic polylike domains.

The embedding f_0.

First note that we have a symplectic embedding g_0 given by the composition

$$E(A, B) := AE(1, \frac{B}{A}) \subset AE(1, 4) \to AB^4(2) = B^4(2A) \subset B^4(R).$$

The first inclusion here follows from the upper bound on $B < 4A$ in condition (iii). The next map $E(1, 4) \to B^4(2)$ can be read off from the classification of ellipsoid embeddings into balls contained in [19], although this particular embedding was also known at least to Opshtein, [22] Lemma 2.1. Finally the inclusion $B^4(2A) \to B^4(R)$ also holds by condition (iii). Taking our map to be the identity in coordinates $z_3, \dots, z_n$ we get an embedding

$$f_0 : E(B, A, C_3, \dots C_n) \to B^4(R) \times \mathbb{C}^{n-2},$$

$$(z_1, \dots, z_n) \mapsto (g_0(z_2, z_1), z_3, \dots, z_n).$$

The embedding f_1.

The construction of the second embedding is more subtle. In particular, to get the restriction to Q we require, we will need to invoke Theorem 1.2.

First we observe again from [19], Theorem 1.1.2, or by inspection of Figure 1.1, that since $C_3 > 4A$ by condition (iii) we have an embedding

$$\tilde{g}_1 : E(A, C_3) := AE(1, \frac{C_3}{A}) \to AB^4(\frac{C_3}{2A}) := B^4(\frac{C_3}{2}) \subset B^4(R)$$

where the final inclusion is also a consequence of condition (iii). As $c < 4A < C_3$ (by condition (iii)) this embedding restricts to an embedding of $E(1, c)$, which is precisely the intersection of our polylike domain with the (z_2, z_3) plane.

Now as $c > 4$ there also exists an embedding

$$E(1, c) \to B^4(\frac{c}{2}) \subset B^4(1 + b) \subset B^4(R)$$

as $c < 2 + 2b$ and $R > 1 + b$ (see the comment directly after the statement of Theorem 3.1). Hence by Theorem 1.2 we may replace our embedding

$\tilde{g}_1 : E(A, C_3) \to B^4(R)$ by an embedding g_1 which restricts to one sending $E(1, c) \to \mathring{B}^4(1+b)$. Extending this map to be the identity in coordinates $z_3, \ldots, z_n$ and composing with a linear map interchanging the first and third coordinates we get another embedding

$$f_1 : E(B, A, C_3, \ldots C_n) \to B(R) \times \mathbb{C}^{n-2},$$

$$(z_1, \ldots, z_n) \mapsto (g_1(z_2, z_3), z_1, z_4, \ldots, z_n).$$

This maps the polylike domain to $\mathring{B}^4(1+b) \times \mathbb{C}^{n-2}$.

Of course, if f_0 and f_1 were isotopic then the images of the polylike domains would also be isotopic. Hence Theorem 3.1 is a consequence of the following.

Theorem 3.2. *Let R, A, B, C_i, b satisfy the inequalities of Theorem 3.1 and $c > 4A - b$ be chosen as above. Let*

$$f_0 : Q(b, 1, c, \ldots, c) \to B^4(R) \times \mathbb{C}^{n-2}$$

be a symplectic embedding which is a restriction of an embedding $E(B, A, C_3, \ldots, C_n) \to B^4(R) \times \mathbb{C}^{n-2}$ of the form

$$f_0(z_1, \ldots, z_n) = (g_0(z_1, z_2), z_3, \ldots, z_n)$$

and let

$$f_1 : Q(b, 1, c, \ldots, c) \to B^4(S) \times \mathbb{C}^{n-2}$$

be a symplectic embedding for some $S < 1 + b$. Then f_0 and f_1 are not isotopic through symplectic embeddings into $B^4(R) \times \mathbb{C}^{n-2}$ which extend to $E(B, A, C_3, \ldots, C_n)$.

The proof of Theorem 3.2 is the subject of subsections 3.2 and 3.3. We observe here though that as $R < 3 < 2+b$ (by the comment directly after the statement of Theorem 3.1), Theorem 3.2 would be a special case of Theorem 1.6 if we knew that the map g_0 were isotopic to the identity.

3.2 Another isotopy obstruction.

3.2.1 Preliminaries.

Here we outline the proof of Theorem 3.2. The proof will follow exactly the same scheme as that of Theorem 1.6. As coordinates $z_4, \ldots z_n$ play no role in the proof, for notational convenience we will restrict to the case of $n = 3$, that is, dimension 6. Then we study embeddings of a polylike domain $Q(b, 1, c)$ which lies in an ellipsoid $E(B, A, C)$. As in section 2 our method is to replace the image of $Q(b, 1, c)$ under f_0 by a smooth domain $W = W_0$ and argue by contradiction assuming there exists a symplectic isotopy W_t for $0 \leq t \leq 1$ with $W_1 \subset B^4(S) \times \mathbb{C}$. The Reeb orbits in ∂W were described in section 2.1 and we make a choice of a specific orbit $\eta_t \subset \partial W_t$ of type $\gamma^2_{1,1}$ and $\sigma_t \subset \partial W_t$ of type $\gamma^2_{1,4}$ for each t. We also choose compatible almost-complex structures J_t on $X_t = \mathbb{C}P^2(R) \times \mathbb{C} \setminus W_t$ exactly as in section 2.2. Now however there are two corresponding moduli spaces of curves we will need to examine, a moduli space $\mathcal{M}_t$ defined as in section 2.3 and a second moduli space $\mathcal{N}_t$. Define

$$\mathcal{M}_t = \{u : \mathbb{C} \to X | \text{degree}(u) = 1, \overline{\partial}_{J_t} u = 0, u \sim \eta_t\}/G$$

where $u \sim \eta$ means that u is asymptotic at infinity to η, and G is the reparameterization group of $\mathbb{C}$, and analogously

$$\mathcal{N}_t = \{u : \mathbb{C} \to X | \text{degree}(u) = 2, \overline{\partial}_{J_t} u = 0, u \sim \sigma_t\}/G.$$

By Lemma 2.3 both moduli spaces have dimension 0. Note that curves in $\mathcal{M}_t$ have area roughly $R - (1 + b)$ and curves in $\mathcal{N}_t$ have area approximately $2R - (4 + b)$, which is less than $R - (1 + b)$ since $R < 3$. Hence Lemma 2.9 implies that both moduli spaces are empty for a suitable choice of J_1 which is the standard product on $X_1 \setminus (B^4(1 + b) \times \mathbb{C})$.

We will choose J_0 as in section 2.3 so that it is invariant under rotations in the z_3 plane. Then in subsection 3.2.3 we prove the following.

Proposition 3.3. *For any choice of invariant J_0 either the moduli space $\mathcal{M}_0$ or the moduli space $\mathcal{N}_0$ is nonempty.*

The uniqueness part of Lemma 2.8 applies again here to show that if these moduli spaces are nonempty then they consist of a single curve which appears transversally, in particular they represent a nontrivial cobordism class. This follows exactly as in Lemma 2.8 for the degree 1 curves, that is, a positivity of intersection argument shows that there exists at most one element of the moduli space in $\{z_3 = 0\}$, and regularity implies that all elements lie in $\{z_3 = 0\}$. The same argument applies to the degree 2 curves once we show that there exists a regular, invariant J_0. The proof that such almost-complex structures exist follows from a stretching argument as in [10], Proposition 3.16.

At this point the proof of Theorem 3.2 breaks into two parts according to whether $\mathcal{M}_0$ or $\mathcal{N}_0$ is nonempty. If $\mathcal{M}_0$ is nonempty for any J_0 then the proof proceeds exactly as Theorem 1.6. Indeed, Lemma 2.9 implies that $\mathcal{M}_1$ is empty for a suitable choice of J_1 and Lemma 2.10 implies that $\mathcal{M}_0$ and $\mathcal{M}_1$ are cobordant. This gives our contradiction.

We do not know if the analogue of Lemma 2.10 applies in full generality for the moduli spaces $\mathcal{N}_t$. However we will see in section 3.2.2 that it does hold for a particular family of J_t. As we already have a proof if $\mathcal{M}_0$ is nonempty for *any* invariant J_0, this will complete the proof.

3.2.2 The moduli space of degree 2 curves.

In this section we analyze the moduli spaces of degree 2 curves

$$\mathcal{N}(J_t) = \{u : \mathbb{C} \to X_t | \text{degree}(u) = 2, \overline{\partial}_{J_t} u = 0, u \sim \sigma_t\}/G$$

where J_t is an almost-complex structure on X_t.

Recall that our isotopy, say ϕ_t, of W extends to one of $E(B, A, C)$. Set $E_t = \phi_t(E(B, A, C))$. We will choose $J_t = J_t^N$ to be an almost-complex structure on X_t stretched to length N along ∂E_t. We may assume that A, B, C are rationally independent. Then there are three distinct closed Reeb orbits on $\partial E(B, A, C)$, namely the intersection of $\partial E(B, A, C)$ with the coordinate planes. We label the closed Reeb orbits in the respective coordinate planes by δ^k for $1 \leq k \leq 3$.

As E_0 and E_1 are invariant under rotations in the z_3 plane we may take J_0^N and J_1^N to be invariant and also, following [10] to be regular.

Lemma 3.4. *For some N sufficiently large, the universal moduli space*

$$\mathcal{N} = \{(u, t) | u : \mathbb{C} \to X_t, \text{degree}(u) = 2, \overline{\partial}_{J_t} u = 0, u \sim \sigma_t, t \in [0, 1]\}/G$$

is sequentially compact.

Proof. We suppose that there exists a sequence of planes $(u_i, t_i) \in \mathcal{N}$ which degenerates to a J_t holomorphic building. A degree 2 plane can degenerate in X_t into either two planes of degree 1 or a single degree 2 plane of smaller area.

Degree 1 planes are necessarily somewhere injective and as in Lemma 2.6 (see also Lemma 2.10) have index at least -1 and area bounded by $2R - (b + 4) < R - (b + 1)$ only if they are asymptotic to either $2\gamma^2$ or $\gamma_{1,1}^2$.

Thus they have area $R-2$ or $R-(b+1)$ and so the sum of the areas is at least $2R-(2b+2)$. As $b<2$ we have $2R-(2b+2)>2R-(b+4)$, a contradiction as the u_i have area $2R-(b+4)$.

Suppose then that the limiting building has a degree 2 curve C in X_t. The area inequalities above imply that C is not a double cover of a degree 1 plane and so it must be somewhere injective and we may assume it has index at least -1 (this is the deformation index amongst J_t holomorphic curves, equivalently it has nonnegative index in a universal moduli space). Now, any symplectization components of our limit are necessarily cylinders (as otherwise, as we are taking a limit of genus 0 curves we will see planar components which must have area at least 1). It is then easy to check that any nontrivial components necessarily have index at least 1, and it follows that our degree 2 plane has index exactly -1, and as planes asymptotic to elliptic orbits have even index, C has a hyperbolic limit.

If the curve is asymptotic to $\gamma^2_{m,n}$ then the index condition gives

$$\text{index} = 12 - (2(m+n) + 1 + 2\lfloor \frac{n}{c} \rfloor) = -1.$$

Hence $m+n+\lfloor \frac{n}{c} \rfloor = 6$. As $c>4$ we see that if $n \leq 4$ then $m+n=6$. In fact, since $m \geq 1$ in all cases we have $m+n \geq 6$. Meanwhile the area inequality gives $0 \leq 2R-(mb+n) \leq 2R-(b+4)$ and the first inequality is a contradiction as $R<3$.

The remaining possibility is if C is a degree 2 curve asymptotic to an orbit $\gamma^3_{m,n}$. Now we have

$$\text{index} = 12 - (2(m+n) + 1 + 2\lfloor nc \rfloor) = -1$$

and hence $m+n+\lfloor nc \rfloor = 6$. As $c>4$ the unique solution is $m=n=1$. For a generic almost-complex structure we do not know whether or not such planes can exist. However we will show that at least they do not exist if N is chosen sufficiently large.

Arguing by contradiction, suppose such planes exist for all N. Then we can take a limit as $N \to \infty$ and the limit will be a holomorphic building with components in $X_t \backslash E_t$ and $E_t \backslash W_t$. The components in $X_t \backslash E_t$ necessarily have even index so by regularity we may assume that the index is nonnegative. An analysis similar to that above implies that these components consist either of two planes of degree 1 or a single plane of degree 2, and all planes must have index 0.

The index formula for planes of degree d in $X_t \setminus E_t$ asymptotic to $r\delta^k$, for $1 \leq k \leq 3$ are given respectively by

$$\text{index} = 6d - (2r + 2 + 2\lfloor \frac{rB}{A} \rfloor + 2\lfloor \frac{rB}{C} \rfloor)$$

$$\text{index} = 6d - (2r + 2 + 2\lfloor \frac{rA}{B} \rfloor + 2\lfloor \frac{rA}{C} \rfloor)$$

$$\text{index} = 6d - (2r + 2 + 2\lfloor \frac{rC}{A} \rfloor + 2\lfloor \frac{rC}{B} \rfloor)$$

Note that our bounds on A and b together with condition (iii) imply that $B > \frac{Ab}{A-1} > 2A$. Therefore setting the index to be 0 we find only either degree 1 planes asymptotic to $2\delta^2$ or degree 2 planes asymptotic to $4\delta^2$. In both cases the total area of the planes in $X_t \setminus E_t$ is $2R - 4A$ which exceeds $2R - (b + c)$ as $c > 4A - b$. This gives a contradiction as required.

□

Lemma 3.4 implies, for large N, that $\mathcal{N}_0(J_0)$ is cobordant to $\mathcal{N}_1(J_1)$. If we assume $\mathcal{N}_0(J_0)$ is nontrivial this implies $\mathcal{N}_1(J_1)$ is also nontrivial for a regular, invariant almost-complex structure $J_1 = J_1^N$. Such almost-complex structures can be shown to exist by following [10], Proposition 3.16. By regularity this means we have a degree 2 curve lying in $X_1 \cap \{z_3 = 0\}$ asymptotic to $\gamma_{1,4}^2$. This is not an immediate contradiction to monotonicity however since J_1 is chosen to be stretched along ∂E_1 rather than the standard product on $X_1 \setminus (B^4(1+b) \times \mathbb{C})$. To derive a contradiction we need to show that such degree 2 curves persist as we deform J_1 to be a product outside of $B^4(1+b) \times \mathbb{C}$. The proof of this is omitted as it is almost identical to the first part of the proof of Lemma 3.4. However we can now work entirely in the 4-manifold $\{z_3 = 0\} = \mathbb{C}P^2(R) \setminus W_1$ which means all curves are asymptotic to orbits in the (z_1, z_2) plane. In particular there is no bubbling to planes asymptotic to $\gamma_{1,1}^3$ orbits, and so we do not need the almost-complex to be stretched along an ellipsoid to arrive at the conclusion.

3.2.3 The proof of Proposition 3.3

Here we prove Proposition 3.3, or rather reduce it to another existence result, namely Proposition 3.6. We recall that the proof in Lemma 2.8 that $\mathcal{M}_0$ was nonempty consisted of essentially writing down an explicit curve. Now, since the map g_0 defining f_0 is nonstandard, a new, indirect, method is required.

On the other hand, as J_0 is invariant under rotations in the z_3 plane it will suffice as in Lemma 2.8 to prove the following in dimension 4.

Proposition 3.5. *Let $W \subset B(R) \subset \mathbb{C}P^2(R)$ be a smoothing of the image of a sympectically embedded polydisk $P(b,1)$ which extends to an ellipsoid $E(B,A)$ with our parameters satisfying the bounds of Theorem 3.1. Then $Y = \mathbb{C}P^2(R) \setminus W$ admits either a finite energy plane of degree 1 asymptotic to an orbit $\gamma_{1,1}$ or a finite energy plane of degree 2 asymptotic to an orbit $\gamma_{1,4}$.*

We note that as we are now working entirely in the (z_1, z_2) plane there is no need for superscripts on our hyperbolic orbits.

Proof. Rescaling slightly we may assume that the image of the polydisk $P(b,1)$ lies in the interior of W. Using coordinates induced from the polydisk, let $L = \{|z_1| = b, |z_2| = 1\}$ be the Lagrangian torus in its distinguished boundary. Now, for any compatible almost-complex structure on $\mathbb{C}P^2(R) \setminus L$, with a cylindrical end symplectomorphic to the complement of the zero-section in the unit cotangent bundle of L, we can study finite energy curves asymptotic to geodesics on L. The key proposition which implies Proposition 3.3 is the following.

Proposition 3.6. *Any almost-complex structure on $Y = \mathbb{C}P^2(R) \setminus W$ extends to an almost complex structure on $\mathbb{C}P^2(R) \setminus L$ such that $\mathbb{C}P^2(R) \setminus L$ admits either a degree 1 finite energy plane of area roughly $R - 1 - b$ or a degree 2 finite energy plane of area roughly $2R - 4 - b$. In either case, the planes have deformation index 1.*

We recall that the degree of a finite energy curve is defined to be its intersection number with the line at infinity $\mathbb{C}P^1(\infty)$. The proof of Proposition 3.6 will involve the use of finite energy foliations following [11] and we postpone this until section 3.3.

Given Proposition 3.6, to complete the proof of Proposition 3.3 we consider a sequence of finite energy planes from Proposition 3.6 with respect to a sequence of almost-complex structures J^N stretched to length N along ∂W.

We divide our proof into cases. First suppose that for each N in a sequence $N \to \infty$ Proposition 3.6 produces finite energy planes of degree 1. Then in the limit we have a finite energy curve u of degree 1 in Y. Proposition 3.3 is completed in this case by the following lemma.

Lemma 3.7. *The curve u has a single end asymptotic to a hyperbolic orbit of type $\gamma_{1,1}$.*

Proof. Suppose that u has e^k punctures asymptotic to multiples of γ^k, now with $k = 1, 2$, the ith one asymptotic to $r_i^k \gamma^k$, $1 \leq i \leq e^k$. Also, suppose u has h punctures asymptotic to hyperbolic orbits with the ith one asymptotic to γ_{m_i,n_i}, $1 \leq i \leq h$. Then as in the area formula of Proposition 2.2, we have

$$\operatorname{area}(u) = R - \sum_{i=1}^{e^1} r_i^1 b - \sum_{i=1}^{e^2} r_i^2 - \sum_{i=1}^{h} (m_i b + n_i)$$

and since we are taking limits of curves of area $R - 1 - b$ this implies

$$1 + b \leq \sum_{i=1}^{e^1} r_i^1 b + \sum_{i=1}^{e^2} r_i^2 + \sum_{i=1}^{h} (m_i b + n_i) \leq R.$$

As $R < 2 + b$ we see that if there is a hyperbolic limit then it is the only limit and is of type $\gamma_{1,1}$.

For elliptic orbits, we first claim that the only possibility is two ends asymptotic to the elliptic orbits γ^1 and γ^2.

Proof of claim. For this, we note that as $R < 3$ and $R < 2b$ (by condition (vi)), the limits cannot cover γ^2 more than twice in total, and cannot cover γ^1 more than once. Furthermore, if we have ends covering γ^2 twice in total, since $b > 1$ the lower bound above implies that there must still be another end, which then contradicts our upper bound.

To conclude, if one end is asymptotic to a cover of γ^1, it is the only end asymptotic to γ^1 and the other ends cover γ^2. Then to satisfy the area inequalities we see that there can only be one more end, which is asymptotic to γ^2. Similarly, if one end is asymptotic to a cover of γ^2 it covers γ^2 exactly once and to satisfy the inequalities we must have another end asymptotic to γ^1. This justifies our claim.

Finally, to exclude these elliptic orbits, recall that as we are taking limits of finite energy planes the limiting building has genus 0, and so only one of the ends can be connected in W to a finite energy curve with an asymptotic limit on L. The other end is then connected to components of area at least 1, and this is a contradiction as the original planes have area $R - 1 - b < 1$. $\square$

In the second case of Proposition 3.3 we now suppose that for each N in a sequence $N \to \infty$ Proposition 3.6 produces finite energy planes of degree 2. We will deduce from this the existence of a degree 2 plane in Y asymptotic to an orbit $\gamma_{1,4}$.

By the same area argument as in Lemma 3.7 the components of the limit in Y can each have only a single negative end, and so the limit contains either a single plane of degree 2 or two planes of degree 1 in Y. We deal with these possibilities separately, starting with the limit containing two planes.

Degree 1 planes are necessarily somewhere injective and have nonnegative index only if they are asymptotic to either an elliptic orbit $r\gamma^1$ or $r\gamma^2$ with $r \leq 2$ or to a hyperbolic orbit of type $\gamma_{1,1}$. As $R < 2b$ by condition (vi), planes asymptotic to $2\gamma^1$ are excluded, and so by Proposition 2.2 all possible planes have area at least $R - (1+b)$. But then the sum of the areas is at least $2(R-(1+b))$ and $2R-(2+2b) > 2R-(4+b)$ since $b < 2$, giving a contradiction as the degree 2 planes from Proposition 3.6 have area $2R-4-b$.

With two planes now excluded, our limit has a single plane of degree 2 in Y. First suppose the plane is asymptotic to $r\gamma^2$. As its area lies between 0 and $2R-4-b$ we have

$$4 + b \leq r \leq 2R$$

which is a contradiction as $4+b > 5$ and $2R < 6$.

Next suppose the plane is asymptotic to $r\gamma^1$. Then the area inequality is

$$1 + \frac{4}{b} \leq r \leq \frac{2R}{b}.$$

But as $b < 2$ the lower bound is greater than 3, and as $R < 2b$ by condition (vi) the upper bound is less than 4. This is another contradiction.

Thus we can conclude that the plane is asymptotic to a hyperbolic orbit, say $\gamma_{m,n}$. A multiply covered curve can be ruled out using area considerations as above, more precisely, the underlying curve has area at least $R-(1+b)$ and so the plane itself has area at least $2(R-1-b)$, which exceeds $2R-4-b$. Hence the plane is somewhere injective and so has nonnegative index. By the index formula in dimension 4 this means

$$m + n \leq 5.$$

We have $\text{area}(u) = 2R-(mb+n)$, and as this is bounded by 0 and $2R-4-b$ we also have the inequality

$$4 + b \leq mb + n \leq 2R.$$

Proposition 3.3 holds if we can show that $m = 1$, for then the lower bound in the area inequality implies that $n \geq 4$ and we get equality as the index inequality says $m + n \leq 5$. Arguing by contradiction then, suppose $m \geq 2$.

If $m = 2$ then $n \geq 4 - b > 2$. Thus as $m + n \leq 5$ we get $n = 3$. But then $mb + n = 2b + 3 > 2R$ by condition (vi). This is a contradiction to the area inequality.

Similarly, if $m \geq 4$ then $mb + n \geq 4b > 2R$, another contradiction.

It remains to exclude the case $m = 3$. Here the lower area bound gives $n \geq 4 - 2b > 0$. The upper area bound gives $3b + n \leq 2R < 4b$ and so $n < b < 2$. We deduce that $n = 1$ and the plane is asymptotic to an orbit $\gamma_{3,1}$.

Recall that u is a component of a holomorphic building arising as the limit of a sequence of degree 2 planes in $\mathbb{C}P^2(R) \setminus L$ of index 1. The sum of the virtual indices of the components of our limit, minus any matching conditions, must also be 1, and we will derive our contradiction from this. The limit has a single component u in Y but may also have components in symplectization $S(\partial W)$ and in $W \setminus L$.

As planar components in W have area at least 1, and as the total area of the building is $2R - 4 - b < 1$, all components of our (genus 0) limiting building in the symplectization $S(\partial W)$ and in $W \setminus L$ are cylinders. Moreover there is a unique component in $W \setminus L$ with its negative end on L. Further, all asymptotic limits of all components are necessarily hyperbolic. Indeed, if a component has elliptic ends we can abstractly glue it to higher level components to produce a degree 2 plane with elliptic limits. This contradicts our area inequalities as above.

Now, the curve u has virtual index 3. We will see in the next section that the curve in $W \setminus L$ has negative end on an indivisible Reeb orbit (that is, the Reeb orbit cannot be written as a sum of shorter orbits). This implies that that the curve is not multiplty covered and so has nonnegative index. A cylindrical component in $S(\partial W)$ with positive end asymptotic to an orbit $\gamma_{m,n}$ and negative end asymptotic to an orbit $\gamma_{m',n'}$ has virtual index

$$\text{index} = 2(m + n - m' - n') + 1.$$

For nontrivial somewhere injective curves this is at least 1 by translation invariance of the symplectization. Hence $m + n - m' - n' \geq 0$. If the cylinder is an r times multiple cover of a nontrivial cylinder as above then the virtual index is $2r(m + n - m' - n') + 1$ which therefore again is at least 1.

Let x be the number of nontrivial symplectization components in the liming building, so we have $x+1$ matching asymptotic orbits. Then invariance of the index gives
$$1 \geq 3 + x - (x+1) = 2$$
and this is our contradiction, completing the proof of Proposition 3.3. □

3.3 Proof of Proposition 3.6.

We recall that we are studying a Lagrangian L which is the distinguished boundary of a polydisk $P(b,1) \subset B^4(R) \subset \mathbb{C}P^2(R)$. Our assumption that the embedding of $P(b,1)$ extends to $E(B,A)$ implies that there is a ball $B^4(A) \subset E(B,A) \subset B^4(R)$ which we may assume is disjoint from L. (Indeed, reducing B if necessary, we can take $L \subset \partial E(B,A)$.) However only the ball of capacity 1 lies in the interior of W. We will study symplectic forms ω_w and corresponding almost-complex structures on $Z = \mathbb{C}P^2(R) \sharp \overline{\mathbb{C}P^1}(w) \setminus L$ given by blowing-up a ball of some capacity w in the interval $[1,A]$, and will prove a refined version of Proposition 3.6. We will always assume that our almost-complex structures leave the exceptional divisor E and the line at infinity $\mathbb{C}P^1(\infty)$ complex. In fact this is precisely the arrangement considered in [11] and we follow those methods closely. Using coordinates on the polydisk we can describe the homology class of an oriented geodesic on L by a pair $(k,l) \in \mathbb{Z}^2$.

Proposition 3.8. *Let Z be as above with $w \in [1,A]$. Then either there exists a degree 1 finite energy plane asymptotic to a geodesic in the class $(-1,-1)$ and disjoint from E, or there exists a degree 2 finite energy plane asymptotic to a geodesic in the class $(-1,-4)$ and again disjoint from E.*

By taking $w = 1$ and extending an almost-complex structure on $\mathbb{C}P^2(R) \setminus W$ to all of Z, this immediately implies Proposition 3.6 (after blowing the ball back down).

In the course of the proof of Proposition 3.8 we will occasionally need the index formula for curves in Z, which is as follows.

Proposition 3.9. *(Hind-Lisi, [11], Proposition 3.1) Let C be a curve in $X \setminus L$ of degree d, with e intersections with E, and with s negative ends asymptotic to geodesics in the classes (k_i, l_i) respectively for $1 \leq i \leq s$.*

The index of C (as an unparametrized curve, allowing the asymptotic ends to move in the corresponding S^1 family of Reeb orbits) is given by

$$\mathrm{index}(C) = s - 2 + 6d - 2e + 2\sum_{i=1}^{s}(k_i + l_i).$$

Proof. (of Proposition 3.8.) We first record the following.

Lemma 3.10. *Let J_t be a 1-parameter family of almost-complex structures on Z tamed by a family of symplectic forms $\omega_{w(t)}$.*

The universal moduli spaces of degree 1 planes asymptotic to $(-1,-1)$ geodesics and disjoint from E, and the universal moduli space of degree 2 planes asymptotic to $(-1,-4)$ geodesics and disjoint from E, are both compact for generic 1-parameter families J_t.

Proof. We will deal only with the case of degree 2 planes. For generic 1-parameter families of almost-complex structures there is no bubbling of holomorphic spheres. Also, as $2R - (b+4) < 1$ there can be no degree 0 curves in the limit. Hence the only possible nontrivial degeneration of a degree 2 plane is into two planes of degree 1, which of course are necessarily somewhere injective. Suppose these are asymptotic to geodesics in the classes (k_1, l_1) and (k_2, l_2) respectively. By Proposition 3.9, the deformation index of such a plane is

$$\mathrm{index} = 5 + 2(k_i + l_i) \geq -1$$

and so $k_i + l_i \geq -3$ for each i. However as the total homology class of geodesics is preserved in any limit we also have $(k_1 + l_1) + (k_2 + l_2) = -1 - 4 = -5$ and so we may assume that $k_1 + l_1 = -3$ and $k_2 + l_2 = -2$. Focusing on the first plane, it has area

$$\mathrm{area} = R + bk_1 + l_1 = R - 3 + (b-1)k_1.$$

As $R < 3$ this is negative unless $k_1 \geq 1$, and in this case the area is at least $R + b - 4$. This implies that the area of the second plane is at most

$$(2R - b - 4) - (R + b - 4) = R - 2b < 0$$

by our condition (vi), a contradiction. □

Combined with automatic regularity for finite energy planes in dimension 4, see [26], this implies that it suffices to construct our planes for a specific almost-complex structure, and we will work with one tamed by the symplectic form ω_A.

An important tool for studying holomorphic curves in $\mathbb{C}P^2(R)\sharp\overline{\mathbb{C}P^1}(A)$ is the following result on foliations of holomorphic curves.

Proposition 3.11. *(see [9], Proposition 4.1) For a generic J, the manifold $\mathbb{C}P^2(R)\sharp\overline{\mathbb{C}P^1}(A)$ is foliated by J holomorphic spheres in the class $[\mathbb{C}P^1(\infty)]-[E]$.*

We are interested in the behavior of this foliation as we stretch the neck along L to produce a finite energy foliation of Z. This process was first described in [16], see also section 2 of [11] for the analysis in this particular case. As in [11] the finite energy foliation has the following description.

Lemma 3.12. *The foliation of Z consists of holomorphic spheres in the homology class $[\mathbb{C}P^1(\infty)] - [E]$ together with an S^1-family of broken curves.*

More precisely, there exists an S^1 family of geodesics on L with each geodesic corresponding to a unique broken curve. The broken curve consists of two finite energy planes which are asymptotic to the geodesic with opposite orientations.

Hence our finite energy foliation contains two S^1 families of finite energy planes asymptotic to the same family of geodesics in L with opposite orientations. As our broken curves are limits of closed curves of degree 1, one family of planes will have degree 1 and the other will have degree 0. Similarly, one family will intersect the exceptional divisor and the other will not. Further, as they are part of a foliation the planes in the S^1 families are disjoint. The two possibilities for broken curves can be derived from the following lemma.

Lemma 3.13. *Let C be the degree 0 component of a broken curve.*

(i) If C is disjoint from E then C is asymptotic to a geodesic in the class $(1,0)$ or $(0,1)$.

(ii) If C intersects E then C is asymptotic to a geodesic in the class $(1,1)$.

Proof. We start with case (i). Let C be asymptotic to a geodesic in the class (k,l) and let C' be the other half of the broken curve. Then C' must intersect the exceptional divisor. Suppose first that $k,l \geq 0$. Then we can replace C

by a (possibly singular) disk

$$D = \{(\frac{z}{b})^k = e^{i\theta} w^l\}$$

where (z, w) denote coordinates on the polydisk $P(b, 1) := \{\pi|z|^2 \leq b, \pi|w|^2 \leq 1\}$. We choose θ such that the boundaries of C and D coincide. We may assume that D is also disjoint from E and so it can be glued to C' to produce a sphere S which lies in the class $[\mathbb{C}P^1(\infty)] - [E]$ and hence has self-intersection 0.

Now, we can also compute the self-intersection number of S by pushing both D and C' onto disks asymptotic to a nearby geodesic. As it is part of the finite energy foliation C' will have no intersections with its perturbation, however D and its perturbation have intersection number kl relative to the boundaries. Also, as the broken curves are part of a foliation, relative to their boundaries we have $C' \cdot C = 0$, and similarly, since $\pi_2(\mathbb{C}^2 \setminus L) = 0$, we have $C' \cdot D = C' \cdot C = 0$. Therefore there are no other contributions to the intersection number and so $kl = 0$ and we are in one of the cases described.

If either k or l are negative then this argument only implies that C is asymptotic to a geodesic in the class $(\pm 1, 0)$ or $(0, \pm 1)$, but as degree 0 planes asymptotic to geodesics in the classes $(-1, 0)$ or $(0, -1)$ have negative index we can exclude these cases by regularity.

For case (ii), we again replace C by a disk D and glue D to C'. However now C' is disjoint from E and so the result is a sphere in the class $[\mathbb{C}P^1(\infty)]$ which has self-intersection number 1. This implies that $kl = \pm 1$. Then for C to have nonnegative index we conclude that the plane is asymptotic to a geodesic in the class $(1, 1)$. □

We have concluded that three kinds of finite energy foliation are possible. If the degree 0 broken curves intersect E then the degree 1 components of the broken curves are disjoint from E and asymptotic to geodesics in the class $(-1, -1)$. Thus Proposition 3.8 is valid in this case.

If the degree 0 broken components are planes disjoint from E and asymptotic to $(1, 0)$ geodesics, then the degree 1 components are asymptotic to $(-1, 0)$ geodesics and have area $R - A - b$, recalling that this component must intersect an exceptional divisor of area A. But the contradicts condition (v) and so may be excluded.

Thus we may assume that the foliation is exactly the one described in [11], that is, the broken curves consist of degree 0 planes asymptotic to geodesics in the class $(0,1)$ and degree 1 planes which intersect E and are asymptotic to geodesics in class $(0,-1)$. These have area $R-1-A$.

It is useful to define the map $\pi : Z \to E$ given by projection along the leaves of the finite energy foliation, which all intersect E in a single point (considering broken finite energy curves as representing a single leaf). Then the broken curves project onto a circle $\Gamma \subset E$.

The next step is to consider limits of high degree holomorphic spheres, whose existence is claimed by the following.

Proposition 3.14. *(see [10] Proposition 2.2, [11], Proposition 2.1) Let J be a regular almost complex structure on $\mathbb{C}P^2(R)\sharp\overline{\mathbb{C}P^1}(A)$ (so all somewhere injective curves are regular).*

Then, there is a co-meagre set $\mathcal{P} \subset X^{2d}$ consisting of $2d$ constraint points so that for each tuple of constraints $p_1, \ldots, p_{2d}$, there is a unique embedded holomorphic sphere S in the class $d[\mathbb{C}P^1(\infty)] - (d-1)[E]$ passing through the points.

We note that such curves S have intersection number 1 with curves in the foliation of Proposition 3.11. As in [11], we fix $2d$ points on L and take a limit of our high degree curves from Proposition 3.14 as the almost-complex structure is stretched along L. The limit contains a union F of finite energy planes in Z. By positivity of intersection, if $p \in E\setminus\Gamma$ then $\pi^{-1}(p)\cap F$ consists of a single point corresponding to the unique intersection of the fiber curve through p with F. This implies two possibilities for the curves F.

Case 1. F consists of a number of curves covering leaves of the foliation together with a single curve F_0 asymptotic to geodesics of the form $(0, \pm l)$. Then $\pi(F_0)$ is equal to $E \setminus \{q_i\}$ where the $\{q_i\}$ correspond to the broken leaves asymptotic to the limiting geodesics.

Case 2. F consists of a number of curves covering leaves of the foliation together with two curves F_0 and F_1 asymptotic to geodesics of the form $(1, l_0)$ and $(-1, l_1)$ respectively. In this case $\pi(F_0) \cup \pi(F_1) = E \setminus \Gamma$.

Case 1 was dealt with in [11]. It turns out that the remaining components of F consist of $2d$ planes covering broken components of degree 0. These components have a total area of at least $2d$, while closed curves in the class $d[\mathbb{C}P^1(\infty)] - (d-1)[E]$ have area $dR-(d-1)A$. Therefore we must have $dR-(d-1)A \geq 2d$ and hence $R \geq 3-1/d$. As $R < 3$ this is a contradiction if d is sufficiently large.

Eliminating this case, we have now reduced to Case 2 provided d is chosen large.

By Proposition 3.9 we see that F_0 and F_1 have odd deformation indices. Hence, as they cannot be multiple covers (as they are planes asymptotic to nondivisible geodesics in the classes $(1, l_0)$ or $(-1, l_1)$) for a generic almost-complex structure we have $\mathrm{index}(F_0) \geq 1$ and $\mathrm{index}(F_1) \geq 1$. In fact, to simplify our calculations, we can actually prove the following.

Lemma 3.15. $\mathrm{index}(F_0) = \mathrm{index}(F_1) = 1$.

Proof. Besides F_0 and F_1, it turns out that in fact all components in Z have virtual index at least equal to their number of negative ends. For example, if a component C with s ends is an r-times multiple cover of a broken component of degree 1, then Proposition 3.9 gives

$$\mathrm{index}(C) = s - 2 + 6r - 2r - 2r = s - 2 + 2r.$$

Next we recall that our components arise as a limit of holomorphic spheres of constrained index 0 (that is, the moduli space of holomorphic spheres passing through the constraint points has virtual index 0). Therefore the sum of the constrained indices of the components of our holomorphic building, minus the number of matched asymptotic ends, should also be 0. The virtual index of components in T^*L is nondecreasing under multiple covers (see again [11], Proposition 3.2 and 3.3) and thus the constrained index can be assumed to be nonnegative. Hence our sum can be 0 only if all components in Z have index exactly equal to their number of ends. In particular F_0 and F_1 have index 1. $\square$

Suppose F_1 has degree d_1 and intersection number e_1 with E. Then computing using Proposition 3.9 we find

$$1 = \mathrm{index}(F_1) = -1 + 6d_1 - 2e_1 + 2l_1 - 2$$

and so $3d_1 - e_1 + l_1 = 2$. Therefore

$$\mathrm{area}(F_1) = Rd_1 - Ae_1 - b + l_1$$

$$= (3d_1 - e_1 + l_1) - (3 - R)d_1 - (A - 1)e_1 - b$$
$$= 2 - (3 - R)d_1 - (A - 1)e_1 - b.$$

Suppose that $d_1 \geq 3$. Then

$$\mathrm{area}(F_1) \leq 2 - 9 + 3R - b$$

which contradicts condition (vi). Next suppose that $d_1 = 1$ and $e_1 \geq 1$. Then

$$\text{area}(F_1) \leq 2 - 3 + R - A + 1 - b = R - A - b$$

which contradicts condition (v). If $d_1 = 1$ and $e_1 = 0$ then by the index formula $l_1 = -1$ and we have a curve of degree 1 as required for Proposition 3.8. Thus we reduce to the case when $d_1 = 2$. Now if $e_1 \geq 1$ we have

$$\text{area}(F_1) \leq 2 - 6 + 2R - A + 1 - b = 2R - A - 3 - b$$

which contradicts the upper bound on R of condition (iii). Hence the curve F_1 may be assumed to have degree 2 and avoid E. Then by the index formula $l_1 = -4$ and we have a degree 2 curve as required, completing the proof of Proposition 3.8. □

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