CONTACT GEOMETRY AND LENS SPACES

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1. INTRODUCTION

There exists a canonical symplectic structure on the cotangent bundle of a differentiable manifold and similarly a canonical contact structure on the positive projectivization of the cotangent bundle under the scaling $\mathbb{R}$ action. Given a Riemannian (or Finsler) metric on the original manifold, this contact structure is contactomorphic to the contact structure determined by the metric on the unit tangent bundle.
In symplectic topology it is a general question to ask to what extent the symplectic or contact geometry of these structures determines the smooth structure of the smooth underlying manifold. For example one might expect symplectic invariants to give interesting new smooth invariants.
In this paper we study the canonical contact structures associated to 3 dimensional Lens spaces $L(r, s)$. These are the first examples of manifolds which may be homotopic but not diffeomorphic. We will see that the contact structures on their unit tangent bundles do indeed distinguish the smooth structures on the Lens spaces.
We note that homotopic but non diffeomorphic Lens spaces still have diffeomorphic tangent bundles. Therefore we do need contact methods to distinguish these structures and in particular we discover examples of different 5 dimensional contact structures on the same manifold (with the same Chern class). Such examples are still quite rare (but see [6]).
One natural approach to this problem is to compute the contact homology of our manifolds (see [3]). This was the approach of Ustilovsky. However the calculations in sections 2 and 3 show that this fails to separate the contact structures. Contact homology is a homology theory with chain groups generated by certain periodic orbits of a Reeb vector field on our contact manifold and with the differential defined by counting holomorphic curves. We will see that nondegenerate Reeb vector fields on our contact manifolds have isomorphic periodic orbits. Moreover these orbits are such that the necessary differentials vanish in all cases.
Motivated by work of Bourgeois [1] we study holomorphic curves corresponding to natural Morse-Bott contact forms in section 4. Relative to these contact forms we can describe fairly explicitly various moduli spaces of holomorphic curves. They do not appear in the differential defining contact homology although would certainly influence the more subtle invariants coming from Symplectic Field Theory (see [2]). In any case, in section 4 we show directly that properties of these moduli spaces of holomorphic curves are enough to distinguish our contact manifolds.

2. FINSLER PERTURBATION OF THE ROUND METRIC

Consider a Riemannian manifold (M, g) and the induced Hamiltonian on the cotangent bundle with the canonical Liouville structure $\omega = dp \wedge dq$,

$$H_o \colon T^*M \to \mathbb{R},\ H_o(\xi) = \|\xi\|_g^* \text{ f.a. } \xi \in T^*M, \tag{1}$$

where $\| \cdot |_G^*$ is the norm of the dual metric.
Note that for any diffeomorphism $\varphi \colon M \to M$ we have the *prolongation*

$$\phi = (D\varphi^{-1})^* \colon T^*M \to T^*M$$

covering φ which is symplectic and also preserves the Liouville 1-form $\lambda = p\,dq$.

Suppose we have a 1-parameter group of isometries $\varphi^t\colon M \xrightarrow{\cong} M$ generated by the Killing field V on M, then the prolongation of φ^t is a Hamiltonian diffeomorphism with Hamiltonian

$$H_1\colon T^*M \to \mathbb{R},\ H_1(q,\xi) = \xi(V(q))\,. \tag{2}$$

Since φ^t is an isometry, we have

$$\phi^t_{H_o} \circ \phi^s_{H_1} = \phi^s_{H_1} \circ \phi^t_{H_o} \quad \text{f.a. } s,t \in \mathbb{R}, \tag{3}$$

or equivalently

$$\{H_o, H_1\} = 0\,.$$

Since H_1 preserves the level sets $H_0^{-1}(c)$ as well as the Liouville 1-form, the symplectic flow $\phi^t_{H_1}$ preserves the contact form $\lambda_{|H_o^{-1}(c)}$ and thus also the contact structure $\xi = \ker \lambda_{|\dots}$.
The Finsler perturbation of the round metric geodesic flow is given by the Hamiltonian

$$H_\alpha := H_o + \alpha H_1,\ \alpha \in \mathbb{R}\,. \tag{4}$$

Lemma 2.1. *If the isometric action is periodic of period* 1, $\varphi_V^{t+1} = \varphi_V^t$ *and* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, *then the only closed periodic orbits of the Hamiltonian equation* $\dot{x} = X_{H_\alpha}(x)$, *corresponding 1-1 to the closed geodesic of the Finsler metric, are the geodesics of the Riemannian metric, which are invariant under the isometric action* φ_V^t.

From now on let us denote by $c\colon \mathbb{R} \to M$ a closed geodesic of the Finsler metric corresponding to H_α, with minimal period $T > 0$, and by

$$x\colon \mathbb{R}/T\mathbb{Z} \to H_\alpha^{-1}(1),\ \dot{x} = X_{H_\alpha}(x),$$

the corresponding orbit of the Hamiltonian system.
The primary aim is to compute the Conley-Zehnder index of x as a closed Reeb orbit for the contact form $\lambda_{|H_o^{-1}(1)}$. This has to be carried out with respect to a suitably chosen symplectic trivialization of the contact structure ξ.
Note that, due to the fact that the Hamiltonians H_o and H_1 are homogeneous of degree 1, the Hamiltonian flow for H_α satisfies

$$\phi^t_{H_\alpha}(\lambda p) = \lambda \phi^t_{H_\alpha}(p) \text{ f.a. } \lambda > 0,\ p \in T^*M\,. \tag{5}$$

Differentiating with respect to λ implies that the real line $\mathbb{R}x(t) \subset x^*T^vT^*M$, where we identify the point $x(t)$ in the cotangent bundle with the vertical tangent vector at this point, is invariant under the flow. Hence, the vectors $(x(t), \dot{x}(t)) \in x^*TT^*M$ span an invariant symplectic subbundle.
We call a trivialization of x^*TT^*M *standard* if it is derived from any trivialization of the vertical bundle along x. By means of the invariant symplectic subbundle from $(x, \dot{x})$ this induces a symplectic trivialization of the contact structure.
In order to compute such a standard trivialization in concrete terms, one can proceed as follows. Given any periodic isometric action $\varphi\colon S^1 \to M$, such that the geodesic c in question appears as a closed orbit of φ, the prolongation ϕ of φ preserves the contact structure along x and its linearization provides a standard trivialization. Thus it remains to find computable isometric actions in view of the geodesic under consideration.

2.1. **Standard Trivialization in the concrete case of $S^3 \subset \mathbb{C}^2$.** Consider $(M,g) = (S^3, g_o)$ the standard unit sphere embedded as $\{\,(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\,\}$ with theinduced round metric.
For the Finsler perturbation we consider the isometric S^1-action

$$\varphi_V^t(z_1,z_2) = (e^{i2t}z_1, e^{i3t}z_2),\ t \in \mathbb{R}/\mathbb{Z}\,. \tag{6}$$

It follows that the only 4 remaining closed geodesic for H_α lie in the the z_1- and z_2-plane,

$$\begin{aligned} c_1^{\pm}(t) &= (e^{(1\pm 2\alpha)ti}, 0), \quad T = \frac{2\pi}{1\pm 2\alpha}, \\ c_2^{\pm}(t) &= (0, e^{(1\pm 3\alpha)ti}), \quad T = \frac{2\pi}{1\pm 3\alpha}, \end{aligned} \tag{7}$$

In order to determine the contact structure along $x^{\pm}_{1,2}$ note that the entire vertical subbundle is contained in the kernel of the Liouville 1-form pdq, and also the complementary complex plane along $c_{1,2}$. It follows that in the coordinates of the ambient vector space $\mathbb{C}^2 \times \mathbb{C}^2 \supset x^*TT^*S^3$, the contact structure $x^*\xi$ is given by

$$\begin{aligned} (x_1^{\pm})^*\xi &= \{0\} \times \mathbb{C} \times \{0\} \times \mathbb{C} \\ (x_2^{\pm})^*\xi &= \mathbb{C} \times \{0\} \times \mathbb{C} \times \{0\} \end{aligned} \tag{8}$$

which is obviously trivialized by the constant trivialization $(1, i)$ in the corresponding factor.
In the case of a lens space S^3/Γ we have to choose suitably rotating trivializations of TS^3, see below, such that they become compatible with the discrete group action of Γ.
Consider the following isometric group actions of $\mathbb{Z}/r$ on the round sphere,

$$\begin{aligned} &\Gamma_{r,s} \colon \mathbb{Z}/r \times S^3 \to S^3, \\ &\sigma(z_1, z_2) = (e^{2\pi i/r} z_1, e^{2\pi i s/r} z_2), \end{aligned} \tag{9}$$

which is free if (r, s) are relatively prime, $1 \le s < r-1$. This gives rise to the lens space

$$L(r, s) = S^3/\Gamma_{r,s}\,.$$

In order to determine the suitable trivializations of $(x^{\pm}_{1,2})^*\xi$, we first note that

$$\begin{aligned} \sigma(c_1^{\pm}(t)) &= c_1^{\pm}(t \pm T/r), \\ \sigma(c_2^{\pm}(t)) &= c_2^{\pm}(t \pm Ts/r), \end{aligned} \tag{10}$$

where T is the respective period. For the trivialization Φ_1^+ along c_1^+ this means, $D\sigma\Phi_1^+(t) = \Phi_1^+(t + T/r)$, etc. Since the contact structure ξ along x is obtained from the plane complementary to the one in which the geodesics c lies, we have $D\sigma = e^{2\pi i s/r} \oplus e^{2\pi i s/r}$ along $c_1^{\pm}$ and $D\sigma = e^{2\pi i/r} \oplus e^{2\pi i/r}$ along $c_2^{\pm}$. (10) leads to the following compatibility condition for the trivializations $\Phi^{\pm}_{1,2}$,

$$\begin{aligned} D\sigma\Phi_1^{\pm}(t) &= \Phi_1^{\pm}(t \pm T/r), \\ D\sigma\Phi_2^{\pm}(t) &= \Phi_2^{\pm}(t \pm sT/r), \end{aligned} \tag{11}$$

where the latter can be transformed into

$$(D\sigma)^k\Phi_2^{\pm}(t) = \Phi_2^{\pm}(t \pm T/r), \tag{12}$$

with

$$ks \equiv 1 \pmod r\,.$$

We obtain for the trivialization component in the respective complex plane,

$$\begin{aligned} \Phi_1^{\pm}(t) &= e^{\pm 2\pi st/T}, \\ \Phi_2^{\pm}(t) &= e^{\pm 2\pi kt/T}\,. \end{aligned} \tag{13}$$

3. Index computation

3.1. A Formula for the Conley-Zehnder Index. Consider $\big(\Psi(t)\big)_{0\le t\le 1} \in \mathrm{Sp}(2,\mathbb{R})$ given by $\Psi_{a,b}(t) = A(t)B(t)$ where

$$A(t) = \begin{pmatrix} \cos a(t) & -\sin a(t) & 0 & 0 \\ \sin a(t) & \cos a(t) & 0 & 0 \\ 0 & 0 & \cos a(t) & -\sin a(t) \\ 0 & 0 & \sin a(t) & \cos a(t) \end{pmatrix} \tag{14}$$

$$B(t) = \begin{pmatrix} \cos b(t) & 0 & \sin b(t) & 0 \\ 0 & \cos b(t) & 0 & \sin b(t) \\ -\sin b(t) & 0 & \cos b(t) & 0 \\ 0 & \sin b(t) & 0 & \cos b(t) \end{pmatrix}, \tag{15}$$

with $a, b \in C^0([0,1],\mathbb{R})$. Compute

$$\begin{aligned} \det(\mathbf{1} - \Psi(t)) &= \det\big(A^{-1}(t) - B(t)\big) \\ &= \Big[\big(\cos a(t) - \cos b(t)\big)^2 + \big(\sin a(t) - \sin b(t)\big)^2\Big] \\ &\quad \cdot \Big[\big(\cos a(t) - \cos b(t)\big)^2 + \big(\sin a(t) + \sin b(t)\big)^2\Big]. \end{aligned}$$

Hence, the path Ψ satisfies $\Psi(1) \in \mathrm{Sp}^*(2,\mathbb{R})$ if and only if

$$a(1) \not\equiv b(1) \pmod{2\pi} \quad \text{and} \quad a(1) \not\equiv -b(1) \pmod{2\pi}. \tag{16}$$

Note that we also have always $\det \Psi(1) \ge 0$, that is, $\Psi(1) \in \mathrm{Sp}^+(2,\mathbb{R})$ if regular.
In order to determine the Conley-Zehnder index $\mu(\Psi)$ of the given path we define

$$\tilde{a} := a(1) - \left\lceil \frac{a(1) - \pi}{2\pi} \right\rceil 2\pi \in (-\pi, \pi], \tag{17}$$

$$\tilde{b} := b(1) - \left\lceil \frac{b(1) - \pi}{2\pi} \right\rceil 2\pi \in (-\pi, \pi]. \tag{18}$$

Which extension of the paths $a(t), b(t)$ has to be chosen such that $\Psi(2) = -\mathbf{1}$ depends on the following case distinctions: $|\tilde{a}| \lessgtr |\tilde{b}|$ and $\tilde{b} \lessgtr 0$. Due to the Lagrangian-orthogonal form of $A(t)$, the value of the index in each different case then depends only on $b(1)$. We obtain following table for the index μ of the path Ψ:

(19)

	$\tilde{b} > 0$	$\tilde{b} < 0$
$\lvert\tilde{a}\rvert < \lvert\tilde{b}\rvert$	$2\left\lceil \frac{b(1)}{\pi} \right\rceil$	$2\left\lfloor \frac{b(1)}{\pi} \right\rfloor$
$\lvert\tilde{a}\rvert > \lvert\tilde{b}\rvert$	$2\left\lfloor \frac{b(1)}{\pi} \right\rfloor$	$2\left\lceil \frac{b(1)}{\pi} \right\rceil$

3.2. Computation for the contractible orbits. Note first that a closed Finsler geodesic on the lens space $L(r,s)$ is contractible if and only if it lifts to a contractible geodesic on S^3. Hence, for the index computation we can work without the consideration of the $\Gamma_{r,s}$-action. When computing the linearization of the Hamiltonian flow $\phi^t_{H_\alpha}$ along the Finsler geodesics $x^{\pm}_{1,2}$, we observe that

$$D\phi^t_{H_\alpha} = D\phi^{\alpha t}_{H_1} \cdot D\phi^t_{H_0}.$$

Since we can use the constant trivialization $(1, i)$ for the contact structure from (8), the corresponding path of symplectic matrices $\Psi(t)$, reparametrized for $0 \le t \le 1$ has exactly the shape $\Psi(t) = A(t)B(t)$ as in (14) with $a(t), b(t)$ depending on $c^{\pm}_{1,2}$ as follows,

$$
\begin{aligned}
\underline{c_1^+}: &\quad a(t) = -\frac{3\alpha}{1+2\alpha}2\pi t, \quad b(t) = \frac{2\pi t}{1+2\alpha},\\
\underline{c_1^-}: &\quad a(t) = -\frac{3\alpha}{1-2\alpha}2\pi t, \quad b(t) = \frac{2\pi t}{1-2\alpha},\\
\underline{c_2^+}: &\quad a(t) = -\frac{2\alpha}{1+3\alpha}2\pi t, \quad b(t) = \frac{2\pi t}{1+3\alpha},\\
\underline{c_2^-}: &\quad a(t) = -\frac{2\alpha}{1-3\alpha}2\pi t, \quad b(t) = \frac{2\pi t}{1-3\alpha},
\end{aligned}
\tag{20}
$$

where the negative sign in $a(t)$ stems from the definition of $a(t)$ and the fact that the flow of H_1 is the isometric action by rotation, in the complementary plane to the geodesic. From this, we obtain

$$
\begin{aligned}
\underline{c_1^+}: &\quad \tilde{a} = -\frac{3\alpha}{1+2\alpha}2\pi, \quad \tilde{b} = -\frac{2\alpha}{1+2\alpha}2\pi,\\
\underline{c_1^-}: &\quad \tilde{a} = -\frac{3\alpha}{1-2\alpha}2\pi, \quad \tilde{b} = \frac{2\alpha}{1-2\alpha}2\pi,\\
\underline{c_2^+}: &\quad \tilde{a} = -\frac{2\alpha}{1+3\alpha}2\pi, \quad \tilde{b} = -\frac{3\alpha\pi}{1+3\alpha}2\pi,\\
\underline{c_2^-}: &\quad \tilde{a} = -\frac{2\alpha}{1-3\alpha}2\pi, \quad \tilde{b} = \frac{3\alpha\pi}{1-3\alpha}2\pi,
\end{aligned}
\tag{21}
$$

and, taking into account that we can choose $\alpha > 0$ arbitrarily small, the following indices for the primitive orbits:

$$
\begin{aligned}
\mu(c_1^+) &= 2\left\lceil (\frac{b(1)}{\pi}) \right\rceil = 4, \quad \mu(c_2^+) = 2\left\lfloor (\frac{b(1)}{\pi}) \right\rfloor = 2,\\
\mu(c_1^-) &= 2\left\lfloor (\frac{b(1)}{\pi}) \right\rfloor = 4, \quad \mu(c_2^-) = 2\left\lceil (\frac{b(1)}{\pi}) \right\rceil = 6,
\end{aligned}
\tag{22}
$$

which is consistent with the fact that the Finsler perturbation is equivalent with the Morse-Bott approach to the round metric which gives a critical family of contractible geodesics of the type of a Grassmannian $\mathrm{Gr}(4,2)$.

3.3. **Computation for the non-contractible orbits.** Taking care of the trivializations adjusted to the $\Gamma_{r,s}$-action we have,

$$\Psi(t/T) = (\Phi^{\pm}_{1,2})^{-1} D\phi^{\alpha t}_{H_1} D\phi^{t}_{H_o}\,.$$

Hence we obtain with (13) for n-fold multiple covers of the primitive orbits

$$
\begin{aligned}
\underline{nc_1^+}: &\quad a(t) = \big(s - \frac{3\alpha}{1+2\alpha}\big)\frac{n}{r}2\pi t, \quad b(t) = \frac{2\pi t}{1+2\alpha}\frac{n}{r},\\
\underline{nc_1^-}: &\quad a(t) = \big(-s - \frac{3\alpha}{1-2\alpha}\big)\frac{n}{r}2\pi t, \quad b(t) = \frac{2\pi t}{1-2\alpha}\frac{n}{r},\\
\underline{nc_2^+}: &\quad a(t) = \big(k - \frac{2\alpha}{1+3\alpha}\big)\frac{n}{r}2\pi t, \quad b(t) = \frac{2\pi t}{1+3\alpha}\frac{n}{r},\\
\underline{nc_2^-}: &\quad a(t) = \big(-k - \frac{2\alpha}{1-3\alpha}\big)\frac{n}{r}2\pi t, \quad b(t) = \frac{2\pi t}{1-3\alpha}\frac{n}{r},
\end{aligned}
\tag{23}
$$

Moreover, note that the multiplicities n have to be chosen according to the homotopy class of c_1^+, that is we have 4 representatives of the same homotopy class, namely

$$c_1^+,\ (r-1)c_1^-,\ sc_2^+,\ (r-s)c_2^-\,. \tag{24}$$

3.4. **The case $s = 1$.** In this case, formula (23) provides for the 4 representatives of the class of nc_1^+,

$$\begin{aligned}
\underline{nc_1^+}: &\quad a(1) = \big(1 - \frac{3\alpha}{1+2\alpha}\big)\frac{2n\pi}{r}, \quad b(1) = \frac{2n\pi}{r(1+2\alpha)},\\
\underline{(r-n)c_1^-}: &\quad a(1) = \big(-1 - \frac{3\alpha}{1-2\alpha}\big)\frac{r-n}{r}2\pi, \quad b(1) = \frac{2\pi}{1-2\alpha}\frac{r-n}{r},\\
\underline{nc_2^+}: &\quad a(1) = \big(1 - \frac{2\alpha}{1+3\alpha}\big)\frac{2n\pi}{r}, \quad b(1) = \frac{2n\pi}{r(1+3\alpha)},\\
\underline{(r-n)c_2^-}: &\quad a(1) = \big(-1 - \frac{2\alpha}{1-3\alpha}\big)\frac{r-n}{r}2\pi, \quad b(1) = \frac{2\pi}{1-3\alpha}\frac{r-n}{r}.
\end{aligned} \tag{25}$$

From this we compute in the case $1 \leq n \leq \frac{r}{2}$,

$$\begin{aligned}
\underline{nc_1^+}: &\quad \tilde{a} = \frac{1-\alpha}{1+2\alpha}\frac{2n\pi}{r}, \quad \tilde{b} = \frac{1}{(1+2\alpha)}\frac{2n\pi}{r},\\
\underline{(r-n)c_1^-}: &\quad \tilde{a} = \frac{1-(3r-1)\alpha}{1-2\alpha}\frac{2n\pi}{r}, \quad \tilde{b} = -\frac{1-2r\alpha}{1-2\alpha}\frac{2n\pi}{r},\\
\underline{nc_2^+}: &\quad \tilde{a} = \frac{1+\alpha}{1+3\alpha}\frac{2n\pi}{r}, \quad \tilde{b} = \frac{1}{1+3\alpha}\frac{2n\pi}{r},\\
\underline{(r-n)c_2^-}: &\quad \tilde{a} = \frac{1-(2r+1)\alpha}{1-3\alpha}\frac{2n\pi}{r}, \quad \tilde{b} = -\frac{1-3r\alpha}{1-3\alpha}\frac{2n\pi}{r},
\end{aligned} \tag{26}$$

Since $r \geq 2$ we obtain the indices

$$\begin{aligned}
\mu(nc_1^+) &= 2\left\lceil\frac{b(1)}{\pi}\right\rceil = 2, \quad \mu(nc_2^+) = 2\left\lfloor\frac{b(1)}{\pi}\right\rfloor = 0,\\
\mu((r-n)c_1^-) &= 2\left\lfloor\frac{b(1)}{\pi}\right\rfloor = 2, \quad \mu((r-n)c_2^-) = 2\left\lceil\frac{b(1)}{\pi}\right\rceil = 4,
\end{aligned} \tag{27}$$

which is consistent with the non-Finsler Morse-Bott point of view that the corresponding critical manifold consists of a disjoint union of S^2's.
In the case that $\frac{r}{2} < n < r$ we obtain instead of (26) the conditions

(28)

$$\begin{aligned}
\underline{nc_1^+}: &\quad \tilde{a} = -\frac{r-n+(n+2r)\alpha}{1+2\alpha}\frac{2\pi}{r}, \quad \tilde{b} = -\frac{r-n+2r\alpha}{(1+2\alpha)}\frac{2\pi}{r},\\
\underline{n(r-1)c_1^-}: &\quad \tilde{a} = -\frac{r-n+((3n-2)r-n)\alpha}{1-2\alpha}\frac{2\pi}{r}, \quad \tilde{b} = \frac{r-n+2(n-1)r\alpha}{1-2\alpha}\frac{2\pi}{r},\\
\underline{nc_2^+}: &\quad \tilde{a} = -\frac{r-n+(3r-n)\alpha}{1+3\alpha}\frac{2\pi}{r}, \quad \tilde{b} = -\frac{r-n+3r\alpha}{1+3\alpha}\frac{2\pi}{r},\\
\underline{n(r-1)c_2^-}: &\quad \tilde{a} = -\frac{r-n+(2rn+n-3r)\alpha}{1-3\alpha}\frac{2\pi}{r}, \quad \tilde{b} = +\frac{r-n+3r(n-1)\alpha}{1-3\alpha}\frac{2\pi}{r},
\end{aligned}$$

We obtain

$$\begin{aligned}
\mu(nc_1^+) &= 2\left\lceil\frac{b(1)}{\pi}\right\rceil = 4, \quad \mu(nc_2^+) = 2\left\lfloor\frac{b(1)}{\pi}\right\rfloor = 2,\\
\mu((r-n)c_1^-) &= 2\left\lfloor\frac{b(1)}{\pi}\right\rfloor = 0, \quad \mu((r-n)c_2^-) = 2\left\lceil\frac{b(1)}{\pi}\right\rceil = 2,
\end{aligned} \tag{29}$$

3.5. **The Case $s \not\equiv \pm 1 \pmod r$.** Here, we can assume wlog that $\alpha = 0$, since already for the round metric, the noncontractible solutions for $s \not\equiv \pm 1$ are non-degenerate as can be also seen from (23).
It turns out that for all cases of (r, s) we always find $(0, 2, 2, 4)$ as index tuples for all homotopy classes.

We have

$$
\begin{aligned}
\underline{nc_1^+}: &\quad a(1) = \frac{sn}{r}2\pi, \quad b(1) = \frac{n}{r}2\pi,\\
\underline{(r-n)c_1^-}: &\quad a(1) = -\frac{s(r-n)}{r}2\pi, \quad b(1) = \frac{r-n}{r}2\pi,\\
\underline{\overline{ns}c_2^+}: &\quad a(1) = \frac{ksn}{r}2\pi, \quad b(1) = \frac{\overline{ns}}{r}2\pi,\\
\underline{\overline{(r-sn)}c_2^-}: &\quad a(1) = -\frac{k\overline{(r-sn)}}{r}2\pi, \quad b(1) = \frac{\overline{r-sn}}{r}2\pi\,.
\end{aligned}
\tag{30}
$$

4. HOLOMORPHIC CURVES

The computations of the previous section show that the contact homology groups of the unit contangent bundles $T^1L(r,s)$ are independent of s, whether we look only at contractible orbits or at the contact homology generated by periodic orbits in any other homotopy class. The contact homology is generated by the periodic orbits themselves since they all have even index and so the boundary map is trivial. Nevertheless there are finite energy curves in $ST^1L(r,s) = T^*L(r,s) \setminus \mathbb{O}$ and we will use these to distinguish the contact structures. We work with homogeneous coordinates $(z_0 : z_1 : z_2 : z_3 : z_4)$ in $\mathbb{C}P^4$. Let Q be the quadric $\{\sum_{i=1}^4 z_i^2 = z_0^2\}$. Then $Q \setminus \{z_0 = 0\}$ can be identified with the affine quadric $Q' = \{\sum_{i=1}^4 w_i^2 = 1\}$ by setting $w_i = z_i/z_0$. The real points $\overline{x} = (x_1,x_2,x_3,x_4)$ of Q' we identify with S^3 and Q' itself can be identified with TS^3 by sending $(w_1,..,w_4)$ to the pair of vectors $\overline{x} = (\frac{\mathcal{R}w_1}{\sqrt{1+\sum(\mathcal{I}w_i)^2}},..,\frac{\mathcal{R}w_4}{\sqrt{1+\sum(\mathcal{I}w_i)^2}})$ and $\overline{y} = (\mathcal{I}w_1,..,\mathcal{I}w_4)$. We check that $\overline{x}$ and $\overline{y}$ are orthogonal and $\overline{x}$ has norm 1, and therefore they define a point on TS^3. Furthermore, under this identification the complex tangencies to the level sets $\sum(\mathcal{I}w_i)^2 = c$ correspond to the standard contact hyperplanes in $|\overline{y}|^2 = c$. We remark however that the complex structure on Q' is not scale invariant, so choosing different values of $c \in (0,\infty)$ defines different complex structures on T^1S^3 compatible with the contact form.

Let

$$
A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos s\theta & -\sin s\theta\\ 0 & 0 & \sin s\theta & \cos s\theta \end{pmatrix},
\tag{31}
$$

where $\theta = \frac{2\pi}{r}$.

The map $\sigma : \overline{x} \mapsto A\overline{x}$ gives a free $\mathbb{Z}_r$ action on S^3 and the quotient S^3/σ is the Lens space $L(r,s)$.

We observe that σ extends to a free $\mathbb{Z}_r$ action on Q' and to a $\mathbb{Z}_r$ action with fixed points on Q.

The surface $S = Q \cap \{z_0 = 0\} = \{z_1^2+z_2^2+z_3^2+z_4^2 = 0\}$ can be identified with $\mathbb{C}P^1 \times \mathbb{C}P^1$. To do this, choose homogeneous coordinates $((s_0 : s_1),(t_0 : t_1))$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and identify this point with the point

$$(s_0t_0 + s_1t_1, -i(s_0t_0 - s_1t_1), s_0t_1 - s_1t_0, -i(s_0t_1 + s_1t_0)) \in S.$$

In these coordinates the action of σ on S becomes

$$\sigma((s_0 : s_1),(t_0 : t_1)) = ((e^{i(s+1)\theta}s_0 : s_1),(e^{-i(s-1)\theta}t_0 : t_1)).$$

The Fubini-Study form on $\mathbb{C}P^4$, normalized such that the lines have area 1, restricts to a symplectic form ω on S which is the standard split form in our coordinates. The normal bundle ν of S in Q has first Chern class ω. We can identify ST^1S^3 with $\nu \setminus S$.

Let L be the unit normal bundle of ν with respect to an Hermitian metric. For a suitable choice of connection $i\alpha$ on L the Reeb vector field R corresponding to the contact form α will be tangent to the fibers of L and the contact hyperplanes coincide with the complex tangencies. This is an example of a Morse-Bott contact form, see [1]. Thinking of L as a fixed length level set in T^*S^3, such a contact form can also be induced from the round metric on S^3. Thus studying holomorphic curves in the symplectization of T^1S^3 with respect to compatible almost-complex structures is exactly the same as studying holomorphic curves in the symplectization of L.

We choose an almost-complex structure on $\xi = \ker \alpha$ which is invariant under the periodic Reeb flow and extend this to an almost-complex structure J on $\nu \setminus S \cong (L \times (0, \infty), d(t\alpha))$ which is scale invariant and satisfies $J(\frac{\partial}{\partial t}) = R$. Since J is R invariant it projects to give an almost-complex structure on S and in fact it extends to give an almost-complex structre on ν with holomorphic fibers. Restricted to the bundle over a holomorphic curve in S, the complex structure J is automatically integrable and defines a holomorphic line bundle over the curve. Thus finite energy curves in $\nu \setminus S$ correspond to curves C in S together with meromorphic sections of a holomorphic line bundle, see again [1]. Zeros correspond to posivite asympotoic limits and poles to negative asymptotic limits. We will work with almost-complex structures projecting to σ-invariant almost-complex structures on S. Then finite energy curves in $ST^1L(r, s)$ correspond to σ invariant sections over C. In order for the finite energy curve to have noncontractible asymptotic limits the curve $C \subset S$ must be connected and the holomorphic curve in $\nu \setminus S$ become multiply covered under the quotient by σ.

To study such sections we first calculate the action of σ on the fibers of ν over the fixed points of the action. Recall that points in S correspond to geodesics on S^3, in fact we can think of S precisely as a symplectic reduction of T^*S^3 under the S^1 action of the geodesic flow. The fixed points correspond to geodesics in S^3 which are invariant under σ.

In our coordinates, the point $((0 : 1), (0 : 1)) \in S$ corresponds to c_1^+ and the action on the corresponding fiber is multiplication by $e^{i\theta}$. The point $((1 : 0), (1 : 0)) \in S$ corresponds to c_1^- and the action on the corresponding fiber is multiplication by $e^{-i\theta}$. The point $((0 : 1), (1 : 0)) \in S$ corresponds to c_2^+ and the action on the corresponding fiber is multiplication by $e^{is\theta}$. The point $((1 : 0), (0 : 1)) \in S$ corresponds to c_2^- and the action on the corresponding fiber is multiplication by $e^{-is\theta}$.

We can now describe σ-invariant finite energy curves in $\nu \setminus S$ with respect to a standard almost-complex structure, that is, one projecting to the integrable complex structure above on S.

The only isolated σ invariant curves in S are $t_0 = 0$, $t_1 = 0$, $s_0 = 0$ and $s_1 = 0$. We study these separately.

$\underline{t_0 = 0}$ Suppose that f is a σ invariant section. Then it has either zeros or poles at the fixed points $s_0 = 0$ and $s_1 = 0$. In local coordinates let $f(z) = \sum a_n z^n$ near $s_0 = 0$. Since σ acts as $e^{i(s+1)\theta}$ on the base and $e^{i\theta}$ on the fiber, for f to be locally σ invariant it must satisfy $f(e^{i(s+1)\theta} z) = e^{i\theta} f(z)$. So if $a_n \neq 0$ we have $n(s+1) \equiv 1(r)$ or $n \equiv (s+1)^{-1}(r)$. We note that if $s \equiv -1(r)$ then such a section must be identically zero.

Near $s_1 = 0$ let $f(z) = \sum b_n z^n$, then $b_n \neq 0$ implies that $n \equiv s(s+1)^{-1}(r)$.

Since $(s+1)^{-1} + s(s+1)^{-1} \equiv 1(r)$ and $c_1(\nu)|_{t_0=0} = 1$ there exist meromorphic sections with $(s+1)^{-1}(r)$ zeros at $s_0 = 0$ and $s(s+1)^{-1}(r)$ zeros at $s_1 = 0$ and a number of poles distributed σ equivariantly at the other points.

Since a meromorphic section is determined up to scale by its zeros and poles, if these zeros and poles are σ equivariant then so is the section itself.

Therefore such sections will project in $ST^1L(r, s)$ to finite energy holomorphic curves with positive asymptotic limits $q_1 c_1^+$ and $q_2 c_2^-$ and contractible negative asymptotic limits, where $q_1 \equiv (s+1)^{-1}(r)$ and $q_2 \equiv s(s+1)^{-1}(r)$. We check that indeed $q_1 c_1^+$ is homotopic to $-q_2 c_2^-$.

$\underline{t_1 = 0}$ Again we must have zeros or poles at $s_0 = 0$ and $s_1 = 0$. Near $s_0 = 0$ if $f(z) = \sum a_n z^n$ then $a_n \neq 0$ implies that $n \equiv s(s+1)^{-1}(r)$. Near $s_1 = 0$ the zero or pole must have order $(s+1)^{-1}(r)$. Since $s(s+1)^{-1} + (s+1)^{-1} \equiv 1(r)$ there exist sections with zeros at $s_0 = 0$ and $s_1 = 0$ and poles elsewhere.

In $ST^1L(r,s)$ the sections correspond to finite energy curves with positive asymptotic limits $q_1c_2^+$ and $q_2c_1^-$ and contractible negative asymptotic limits where $q_1 \equiv s(s+1)^{-1}(r)$ and $q_2 \equiv (s+1)^{-1}(r)$.
We observe that the positive asymptotic limits here are homotopic to those in the case of $t_0 = 0$.
$\underline{s_0 = 0}$ Now sections will have poles at $t_0 = 0$ and $t_1 = 0$.
A similar analysis to those above shows that we find finite energy holomorphic curves in $ST^1L(r,s)$ with positive asymptotic limits $q_1c_1^+$ and $q_2c_2^+$ and contractible negative limits where $q_1 \equiv -(s-1)^{-1}(r)$ and $q_2 \equiv s(s-1)^{-1}(r)$.
$\underline{s_1 = 0}$ Sections here project to finite energy holomorphic curves in $ST^1L(r,s)$ with positive asymptotic limits $q_1c_2^-$ and $q_2c_1^-$ and contractible negative limits where $q_1 \equiv s(s-1)^{-1}(r)$ and $q_2 \equiv -(s-1)^{-1}(r)$.
The positive asymptotic limits are homotopic to those in the case of $s_0 = 0$.
Proof of main result
Now suppose that there exists an orientation preserving contactomorphism ϕ from $T^1L(r,s)$ to $T^1L(r,s')$. For convenience we suppose that $s, s' \neq \pm 1(r)$. Then ϕ will push forward our Morse-Bott contact form α_s on $T^1L(r,s)$ to a form β on $T^1L(r,s')$ generating the same contact structure as the Morse-Bott form $\alpha_{s'}$ on $T^1L(r,s')$. Suppose that $\phi_*[c_2^+] = n[c_2^+]$. Then ϕ will map periodic orbits in the class $m[c_2^+]$ in $T^1L(r,s)$ to periodic orbits in the class $mn[c_2^+]$ in $T^1L(r,s')$ for all $0 \leq m < r$.

Lemma 4.1. *With respect to any translation invariant almost-complex structure compatible with the contact form $\alpha_{s'}$ there exist finite energy spheres in $ST^1L(r,s')$ with two positive punctures asymptotic to orbits homotopic to $\pm ns(s+1)^{-1}[c_2^+]$ and a negative puncture asymptotic to a contractible orbit. Also, there exist finite energy spheres with two positive punctures asymptotic to orbits homotopic to $\pm ns(s-1)^{-1}[c_2^+]$ and a negative puncture asymptotic to a contractible orbit. Here $ns(s+1)^{-1}$ and $ns(s-1)^{-1}$ denote the smallest positive integers which take the given value modulo r.*

Proof We show the existence of the first class of spheres. With respect to the contact form β we have curves in $ST^1L(r,s')$ with the same asymptotic limits as those described in the lemma. It is required to show that such curves also exist given a complex structure compatible with $\alpha_{s'}$. To this end, we look at the moduli space of finite energy curves from $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ with two positive punctures at 0 and 1 having the required asymptotics, say q_1 and q_2, and one negative puncture at ∞ asymptotic to a contractible orbit. With respect to the complex structure compatible with β it consists of the images under projection and ϕ of meromorphic sections in $\nu \setminus S$ over $t_0 = 0$ or $t_1 = 0$. This moduli space has dimension 4. We can see this directly from the construction. The negative asymptotic limit must lie in $t_0 = 0$ or $t_1 = 0$ and so there exists a 2-parameter family of such limits. Once this is fixed the curve is defined up to the (2-dimensional) phase. By fixing the positive asymptotic limits we will restrict to the image of meromorphic sections over $t_0 = 0$. We prefer to work with a 0-dimensional moduli space and so impose additional constraints on our curves as follows. First we fix an embedded sphere T in S transverse to $t_0 = 0$ and intersecting in a single point. Then let x_0 be a fixed point on the noncontractible periodic orbit giving the asymptotic behaviour at 0. Let $a : \mathbb{C}P^1 \setminus \{0,1,\infty\} \to \mathbb{R}$ be the vertical (radial) component of a curve in $ST^1L(r,s')$ and v the horizontal $T^1L(r,s')$ component. Denote by J an almost-complex structure on $ST^1L(r,s')$ which outside of a compact set is translation invariant and compatible with β.

Definition 4.2. *$\mathcal{M}(J)$ is the moduli space of J-holomorphic finite energy planes $f : \mathbb{C}P^1 \setminus \{0,1,\infty\} \to ST^1L(r,s')$ such that f is asymptotic to q_1 at 0, q_2 at 1 and a contractible orbit at ∞. Furthermore,*
(1) $da^{-1}(0) \cap a^{-1}(0) \neq 0$;
(2) $\lim_{x\to 0^+} v(x) = x_0$;
(3) *the projection to S of the preimage of the contractible orbit in T^1S^3 intersects T.*

We check that $\mathcal{M}(J)$ has dimension 0. With respect to the complex structure compatible with β, if the surface is transverse to $t_0 = 0$ the final condition fixes the position of our pole (so determining the curve up to scale), and the two remaining conditions determine the modulus and argument of the scale respectively. Therefore with respect to this complex structure the moduli space contains a single curve.
We first establish the following.

Proposition 4.3. *Let J_t for $0 \leq t \leq 1$ be a generic family of almost-complex structures on $ST^1L(r,s')$ which are identical outside of a compact set, where they are translation invariant and compatible with β. Then $\cup_t \mathcal{M}(J_t)$ is a compact 1-dimensional manifold.*

Proof of Proposition
Suppose that $t_n \to t_0 \in [0,1]$ and we are given a sequence $f_n \in \mathcal{M}(J_{t_n})$. It is required to show that $\{f_n\}$ has a convergent subsequence. But the compactness theorem of Symplectic Field Theory, see [2], shows that we have sequential compactness modulo degeneration into a multiple level holomorphic curve. More specifically, the limit can be thought of as a map from a punctured nodal curve whose components map to $ST^1L(r,s')$ and are either J_{t_0}-holomorphic or holomorphic with respect to the translation invariant almost-complex structure compatible with β. The punctures are asymptotic to closed Reeb orbits of β and there are natural matching conditions at the nodes, such that if we abstractly glue copies of $ST^1L(r,s')$ with its translation invariant structure to the positive and negative ends of $(ST^1L(r,s'), J_{t_0})$ then the holomorphic components mapping to the different copies of $ST^1L(r,s')$ fit together to give a genus 0 curve with the same asymptotic limits as the curves in $\mathcal{M}(J_t)$. It is required to show that the limit in fact consists of a single nontrivial J_{t_0}-holomorphic component.
Chern classes are preserved in the limit and so the total deformation index of such a limit, where we deform in the moduli space of multiple level curves with matching asymptotic limits and which satisfy our constraints, must be 0.
Remark We need to clarify the meaning of the constraints for multiple level curves. The point $0 \in \mathbb{R}$ applies to the holomorphic component mapping to $(ST^1L(r,s'), J_{t_0})$ and so condition (1) applies similarly. To understand condition (2) we can think of our J_{t_n}-holomorphic curves as graphs

$$f : \mathbb{C}P^1 \setminus \{0,1,\infty\} \to \mathbb{C}P^1 \setminus \{0,1,\infty\} \times ST^1L(r,s')$$

which are holomorphic with respect to the product structures on the range. The projection is holomorphic and the limiting curves will also project holomorphically to $\mathbb{C}P^1 \setminus \{0,1,\infty\}$. Thus the limit consists of a holomorphic curve defined on $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ together with various components projecting to points in $\mathbb{C}P^1$, possibly including 0,1 and ∞. The components projecting to 0 fit across common asymptotic limits to form a surface with two asymptotic ends, one asymptotic to q_1 and the other matching the limit of the $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ component at 0. Ignoring any planar components and removing the corresponding singularities on the remaining components, we are left with a finite sequence of cylinders, or images of $\mathbb{C}^*$, with matching asymptotic limits end to end along which they can abstractly be glued to form a cylinder with one end asymptotic to q_1 and the other matching the limit of the $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ component at 0. Then the radial directions in the $\mathbb{C}^*$ give a formula for associating a point on the asymptotic limit of the $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ component at 0 to a point on q_1. Choosing the positive real points gives a point on the asymptotic limit of the $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ component at 0 as before, and so a point on q_1. The analogue of condition (2) is that this point should be x_0. The analogue of condition (3) is that after identifying all matching asymptotic limits the remaining negative limit should lie on a contractible orbit which lies in T when lifted to S. The key point here is that if a sequence of curves f_n satisfying conditions (1), (2) and (3) converges to a level curve f_0, then f_0 will also satisfy the conditions in the sense here. This follows from the convergence of decorated curves, [2].

Returning to the proof of the proposition, if the family J_t is sufficiently generic then all components of the limit have index at least -1. However all holomorphic curves appearing in $ST^1L(r,s')$ have even index and so no deformation indices can be negative. Therefore no component in our limit can have a positive index in the moduli space of similar components with *exactly* the same asymptotic ends. We note that since our asymptotic limits have minimal length in their homology classes none of our components can be nontrivial multiple covers. Thus all components mapping into $ST^1L(r,s')$ with its translation invariant structure must themselves be translation invariant (otherwise they could be deformed while still satisfying matching asymptotic conditions) and hence must be trivial cylinders. This is equivalent to the moduli space being compact as required for the proposition.

□

If J_0 denotes the translation invariant almost-complex structure on $ST^1L(r,s')$ compatible with β then we have seen that $\mathcal{M}(J_0)$ consists of a single curve. Proposition 4.3 implies that $\mathcal{M}(J)$ is cobordant to $\mathcal{M}(J_0)$ for any J agreeing with J_0 outside of a compact set. In particular $\mathcal{M}(J)$ is nonempty, it consists of an odd number of curves.

Now we begin to deform the complex structure on $ST^1L(r,s')$ to structures J_N such that we can holomorphically embed copies of $[-N,N]\times T^1L(r,s')$ with its almost-complex structure compatible with $\alpha_{s'}$. We do this such that $\{0\}\times T^1L(r,s')$ maps to itself. Then by the above, for all values of N there still exist finite energy planes with the same asymptotic properties (in fact an odd number).

Again following [4] we now take a limit as $N\to\infty$. According to [2] the result is a nodal curve but the limit is now potentially more complicated than in Proposition 4.3. The components map to manifolds W_k, where $-m\le k\le n$ for some m, n. Here $W_k = ST^1L(r,s')$ with a complex structure compatible with β for $-m<k<-m_1$; $W_{-m_1} = ST^1L(r,s')$ with a complex structure compatible with β for large negative t and $\alpha_{s'}$ for large positive t; $W_k = ST^1L(r,s')$ with a complex structure compatible with $\alpha_{s'}$ for $-m_1<k<n_1$; $W_{n_1} = ST^1L(r,s')$ with a complex structure compatible with $\alpha_{s'}$ for large negative t and β for large positive t; $W_k = ST^1L(r,s')$ with a complex structure compatible with β for $n_1<k<n$. We identify our $[-N,N]\times T^1L(r,s)$ with increasing subsets of W_0.

We can form a tree from the limiting components. The components correspond to vertices and branches join components in W_i and W_{i+1} such that the positive asymptotic limits of the component in W_i and the negative limits of the component in W_{i+1} have a Reeb orbit in common. This common asymptotic limit results from a degenerating circle in $\mathbb{C}P^1\setminus\{0,1,\infty\}$.

Since our J_N-holomorphic curves are all of genus 0 with two positive punctures, the components of the limit abstractly fit together to form a connected genus 0 curve with two positive punctures. All components must have at least one positive puncture (by the maximum principle) and so we observe that exactly one curve can have two positive punctures and none can have more than two. As in Proposition 4.3 the limiting component with two positive punctures inherits a parameterization as a map from $\mathbb{C}P^1\setminus\{0,1,\infty\}$ perhaps minus some additional punctures coming from nodes. The two positive punctures must be asymptotic to the same required noncontractible orbits (at least up to homotopy). In fact, the components in W_n have a total of two positive punctures asymptotic to q_1 and q_2 and the components in W_{-m} have a single negative puncture which after lifting and projection lies in T. As in Proposition 4.3, the total deformation index of our limit, where we deform in the moduli space of multiple level curves with matching asymptotic limits and which satisfy our constraints, must be 0. We claim that all components not mapping into W_0 are cylinders, the result then readily follows.

To justify the claim, we first observe that components mapping into W_k for $k\neq -m_1,0,n_1$ are invariant under translation and so must be trivial cylinders. A component in W_0 not equal to the image of our $\mathbb{C}P^1\setminus\{0,1,\infty\}$ is invariant under the S^1 action coming from the argument of the phase since this action preserves the condition (1) and does not affect condition (2) for these components. Thus such components must also be trivial cylinders, and

the component in W_0 satisfying condition (1) must be the image of our $\mathbb{C}P^1 \setminus \{0,1,\infty\}$, which again may have additional negative punctures.
Therefore, if the claim is false, either the image of $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ in W_0 has additional negative punctures or there exists a noncylindrical component in either W_{-m_1} or W_{n_1} with a single positive asymptotic limit but multiple negative ends. In either case another component must be a holomorphic plane (and in fact it must map to W_{-m_1}). But holomorphic planes, even with a fixed asymptotic limit, always appear in a moduli space of dimension at least 2. (All planes lift to ST^1S^3 or $\nu \setminus S$ where, with the standard complex structure, they are equivalent to the data of a holomorphic sphere in S and a holomorphic section of ν with a fixed zero. But the phase acts nontrivially on such sections fixing the zero.) Thus we obtain a positive dimensional space of deformations and therefore a contradiction as in Proposition 4.3. This proves the claim and therefore Lemma 4.1.

□

The above Lemma has constructed finite energy planes in $T^1L(r,s')$ with a standard complex structure under the assumption that there exists an orientation preserving contactomorphism from $T^1L(r,s)$ to $T^1L(r,s')$. But the considerations prior to Lemma 4.1 have already determined which such curves exist. So, since we know in particular which homotopy classes in $T^1L(r,s')$ can be asymptotic limits for our curves the above lemma implies that if a contactomorphism exists then either

$$s(s+1)^{-1} \equiv \pm n^{-1}s'(s'+1)^{-1}(r)\text{and}s(s-1)^{-1} \equiv \pm n^{-1}s'(s'-1)^{-1}(r)$$

or

$$s(s+1)^{-1} \equiv \pm n^{-1}s'(s'-1)^{-1}(r)\text{and}s(s-1)^{-1} \equiv \pm n^{-1}s'(s'+1)^{-1}(r).$$

Therefore

$$\frac{s+1}{s-1} \equiv \pm(\frac{s'+1}{s'-1})^{\pm 1}(r).$$

Checking the possibilities this implies that $s \equiv s'^{\pm 1}(r)$ and so the Lens spaces are diffeomorphic.

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