Lagrangian unknottedness in Stein surfaces

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Abstract *We show that the space of Lagrangian spheres inside the cotangent bundle of the 2-sphere is contractible. We then discuss the phenomenon of Lagrangian unknottedness in other Stein surfaces. There exist homotopic Lagrangian spheres which are not Hamiltonian isotopic, but we show that in a typical case all such spheres are still equivalent under a symplectomorphism.*

1 Introduction

Studying the space of Lagrangian submanifolds is a fundamental problem in symplectic topology. Lagrangian spheres appear naturally in the Lefschetz pencil picture of symplectic manifolds.

In this paper we demonstrate the uniqueness up to Hamiltonian isotopy of the Lagrangian spheres in some 4-dimensional Stein symplectic manifolds. The most important example is the cotangent bundle of the 2-sphere, T^*S^2, with its standard symplectic structure. In this case we will go on to study the space of all Lagrangian spheres in T^*S^2, showing that it is contractible.

Finally, we study an example of a Stein manifold in which a particular homotopy class (even isotopy class) contains Lagrangian spheres which are not Hamiltonian isotopic. We show that the spheres in this class are still unknotted in a weaker sense, namely they are all equivalent under a global (non Hamiltonian) symplectomorphism built by composing a Hamiltonian diffeomorphism

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with a product of symplectic Dehn twists.

We recall that if a convex symplectic manifold has a boundary of contact-type, then we can perform surgery operations on the manifold by adding handles to the boundary. In the 4-dimensional case these handles can be of index 1 or 2. Our first examples are symplectic manifolds formed by adding 1-handles to a unit cotangent bundle T^1S^2. Questions regarding Lagrangian isotopy classes are independent of which metric we use to define a unit tangent bundle or of any choices involved in adding 1-handles.

Theorem 1 *Let M be T^*S^2 or the result of adding any number of 1-handles to T^1S^2 and $L \subset M$ be a Lagrangian sphere. Then there exists a Hamiltonian diffeomorphism of M mapping L onto the zero-section.*

We will establish this theorem by utilizing an existence result for almost-complex structures on $S^2 \times S^2$ with convenient properties, taken from [18], and a fact about diffeomorphisms of the 2-sphere.

In fact more is true. We let $\mathcal{L}$ denote the space of Lagrangian spheres in T^*S^2 endowed with the topology of smooth convergence.

Theorem 2 *The topological space $\mathcal{L}$ is contractible.*

It is a consequence of a general theorem of J. Coffey [4], combined with the result of [18], that the space of parameterized Lagrangian spheres in $S^2 \times S^2$ is homotopic to $SO(3) \times SO(3)$. A theorem of Y. Eliashberg and L. Polterovich, see [11], says that the space of Lagrangian planes in a standard $\mathbb{R}^4$, equal to a fixed plane outside of a compact set, is also contractible. The proof here involves parameterized versions of the arguments in Theorem 1. In both cases we need a result about diffeomorphisms of the 2-sphere.

Theorem 3 *The subset of fixed-point free maps contained in the diffeomorphism group of S^2 is contractible.*

In section 2 we prove our result on the diffeomorphisms of S^2. In section 3, by using the conclusions of [18], we reduce our theorem in the case of $M = T^*S^2$

to the statements in section 2. In section 4 we will deal with the addition of handles. This involves slightly generalizing the results from [18] so we will review them again there.

We now consider the addition of 2-handles. Let W be the Stein manifold formed by adding to T^1S^2 a single 2-handle along the Legendrian curve in a single fiber of the boundary. As a Stein manifold it carries a symplectic structure which has a conformally expanding vector field whose flow exists for all time. The symplectic structure is the Kähler form associated to a plurisubharmonic exhaustion function and all such forms are equivalent up to symplectomorphism (see [10]). Alternatively W can be realized as the plumbing of two copies of T^1S^2. The resulting symplectic manifold W has two Lagrangian spheres L_1 and L_2 coming from the zero-sections in the copies of T^1S^2 (or the original zero-section and the stable manifold of the index 2 critical point in the added handle). Again we will establish a uniqueness result for Lagrangian spheres in W.

Theorem 4 *Let L be a Lagrangian sphere in W, the plumbing of two copies of T^*S^2, which is homotopic to one of the zero-sections L_1. Then there exists a symplectomorphism ϕ of W such that $\phi(L) = L_1$.*

The proof combines Theorem 1 with some previous work of the author and is described in section 5.

Thus any Lagrangian spheres which are homotopic to L_1 but are knotted in the Hamiltonian sense must arise from global symplectomorphisms applied to L_1. Such symplectomorphisms do indeed exist. Recall that associated to any Lagrangian sphere L is a compactly supported symplectomorphism τ_L called a generalized Dehn Twist. It is well-defined up to Hamiltonian symplectomorphism. The square τ_L^2 is smoothly but not necessarily symplectically isotopic to the identity. Thus $\tau_{L_2}^{2r}(L_1)$ is a Lagrangian sphere in W which is smoothly isotopic to L_1 for any integer r. However, as demonstrated by P. Seidel in [29], a Floer homology computation shows that none of the $\tau_{L_2}^{2r}(L_1)$ are Hamiltonian isotopic. A natural question is whether these are the only examples of such

Lagrangian knots, and we will show that this is indeed the case.

Theorem 5 *Let L be a Lagrangian sphere in W. Then there exists a composition of Dehn twists τ such that $\tau(L)$ is Hamiltonian isotopic to L_1 or L_2.*

This will be proven in section 6.

In a Stein manifold a Lagrangian isotopy can be composed with a conformally contracting vector field (the negative gradient of the plurisubharmonic exhaustion) so as to lie in an arbitrarily small neighborhood of the union of the stable manifolds of the critical points. Also, a theorem of Weinstein, [34], says that a Lagrangian sphere (or two Lagrangian spheres intersecting transversally in a single point) have tubular neighborhoods unique up to symplectomorphism. Thus Theorem 1 about Lagrangian spheres in T^*S^2 implies the following.

Theorem 6 *Let L_1 be a Lagrangian sphere in a symplectic 4-manifold M. Then any other Lagrangian sphere $L \subset M$ which is sufficiently C^0 close to L_1 is Hamiltonian isotopic to L_1.*

Theorem 5 similarly gives the following.

Theorem 7 *Let L_1 and L_2 be two Lagrangian spheres in a symplectic 4-manifold M, intersecting transversally in a single point. Then for any other Lagrangian sphere $L \subset M$ which is sufficiently C^0 close to $L_1 \cup L_2$ there exists a composition τ of the Dehn twists τ_{L_1} and τ_{L_2} about L_1 and L_2 such that $\tau(L)$ is Hamiltonian isotopic to L_1 or L_2.*

Similar methods can generalize the unknottedness result of Theorem 5 to a larger class of Stein manifolds, but it is unclear whether or not it is true in general that homotopic Lagrangian spheres are equivalent under a global symplectomorphism composed of a Hamiltonian flow and Dehn twists.

As yet we are unable to prove any similar results for Lagrangian surfaces of higher genus. However in the case of $\mathbb{R}P^2$ we can prove the following, see section 3.1.

Theorem 8 *The space of Lagrangian submanifolds homotopic to the zero-section in $T^*\mathbb{R}P^2$ is connected.*

It is interesting to note that this can be established without the detailed analysis required for Theorem 3, see Lemma 26.

A natural compactification of the (unit) cotangent bundle of $\mathbb{R}P^2$ is $\mathbb{C}P^2$. Again the Lagrangian is unique.

Theorem 9 *Let L be a Lagrangian $\mathbb{R}P^2$ in $\mathbb{C}P^2$. Then there exists a Hamiltonian isotopy taking L onto the standard embedding.*

Perhaps our methods can be extended to cover this case, but, at least to show connectedness of the space of Lagrangians, the theorem can be established by other methods. For example, the surgery technique described by M. Symington in [32] replaces a Lagrangian $\mathbb{R}P^2$ by a symplectic sphere, transforming $\mathbb{C}P^2$ into an $S^2 \times S^2$. This is a symplectic cut, see [25], where a tubular neighborhood of our Lagrangian $\mathbb{R}P^2$ is replaced by a tubular neighborhood of a symplectic sphere S of self-intersection -4. Then the theorem can be established by using results of F. Lalonde and D. McDuff, see [24], on uniqueness of symplectic structures on $S^2 \times S^2$, and M. Abreu and D. McDuff, see [1], on uniqueness of symplectic spheres. This argument has been worked through in detail by T-J. Li and W. Wu in [35], Theorem 5.9, see section 5.1.1.

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2 Diffeomorphisms of the two-sphere

In this section we let f denote a diffeomorphism of the 2-sphere S^2 and for a point $x \in S^2$ we denote its antipodal point by $-x$.

Definition 10 *A diffeomorphism f is nowhere antipodal if $f(x) \neq -x$ for all $x \in S^2$.*

The aim of the section is to prove the following theorem. It is equivalent to Theorem 3, noting that composition with the antipodal map gives a bijection between fixed point free and nowhere antipodal diffeomorphisms.

Theorem 11 *Suppose that a smooth family of diffeomorphisms f_p depending upon a parameter $p \in S^k$, $k \geq 0$, are all nowhere antipodal and $f_1 = \mathrm{id}$ for a point $1 \in S^k$. Then there exists a family of isotopies $f_{p,t}$, $0 \leq t \leq 1$, with $f_{p,0} = f_p$ and $f_{p,1} = \mathrm{id}$ for all p, $f_{1,t} = \mathrm{id}$ for all t and such that $f_{p,t}$ is nowhere antipodal for all p, t.*

Remark 12 *Note that nowhere antipodal diffeomorphisms of the 2-sphere are necessarily orientation preserving. Then by analogy we can consider orientation preserving isometries. These can be identified with the rotation group $SO(3)$, and this in turn is can be identified with*

$$\mathbb{R}P^3 \equiv S^2 \times [0,\pi]/(x,0) \sim (y,0), (x,\pi) \sim (-x,\pi).$$

Under this identification the S^2 factor gives an oriented axis of rotation and the $[0,\pi]$ factor is the angle. In this picture, nowhere antipodal rotations get identified with the subset

$$S^2 \times [0,\pi)/(x,0) \sim (y,0) \simeq B^3,$$

which is contractible.

Proof of theorem 11

For economy of notation, and also in the interests of readability, we will give a complete proof of the theorem in the case when $k = 0$ so that all subscripts p can be forgotten. However we will be careful throughout to ensure that all constructions and genericity assumptions apply equally well to the parameterized situation, it is left to the reader to confirm this.

Let E denote an equator on S^2. The complement of E consists of two open disks H_1 and H_2 with $-H_1 = H_2$.

We observe that any nowhere antipodal diffeomorphism g with the property that $g(E) = E$ is indeed isotopic to the identity through nowhere antipodal diffeomorphisms g_t. To construct such an isotopy, we first isotope g to the identity in a neighbourhood of E (using the contractibility of nowhere antipodal diffeomorphisms of S^1). Now the resulting map restricts to a compactly supported diffeomorphism of H_1 and H_2. But by a theorem of Smale, [30], compactly supported diffeomorphisms of the disk are isotopic to the identity (see for instance [33], page 205). Combining these isotopies we get the required isotopy of g. It is nowhere antipodal since $-H_1 = H_2$.

Given this, it suffices to find a nowhere antipodal isotopy from f to a diffeomorphism preserving an equator E.

We will construct our isotopy by applying the following proposition.

Proposition 13 *Let $\Phi : (-1,1) \times S^1 \to S^2$ be a smooth embedding and $L_s = \Phi(\frac{2}{\pi}\arctan(s) \times S^1)$, $-\infty < s < \infty$ be a foliation of $\Phi((-1,1) \times S^1)$ by circles. Suppose that there exist K, N such that $f(L_{s+N})$ is transverse to $-L_s$ for all $s > -K$. Then there exists a nowhere antipodal isotopy f_t such that $f_0 = f$ and $f_t(L_s) = f(L_{s+tN})$ for all $0 \leq t \leq 1$, $s > -K$. Further $f_t(z) = f(z)$ for all z outside of the image of Φ and for all t.*

Remark 14 *Suppose that Φ extends to a diffeomorphism of $S^2 = [-1,1] \times S^1/(\pm 1, \theta) \sim (\pm 1, \theta')$ such that $\Phi(-1,\theta) = -f\Phi(1,\theta)$. Let $z = f\Phi(1,\theta)$. Then for any given fixed K there exists a δ such that L_s is disjoint from $B_\delta(\Phi(-1,\theta))$ (the ball of radius δ centered at $\Phi(-1,\theta)$) for all $s > -K$. In other words, $-L_s$ is disjoint from $B_\delta(-\Phi(-1,\theta)) = B_\delta(z)$ for all such s. On the other hand, if N is chosen sufficiently large then $f(L_{s+N}) \subset B_\delta(z)$ for all $s > -K$. Therefore, given any K, there exists an $N = N(K)$ such that the hypotheses of Proposition 13 are satisfied.*

Proof of Proposition 13

As the condition of being nowhere antipodal is an open one, we may assume any necessary genericity properties for the diffeomorphisms f with respect to the foliation L_s. Specifically for any r, s we will assume that $f(L_r) \cap -L_s$ consists of an isolated set of points and any tangencies are of finite order.

Remark 15 *We have two foliations of subsets of S^2, namely $\{f(L_s)\}$ and $\{-L_s\}$. In the generic case the corresponding line fields will be tangent on a set of codimension 1 and these tangents will be of order 2 except at isolated points when they have order 3. In a high parameter family of foliations though we do expect tangencies of higher order, but still expect the assumption above to hold true.*

Suppose that $N > 0$. Let a_r be a family of diffeomorphisms of S^2 depending on a parameter $r \in \mathbb{R}$ such that $a_r(L_s) = L_{s+r}$ and a_r extends as the identity outside of the image of Φ and $a_0 = \text{id}$. Then we will define f_t on the image of Φ by

$$f_t(z) = h_t f a_{Nt}(z)$$

where h_t is a diffeomorphism of the image of $f\Phi$ which preserves the foliation $\{f(L_s)\}$ and extends by the identity to a diffeomorphism of S^2. We set $h_{t,s} = h_t|_{f(L_{s+Nt})}$. In order that $f_0 = f$ we require all $h_{0,s} = \text{id}$.

Then we need to find smoothly varying $h_{t,s}$ such that $h_{t,s}(f(a_{Nt}(z))) \neq -z$ for all s, all $z \in L_s$, and $0 \leq t \leq 1$.

Fixing N, for s very large and $z \in L_s$ we notice that $a_{Nt}(z)$ must very close to z (as the a_{Nt} extend as the identity outside of the image of Φ). We may assume it is so close to z that $fa_{Nt}(z)$ is very close to $f(z)$ and hence disjoint from $-z$ (as f is nowhere antipodal). Therefore we may choose $h_{t,s} = \text{id}$ for s sufficiently large. We must show that we can extend these diffeomorphisms for all parameters s.

We observe that once we have defined the h_{t,s_0} for some s_0 we can smoothly extend the functions to define $h_{t,s}$ for s slightly less than s_0, again relying on the fact that nowhere antipodal is an open condition. Similarly we can always define $h_{t,s}$ for t close to 0 by extending the identity map.

The following lemma will be useful.

Lemma 16 *Suppose that there exist $h_{t,s}$ for all $0 \leq t \leq 1$ and all $s \geq s'$ such that the corresponding maps f_t restricted to $\bigcup_{s \geq s'} L_s$ are nowhere antipodal. Then the isotopies f_t can be extended to nowhere antipodal isotopies of S^2 mapping the circles $\{L_s\}$ into the circles $\{f(L_s)\}$.*

We remark that the maps f_t are not required to map L_s into $f(L_{s+tN})$. One application of the lemma is that it allows us to conclude the proof of Proposition 13 once the $h_{t,s}$ have been defined for $s > -K$.

Proof of Lemma 16 Define a vector field X on $\bigcup_{s'+N \geq s \geq s'} f(L_s)$ by $X(f_t(x)) = \frac{d}{dt}(f_t(x))$. Then X can be extended to all of S^2 by setting $X = 0$ outside of $\bigcup_{s \geq s'-\epsilon} f(L_s)$ and defining X over $\bigcup_{s' \geq s \geq s'-\epsilon} f(L_s)$ in such a way that the corresponding flow ϕ_t takes the circles $f(L_s)$ into other such circles. The lemma will be established by setting $f_t = \phi_t \circ f$ once we check that such an isotopy is nowhere antipodal.

Again let $x \in L_{s'}$. Then $f_t(x) \neq -x$ for all $0 \leq t \leq 1$ and so there exists a δ such that $\phi_t(f(x)) = f_t(x) \notin B_\delta(-x)$, a δ-ball about $-x$, for all $0 \leq t \leq 1$. (By compactness, the same δ can be chosen for all $x \in L_{s'}$.) If ϵ is sufficiently small and $t < 0$ then $\phi_t(f(x))$ is very close to $f(x)$ and so we may assume that in fact $\phi_t(f(x)) \notin B_\delta(-x)$ for all $-\infty < t \leq 1$ and $f_t^{-1}(f(x)) = f^{-1}\phi_{-t}(f(x)) \in B_\delta(x)$ for all $t > 0$. Thus the points which flow through $f(x)$ also avoid their antipodal points and the extension of f_t is nowhere antipodal as required. This completes the proof of Lemma 16. □

Returning to Proposition 13, for any s, as t increases from 0 to 1 there is a varying collection of points $I_{t,s} = f(L_{s+tN}) \cap -L_s$. The diffeomorphisms $h_{t,s}$ can be extended arbitrarily over $f(L_{s+Nt})$ once they define the inverse image of these intersections. That is, we only need to check that $h_{t,s}(f(a_{Nt}(z))) \neq -z$ for all $z \in -I_{t,s}$.

For a fixed value of s the $I_{t,s}$ will consist of a set of points varying with t. For each t, we are assuming that $I_{t,s}$ is a finite set of points in $f(L_{s+tN})$. If we identify all $f(L_{s+tN})$ then as t varies the only qualitative changes in $I_{t,s}$ are collections of points appearing or vanishing.

For each s we can define a map

$$T_s : -L_s \cap (\cup_{t=0}^{1} f(L_{s+tN})) \to [0,1]$$

by mapping $z \in -L_s$ to the unique r such that $z \in f(L_{s+rN})$.

Lemma 17 *Suppose that $T_{s''}$ is a Morse function without critical values at* 0 *or* 1*. Equivalently, $-L_{s''}$ is transverse to $f(L_{s''})$ and $f(L_{s''+N})$ and any tangencies with other $f(L_{s''+tN})$ are of order* 2*. Then if $h_{t,s'}$ is defined for some $s' > s''$ with $s' - s''$ sufficiently small we can also define a continuous family of $h_{t,s}$ for all $s'' \le s \le s'$.*

Proof of Lemma 17 We know that $h_{t,s'}(f(a_{Nt}(z))) \neq -z$ for all $z \in L_{s'}$ and all t, in particular for $z \in -I_{t,s'}$. To reduce notation, let us assume slightly more generally that $h_{t,s'}(f(a_{Nt}(z))) \notin I_{t,s'}$ for all $z \in -I_{t,s'}$ and all t.

First we consider the case when the critical points of $T_{s''}$ all have distinct values. We recall that Morse functions without critical points on the boundary and with distinct critical values are stable up to reparameterization, that is, if $T_{s''}$ is Morse then so are all T_s for s sufficiently close to s''. Moreover, fixing such an s' with $s' - s''$ sufficiently small there exist continuous families of diffeomorphisms $\psi_s : -L_s \to -L_{s'}$ and ϕ_s of $[0,1]$ such that $T_s = \phi_s^{-1} T_{s'} \psi_s$ for all $s'' \le s \le s'$.

Note that ψ_s can be extended to a diffeomorphism from $\cup_{t=0}^{1} f(L_{s+tN})$ to $\cup_{t=0}^{1} f(L_{s'+tN})$ (preserving the foliation $\{f(L_r)\}$). We still denote this extended map by ψ_s. Then ψ_s necessarily maps each of the sets $I_{t,s}$ onto $I_{\phi_s(t),s'}$. Indeed, $\psi_s(f(L_{s+tN})) = f(L_{s'+\phi_s(t)N})$.

We can now define our $h_{t,s}$ by

$$h_{t,s}(f(a_{tN}(z))) = \psi_s^{-1} h_{\phi_s(t),s'} f a_{\phi_s(t)N}(-\psi_s(-z))$$

for $z \in L_s$ and check that if $z \in -I_{t,s}$ then $h_{t,s}(f(a_{tN}(z))) \notin I_{t,s}$.

In the case when $T_{s''}$ has critical points with the same critical values, we simply divide $-L_{s''}$ into subintervals on which $T_{s''}$ is Morse and stable and define the $h_{t,s}$ on each subinterval as above, adjusting our maps to ensure they match at the boundaries. □

Summarizing our situation so far, the goal is to define $h_{t,s}$ for all $s > -K$ and all $0 \leq t \leq 1$. The maps can easily be defined for s very large and all t, and for all s when $t = 0$ (here they are the identity). Further, by Lemma 17, if s'' is the infimum of the set of s' such that $h_{t,s}$ can be continuously defined on $s \geq s'$ then $T_{s''}$ is either not Morse, or is Morse with boundary critical points.

Suppose that there exists a finite infimum s'' of the above set. For convenience, assume that there is a single point in S^2 and a single t parameter, for which $f(L_{s''+tN})$ is tangent to $-L_{s''}$ to high order, or for which $f(L_{s''+tN})$ is tangent to $-L_{s''}$ and $t = 0$ or $t = 1$. In other words, $T_{s''}$ has a single degenerate critical point with value t.

By hypothesis we are assuming that $f(L_{s''+N})$ is transverse to $-L_{s''}$. If $f(L_{s''})$ is tangent to $-L_{s''}$ then diffeomorphisms can still be defined as in Lemma 17 to give a continuous extension of $h_{t,s}$ to $s \geq s''$ which give nowhere antipodal maps at least for t away from 0. But as all $h_{0,s}$ are the identity anyway, the resulting f_t will in fact be nowhere antipodal even for t close to 0 (maps with antipodal points can only arise through nontrivial isotopies).

So finally we consider the situation when $f(L_{s''})$ and $f(L_{s''+N})$ are transverse to $-L_{s''}$ and there exists a σ with $0 < \sigma < 1$ and $f(L_{s''+\sigma N})$ tangent to high order with $-L_{s''}$, say at a point z. By this we mean that we can choose local coordinates (x, y) in a neighborhood U of z in S^2 such that $-L_{s''} = \{y = x^n\}$ and $f(L_{s''+tN}) = \{y = t - \sigma\}$ for some integer $n > 1$, the order of the tangency.

Nevertheless the $I_{t,s}$ still consist of isolated sets of points. We suppose that $h_{t,s'}$ can be defined for some $s' > s''$ and s'' is the largest critical parameter less than s'.

A short digression. Figure 1 illustrates a possible scenario when $n = 3$ (the typical case which arises for a single generic f). Setting $\epsilon = s - s''$ close to 0, suppose that the curves $-L_s$ are modeled by graphs $f_\epsilon(x) = x^3 - \epsilon x + \epsilon$. So when $s > s''$ the $I_{t,s} \cap U$ consists of a single point when t is sufficiently far from σ, but may consist of three points for some t close to σ. On the other hand,

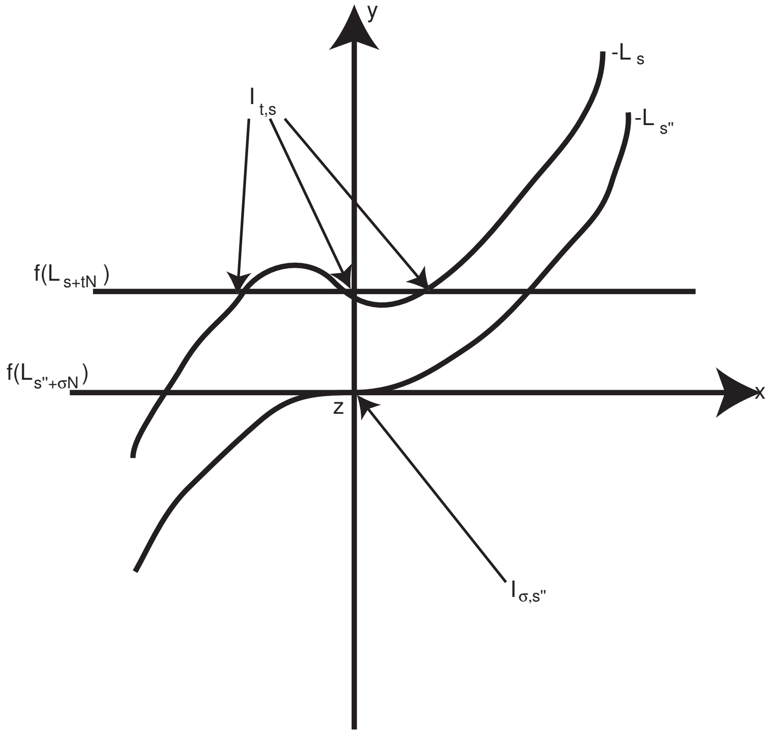

Figure 1: Curves near the point z

$I_{t,s} \cap U$ is always a single point when $s < s''$. Thus, for certain fixed t, as s decreases we find a continuous family of points $-x(s), -y(s) \in I_{t,s} \subset f(L_{s+tN})$ which converge (come together) as $s \to s''$.

Let $f_{t,s} = f_t|_{L_s}$. A problem would then arise if for some $s' > s''$ we find an interval $[x(s'), y(s')] \in L_{s'}$ and t' close to σ such that $f_{t',s'}([x(s'), y(s')]) \subset [-x(s'), -y(s')] \subset f(L_{s'+t'N}) \cap U$. Such $f_{t',s'}$ could certainly not be extended to $s \geq s''$ without finding an s for which either $f_{t',s}(x(s)) = -x(s)$ or $f_{t',s}(y(s)) = -y(s)$, a contradiction if our f_t are to be nowhere antipodal. However, letting t decrease with s' now fixed, the horizontal line $f(L_{s'+tN})$ moves down our figure, and, as $\sigma > 0$, will eventually move into the negative half-space where $I_{t,s'}$ is a single point. In other words, we also find a contracting interval $[-x(t), -y(t)] \subset f(L_{s'+tN})$ with $-x(t), -y(t) \in I_{t,s'}$, where $x(t') = x(s')$ and $y(t') = y(s')$, and so such an $f_{t',s'}$ could also not be extended to smaller t. But we already know this to be possible in our case since for $s > s''$ the $h_{t,s}$ exist for all t. So the potential obstruction does not in fact arise. **End of digression.**

We formalize the intuition of the above digression in the following lemma.

We choose a small neighborhood U of z such that $-L_s \cap U$ consists of a small interval V_s for all $s'' \leq s \leq s'$, and which contains all of the points in $I_{t,s}$ for t close to σ and s close to s'' which converge to z as $t \to \sigma$ and $s \to s''$. We note that $I_{\sigma,s''} \cap U = z$, but for s close to s'' and t close to σ, $I_{\sigma,s} \cap U$ may contain several points.

Lemma 18 *We may isotope our maps $h_{t,s'}$ such that $f_t(L_{s'} \cap -U)$ is disjoint from U for all t.*

Remark 19 *The construction here is sufficiently canonical that it should be clear no additional obstructions arise in higher parameter families.*

Proof of Lemma 18 There is a well defined map π from U to $L_{s'}$ (defined using our original f_t, before the current isotopy) which takes a point $w \in U$ to the unique $x \in L_{s'}$ with $f_t(x) = w$ for some t. The image of U is an interval $W \subset L_{s'}$. If $-V_{s'}$ is disjoint from W then there is nothing to do, otherwise by choosing U sufficiently small we may assume that $W \cup -V_{s'}$ is an interval inside $L_{s'}$. For each $x \in -V_{s'}$ we have $\pi(-x) \in W$ and if we orient the interval $W \cup -V_{s'}$ we observe that either $x > \pi(-x)$ for all $x \in V_{s'}$ or $x < \pi(-x)$ for all $x \in V_{s'}$. For otherwise, by the intermediate value theorem, we can find a $y \in V_{s'}$ for which $f_t(y) = -y$, a contradiction. Suppose that the first scenario arises. Then we can redefine the $h_{t,s'}$ to be unchanged on parts of their domain away from U but to move the image of $-V_{s'}$ (under fa_{Nt}) in the positive direction in order to displace it from U. As points of $-V_{s'}$ move only in the positive direction, we notice that the new $f_t = h_{t,s'} fa_{Nt}$ are still nowhere antipodal. □

Given Lemma 18, to complete the proof of Proposition 13 we can now mimic the proof of Lemma 17 by defining, for $s'' \leq s \leq s'$, diffeomorphisms ψ_s from $\cup_{t=0}^{1} f(L_{s+tN})$ to $\cup_{t=0}^{1} f(L_{s+tN})$ which preserve the foliations $\{f(L_r)\}$ and $\phi_s : [0,1] \to [0,1]$ by the formula $\psi_s(f(L_{s+tN})) = f(L_{s'+\phi_s(t)N})$. Making the same assumptions as in Lemma 17, as $T_{s''}$ is Morse away from $-L_{s''} \cap U$, we may assume that ψ_s maps $-L_s \setminus U$ to $-L_{s'} \cap U$ and hence that $\psi_s(I_{t,s} \setminus U)$ maps to $I_{\phi_s(t),s'} \setminus U$. For $s' - s''$ sufficiently small, we may also assume that the diffeomorphisms ψ_s preserve U itself. Then we check that the formula

$$h_{t,s}(f(a_{tN}(z))) = \psi_s^{-1} h_{\phi_s(t),s'} fa_{\phi_s(t)N}(-\psi_s(-z))$$

still works to define our $h_{t,s}$ for $s'' \leq s \leq s'$. Indeed, if $z \in -(I_{t,s} \setminus U)$ then $h_{t,s}(f(a_{tN}(z))) \neq -z$ as before, but if $z \in -(I_{t,s} \cap U)$ then, by applying the isotopy of Lemma 18, we may assume that $h_{t,s}(f(a_{tN}(z))) \notin U$ and so again $h_{t,s}(f(a_{tN}(z))) \neq -z$.

This completes the proof of Proposition 13, that is, the maps $h_{t,s}$ can be defined for all t for both an open and closed subset of s, including large s. Thus they can be defined everywhere and we have our isotopy. □

We apply Proposition 13 in various situations to complete the proof of Theorem 11.

Let n denote the north pole in S^2, and define $z = f(n)$. Let γ be a great circle intersecting n, z and $-z$. Then we can define E to be the great circle perpendicular to γ and intersecting the two midpoints of γ between n and $-z$.

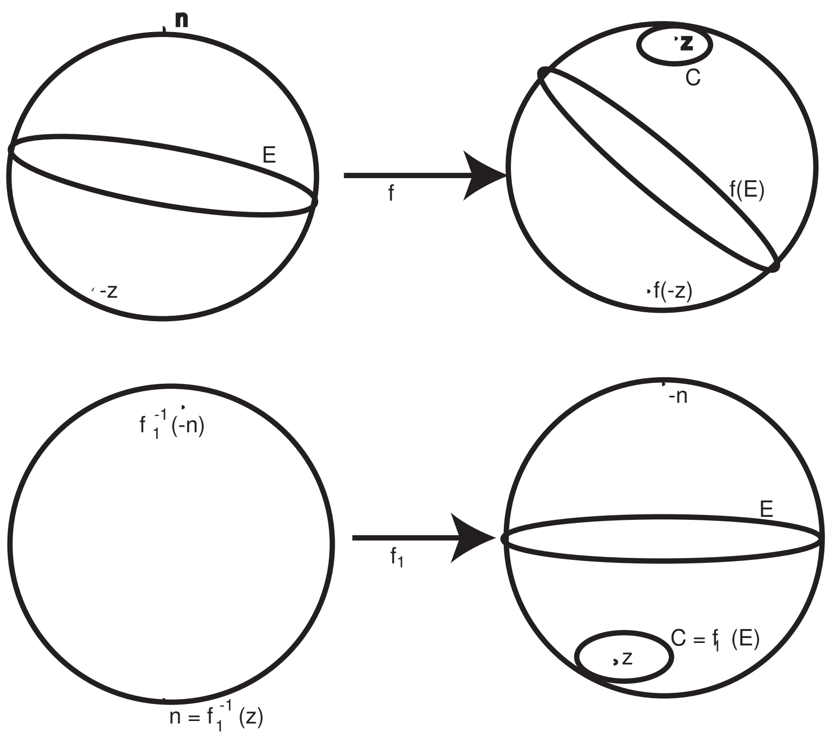

Figure 2: The diffeomorphisms f and f_1

We note that $n \neq -z$ (since f is nowhere antipodal). In the case when $z = n$ there is of course a family of great circles intersecting $n = z$ and $-z$, but all produce the same E, which in this case is just the standard equator. By definition n and $-z$ lie on opposite sides of E. As the antipodal map preserves E but maps one hemisphere to the other, z must lie on the same side of E as n, while $-n$, the south pole, lies on the same side as $-z$.

We can choose an embedding $\Phi_1 : (-1,1) \times S^1 \to S^2$ with $s \times S^1 \to n$ as $s \to 1$ and $s \times S^1 \to -z$ as $s \to -1$. Then we may suppose that Φ_1 extends to a diffeomorphism of spheres and so by Remark 14 the hypotheses of Proposition 13 are satisfied (for any K given a suitably large N) and we can find a smooth isotopy of nowhere antipodal diffeomorphisms from f to a new map f_1 with $f_1(n) = z$ and $f_1(E) = C$, a small circle around z. We have tried to illustrate the situation in Figure 2.

Repeating this argument, as f_1 is nowhere antipodal $f_1^{-1}(-n) \neq n$ and so we can choose another embedding $\Phi_2 : (-1,1) \times S^1 \to S^2$ such that now $s \times S^1 \to f_1^{-1}(-n)$ as $s \to 1$ and $s \times S^1 \to n$ as $s \to -1$. Then $f_1 \Phi_2((-1,1) \times S^1)$ is a cylinder converging at its positive end to $-n$ and at its negative end to z, see again Figure 2. As E separates these two points we can also choose Φ_2 such that $f_1 \Phi_2(\{0\} \times S^1) = E$ and $f_1 \Phi_p(\{-K\} \times S^1) = C$ for a $-K$ close to -1. Thus applying Proposition 13 again gives an isotopy $f_{1,t}$ of f_1 through nowhere antipodal diffeomorphisms such that there will be a moment t_0 when the corresponding diffeomorphism f_{1,t_0} maps E to itself. By our first observation this is enough to complete the proof of Theorem 11. $\square$

3 Lagrangian spheres in T^*S^2

Let L be a Lagrangian sphere in T^*S^2. This has self-intersection number -2 and so must be homotopic to the zero-section. By scaling in the fibers we may assume that $L \subset T^1S^2$, the unit disk bundle defined using a round metric. We will identify T^1S^2 with the complement of the diagonal Δ in $S^2 \times S^2$ with its standard split symplectic form $\omega = \omega_0 \oplus \omega_0$. Under this identification, the zero-section in T^1S^2 becomes the antidiagonal $\overline{\Delta}$. Thus Theorem 1 in this case is equivalent to the following.

Theorem 20 *Given a Lagrangian sphere $L \subset S^2 \times S^2 \setminus \Delta$ homotopic to $\overline{\Delta}$, there exists a Hamiltonian isotopy of $S^2 \times S^2$ which fixes Δ and maps L onto $\overline{\Delta}$.*

Given an almost-complex structure J on $S^2 \times S^2$ tamed by ω, Gromov showed in [12] that there exist unique foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ by J-holomorphic curves in the classes $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$. With respect to the standard almost-complex structure $J_0 = i \oplus i$, these foliations are exactly $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$. The key lemma which we need from [18] is the following.

Lemma 21 *There exists a tame almost-complex structure J on $S^2 \times S^2$ such that each curve in the corresponding foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects L transversally in a single point. The almost-complex structure J can be taken to agree with J_0 near Δ.*

The second statement was not included in [18] but is clearly true from the proof.

There exists a family of tame almost-complex structures J_t, $0 \leq t \leq 1$ on $S^2 \times S^2$ with $J_1 = J$ and, for all t, $J_t = J_0 = i \oplus i$ near Δ. In particular, Δ is a J_t-holomorphic curve for all t. By the positivity of intersections for J_t-holomorphic curves, each holomorphic curve in the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects Δ transversally in a single point. Therefore we can make the following definition.

Definition 22 *$\mathcal{F}_i(t,x)$ is the J_t-holomorphic sphere in the foliation $\mathcal{F}_i$ which intersects Δ at the point x.*

We define a diffeomorphism $f : \Delta \to \Delta$ by $f(x) = y$, where $y \in \Delta$ is the unique point such that $\mathcal{F}_1(1,y) \cap \mathcal{F}_0(1,x) \in L$. Then, as L is disjoint from Δ, we have $f(x) \neq x$ for all $x \in \Delta$. Equivalently, this means that $-f^{-1}$ is nowhere antipodal.

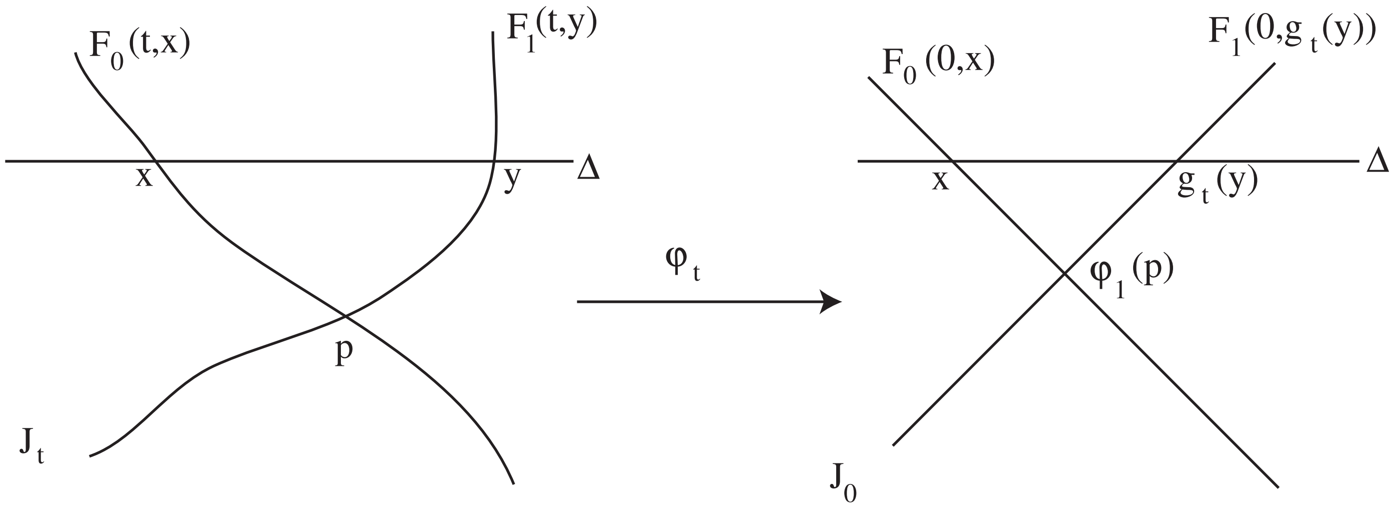

Figure 3: The map ϕ_t

As in the previous section, for a point $x \in \Delta$ we denote its image under the antipodal map by $-x$. Then $\mathcal{F}_0(0,x) \cap \mathcal{F}_1(0,-x) \in L$ for all $x \in \Delta$.

We can apply the Theorem 11 without the parameter p (or in the case $k=0$) to get the following.

Lemma 23 *There exists an isotopy $g_t : \Delta \to \Delta$, $0 \le t \le 1$, of nowhere antipodal maps with $g_0 = \mathrm{id}$ and $g_1 = -f^{-1}$.*

We now define maps $\phi_t : S^2 \times S^2 \to S^2 \times S^2$ to be the unique diffeomorphisms sending $\mathcal{F}_0(t,x)$ to $\mathcal{F}_0(0,x)$ and $\mathcal{F}_1(t,y)$ to $\mathcal{F}_1(0,g_t(y))$ for all $x, y \in \Delta$. The map is illustrated in Figure 3.

Then $\phi_0 = \mathrm{id}$, $\phi_1(L) = \overline{\Delta}$ and $\phi_t(\Delta)$ is disjoint from $\overline{\Delta}$ for all t. For the second point, note that $\mathcal{F}_0(1,x) \cap \mathcal{F}_1(1,y) \in L$ if and only if $y = f(x)$ or, equivalently, $g_1(y) = -f^{-1}(y) = -x$. The third point is equivalent to g_t being nowhere antipodal.

Let $L_t = \phi_t^{-1}(\overline{\Delta})$, so L_t gives a smooth isotopy from L to $\overline{\Delta}$ in $S^2 \times S^2 \setminus \Delta$.

Now, as the coordinate foliations are holomorphic, $\phi_{t*}(J_t)$ is tamed by the split form ω, and we see from this that $\phi_t(\Delta)$, which is $\phi_{t*}(J_t)$-holomorphic, is a symplectic submanifold for all t.

For fixed t, set $\omega_s = s\phi_t^*(\omega) + (1-s)\omega$. This is a symplectic form for all $0 \le s \le 1$. It is clearly closed and is symplectic since it tames J_t. We note that Δ is symplectic for all ω_s and, if $t = 0$ or $t = 1$, L_t is Lagrangian with respect to all ω_s. Hence by an application of Moser's theorem we can find diffeomorphisms ψ_t of $S^2 \times S^2$ such that $\psi_t^*(\omega) = \phi_t^*(\omega)$. The ψ_t can be chosen to vary smoothly with t, to fix Δ and such that $\psi_0 = \mathrm{id}$ and ψ_1 fixes L. To see this, we recall that Moser's method involves writing $\omega_s = \omega_0 + d\alpha_s$ and studying the flow of the vector field X_s defined by $X_s \rfloor \omega_s = \frac{d\alpha_s}{ds}$. The definition implies that $\mathcal{L}_{X_s}\omega_s = d(\frac{d\alpha_s}{ds}) = \frac{d\omega_s}{ds}$. We have the freedom in this construction to add any smooth family of exact 1-forms β_s to the α_s. These β_s can be chosen such that $\alpha_s + \beta_s$ vanishes on the symplectic normal bundle to Δ and, if $t = 0$ or $t = 1$, on the tangent bundle to L_t. Then the flow fixes Δ and, if $t = 0$ or $t = 1$, also fixes L_t.

Thus $\psi_t(L_t)$ is a Lagrangian isotopy from L to $\overline{\Delta}$ inside $S^2 \times S^2 \setminus \Delta$ as required.

To show that the space $\mathcal{L}$ of Lagrangian spheres is contractible, by applying a result of R. S. Palais [28], the Corollary following Theorem 15, it suffices to show that $\pi_k(\mathcal{L}) = 0$ for all integers $k \ge 0$. Thus Theorem 2 reduces to the following.

Theorem 24 *Given a family of Lagrangian spheres $L_p \subset S^2 \times S^2 \setminus \Delta$ for $p \in S^k$ there exists a family of Hamiltonian isotopies of $S^2 \times S^2$ which fix Δ and map L_p onto $\overline{\Delta}$.*

This follows exactly as Theorem 1 for T^*S^2 by applying the full parameterized version of Theorem 11 once we establish the analogue of Lemma 21, that is, we need to show the following.

Lemma 25 *There exists a family of tame almost-complex structures J_p on $S^2 \times S^2$ such that each curve in the corresponding foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects L_p*

transversally in a single point. The almost-complex structures J_p can be taken to agree with J_0 near Δ.

Proof of lemma 25 We briefly recall the construction of the almost-complex structures in [18]. Associated to each $p \in S^k$ and positive integer N there exists a tame almost-complex structure $J_{p,N}$ on $S^2 \times S^2$ which corresponds to stretching the neck to length N along the boundary of a small tubular neighborhood of L_p. It is easy to arrange that the $J_{p,N}$ vary smoothly with p. For fixed p it was shown in [18] that, after taking a subsequence as $N \to \infty$, reparameterizations of $J_{p,N}$-holomorphic spheres in the corresponding foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ converge smoothly to finite energy planes in T^*L_p. For a suitable choice of the $J_{p,N}$ these finite energy planes must be transverse to L_p, in particular the $J_{p,N}$ holomorphic foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are transverse to L_p for N sufficiently large. We claim that there exists an N such that the $J_{p,N}$-holomorphic foliations are transverse to L_p for all p, thus establishing the lemma.

Suppose that the claim is false. Then for all j there exists a point $q_j \in S^k$ and a $J_{q_j,j}$-holomorphic sphere C_j tangent somewhere to L_{q_j}. A subsequence of $\{q_j\}$ will converge to some $p \in S^k$. Now, there exist diffeomorphisms $a_j : S^2 \times S^2 \to S^2 \times S^2$ such that $a_j(L_{q_j}) = L_p$ and a_j is an $(J_{q_j,j}, J_{p,j})$-biholomorphism on the tubular neighborhood of L_{q_j}. Furthermore, after taking the subsequence, the a_j can be chosen to converge C^∞ uniformly to the identity and so $I_j = a_{j*}(J_{q_j,j})$ is a sequence of almost-complex structures on $S^2 \times S^2$ agreeing with $J_{p,j}$ near L_p and which are tame for j large. We apply the compactness theorem from [3] exactly as in [18] to the I_j-holomorphic foliations $\mathcal{F}_0$ and $\mathcal{F}_1$. The same proof shows that reparameterizations converge to finite energy planes in T^*L_p transverse to L_p. But this gives a contradiction as required since the I_j holomorphic spheres $a_j(C_j)$ are tangent to L_p. $\square$

3.1 Lagrangian projective planes

Here we remark that the same methods as above can be used to derive Theorem 8.

The involution σ of $S^2 \times S^2$ interchanging the two factors has fixed-point set equal to Δ and restricts to the antipodal map on $\overline{\Delta}$. Quotienting out by σ, we observe that $S^2 \times S^2 \setminus \Delta$ is a double-cover of a unit cotangent bundle of $\mathbb{R}P^2$ and Lagrangian projective planes in $T^*\mathbb{R}P^2$ homotopic to the zero-section therefore correspond to σ-invariant Lagrangian spheres in $S^2 \times S^2 \setminus \Delta$ homotopic to $\overline{\Delta}$. Therefore Theorem 8 can be established by showing that if the initial Lagrangians L_p are σ-invariant then it is possible to repeat the proof above and find a family $L_{p,t}$ of Lagrangian spheres for $0 \le t \le 1$ with $L_{p,0} = L_p$ and $L_{p,1} = \overline{\Delta}$ and such that all $L_{p,t}$ are σ-invariant. As in section 2, for convenience let us drop the subscript p consider the case of a single invariant Lagrangian L.

As L is σ-invariant we may assume that the almost-complex structures involved in the proof are all σ-invariant, in particular $\sigma\mathcal{F}_0(t,x) = \mathcal{F}_1(t,x)$ for all x and t. Then Moser's argument at the end of the proof will give a family of σ-invariant Lagrangians provided that the $L_t = \phi_t^{-1}(\overline{\Delta})$ are σ-invariant. This holds provided that the maps g_t can be chosen such that $(g_t^{-1}A)^2 = \text{id.}$, where A is the antipodal map. We recall that $g_1 = -f^{-1}$ where f is the diffeomorphism of Δ defined such that $\mathcal{F}_0(1,x) \cap \mathcal{F}_1(1,f(x)) \in L$ for all x. As L is disjoint from Δ the map f is fixed point free and, as L is σ-invariant, $f^2 = \text{id.}$.

The following can be established without relying on Theorem 3. We identify Δ with the 2-sphere S^2.

Lemma 26 *There exists a family f_t of fixed point free diffeomorphisms of S^2 for $0 \le t \le 1$ with $f_0 = A$ and $f_1 = f$ and $f_t^2 = \text{id.}$ for all t.*

Setting $g_t = -f_t^{-1}$ we then have an isotopy satisfying the conclusions of Lemma 23 but also generating σ-invariant Lagrangians. Thus, given the above, Lemma 26 immediately implies Theorem 8.

Proof of Lemma 26

First observe that any such f is an isometry of S^2 with respect to the round metric given by pulling back a round metric from $\mathbb{R}P^2 = S^2/x \sim f(x)$. We also observe that a round metric on S^2 has a unique orientation reversing isometry which is fixed point free and squares to the identity, for the standard metric this is just the antipodal map A. The space of round metrics on S^2 is contractible by Smale's theorem [30] again. Thus we have a surjective map from a contractible space to the space of fixed point free diffeomorphisms which square to the identity, and therefore our space of diffeomorphisms is connected as required. $\square$

4 Manifolds with 1-handles

We will now consider the class of convex symplectic manifolds constructed by adding 1-handles to the unit cotangent bundle T^1S^2 in order to establish Theorem 1 in this case. Our first observation is that any such manifold M can be symplectically embedded in $(S^2 \times S^2, \omega)$, see Figure 4, after perhaps scaling the symplectic form. This follows from the methods of [10]. We can arrange that the zero-section in T^1S^2 again becomes identified with $\overline{\Delta}$ and the boundary of M is a smooth hypersurface Σ of contact-type in $S^2 \times S^2$. More precisely one can think of M as a Stein manifold having a bounded plurisubharmonic exhaustion function which is zero on the zero-section in T^*S^2 and whose other critical points are nondegenerate and have Morse index 1. The symplectic form on M is the Kähler form of the plurisubharmonic exhaustion. Now, as in [7] or [8], 2-handles can be added to M to cancel the 1-handles and produce a Stein manifold symplectomorphic to $T^1S^2 = S^2 \times S^2 \setminus \Delta$. We will later use the fact that Σ is now a level-set of a plurisubharmonic exhaustion on $S^2 \times S^2 \setminus \Delta$.

We plan to find families of almost-complex structures J_t on $S^2 \times S^2$ and diffeomorphisms $f_t : \Delta \to \Delta$ such that $\mathcal{F}_0(t,x)$ intersects $\mathcal{F}_1(t, f_t(x))$ on embedded spheres $L_t \subset M$ with $L_0 = \overline{\Delta}$ and $L_1 = L$. The notation here is from Definition 22. The almost-complex structures can be constructed by deforming J_0 in a neighbourhood of Σ and, for t close to 0 or 1, also in a neighbourhood of $\overline{\Delta}$ or L.

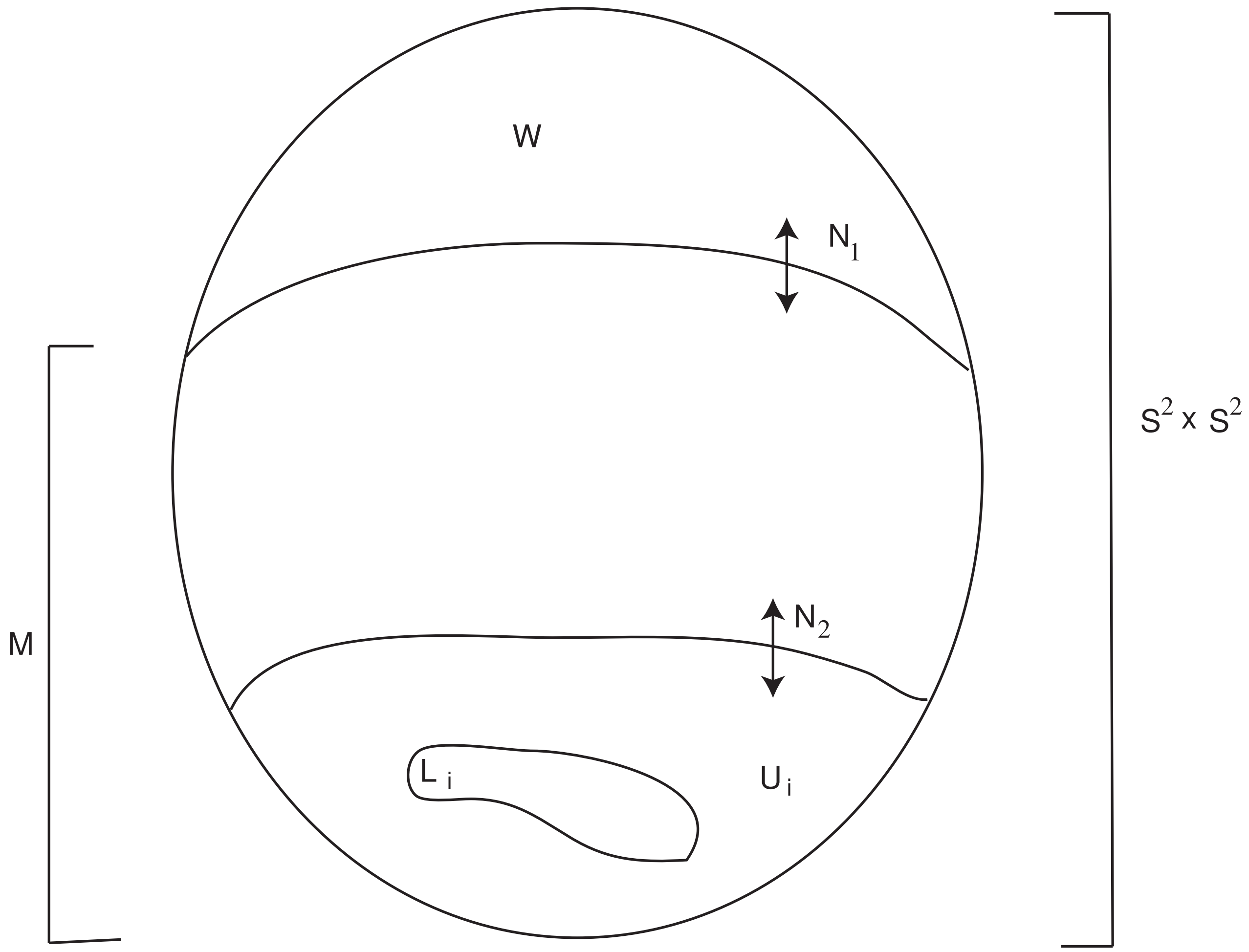

Figure 4: Subsets of $S^2 \times S^2$

Suppose that we perform the operation of stretching-the-neck along Σ. That is, we symplectically identify a neighbourhood of Σ in $S^2 \times S^2$ with $((-\epsilon, \epsilon) \times \Sigma, d(e^t\alpha))$, where α is a fixed contact form on Σ. We can then produce a manifold A_N by replacing this neighbourhood by $(-N, N) \times \Sigma$. Our original almost-complex structure can be extended over $(-N, N) \times \Sigma$ to be translation invariant and the symplectic form can be extended over $(-N, N) \times \Sigma$ such that A_N is symplectomorphic to $(S^2 \times S^2, \omega)$ via a symplectomorphism equal to the identity outside $(-N, N) \times \Sigma$ (for this see [23]). Under this symplectomorphism we can think of stretching the neck as studying a family of almost-complex structures J_N on $S^2 \times S^2$ which degenerate along Σ as $N \to \infty$.

At the same time, we can deform the almost-complex structure along the boundary of tubular neighborhoods U_0 or U_1 of $L_0 = \overline{\Delta}$ or $L_1 = L$ respectively, see again Figure 4. Stretching to length N_1 and N_2 on the contact hypersurfaces Σ and ∂U_i respectively we obtain almost-complex structures $J_{N,0}$ and $J_{N,1}$, where $N = (N_1, N_2)$. There exist smooth families of almost-complex structures $J_{N,t}$ connecting $J_{N,0}$ and $J_{N,1}$ which are fixed on the tubular neighborhoods of Σ and in the complement of M.

The following summarizes work of H. Hofer, K. Wysocki and E. Zehnder, see [23], applied to our situation. We recall that the completion of a symplectic manifold with a contact boundary $(P, \xi = \{\lambda = 0\})$ denotes the union of the symplectic manifold with a cylindrical end equal to the positive $((0, \infty) \times P, d(e^t\lambda))$ or negative $((-\infty, 0) \times P, d(e^t\lambda))$ symplectization of the boundary, depending upon whether the boundary is convex or concave respectively.

Lemma 27 *Given a sequence $N = (N_1, N_2)$ in which both entries tend towards infinity, there exists a subsequence such that for $i, j = 0, 1$ the corresponding $J_{N,i}$-holomorphic curves in $\mathcal{F}_j$ will converge to unions of finite energy curves as $N \to \infty$. The limiting finite energy curves can be chosen to extend to foliations of three symplectic manifolds with cylindrical ends, namely the completion W of the complement of M in $S^2 \times S^2$, the completion of U_i, which will be a copy of T^*S^2, and the completion of $M \backslash U_i$, which has two ends symplectomorphic to the* *positive symplectization of Σ and the negative symplectization of the boundary of U_i.*

A priori these foliations will depend upon the subsequence $(N_1, N_2) \to \infty$. Similarly we can let just N_1 or N_2 tend towards infinity. In the first case we produce foliations of W and the completion of M. In the second case we produce foliations of the completions of U_i and $S^2 \times S^2 \setminus U_i$.

Outline of the proof of Lemma 27

The relevant compactness result here as $N \to \infty$ is contained in [3]. Specifically it says that given a fixed point $p \in S^2 \times S^2$ there exists a subsequence of N such that the J_N-holomorphic curves through p converge to a holomorphic building, that is, a union of finite energy curves in our completed symplectic manifolds (with respect to a compatible almost-complex structure translation invariant near the ends). If we choose a countable dense collection of p in each of the three manifolds then taking a diagonal subsequence we may find a sequence of N such that the J_N-holomorphic curves through all p in the collection converge. Note that any point in one of our competed manifolds can be identified with a point in each A_N provided that N is sufficiently large. The result is finite energy curves through a dense set of points in each of our completions. These curves either coincide or are distinct by positivity of intersection, see [26], since, as the curves are limits of the same sequence of N, any intersections of curves with different images will also be seen as intersections of J_N-holomorphic curves, which are known to form a foliation. Finally, limits of the finite energy curves through the dense set of points produce curves through every point, and positivity of intersection again implies that these form a foliation.

Further facts about finite energy curves, such as definitions, asymptotic convergence to Reeb orbits and Fredholm properties can be found in the series of papers [20], [21], [22]. A brief summary containing the facts we need here appears in section 2 of [18]. □

The foliations of the completion of U_i were determined in [18], Lemma 10. For U_i and its almost-complex structure suitably chosen, the Reeb flow on ∂U_i

is foliated by closed orbits, and exactly one curve in each foliation is asymptotic to each closed orbit. Also, each curve in the foliation from $\mathcal{F}_0$ intersects in a single point each curve in the foliation from $\mathcal{F}_1$ provided that the curves have different asymptotic limits. Another result coming from the analysis in [18], see Lemmas 8 and 9, is that the curves in both foliations are transverse to the zero-section, and it follows that, for a fixed N_1, the curves in the foliations of $S^2 \times S^2$ are transverse to L_i for N_2 sufficiently large.

We now look at the resulting foliations of W coming from taking limits of curves in $\mathcal{F}_0$ or $\mathcal{F}_1$. A priori of course these foliations could depend on the almost-complex structure J on M as well as the original homology class. We fix once and for all the restriction of the almost-complex structure to W. There is a natural topology on almost-complex structures on M which are allowed to degenerate along one of the ∂U_i. Namely we use the topology of smooth uniform convergence to define the restriction of the topology to the nonsingular and singular (degenerate) structures, and say that under neck stretching the nonsingular structures converge to the singular limit. We will use $F_i(J)$ to denote the limiting foliation of W resulting from stretching curves in $\mathcal{F}_i$, suppressing the possible dependence on the sequence $N_1 \to \infty$. In the case when J is singular this corresponds to taking limits of curves in $\mathcal{F}_i$ as $N_1 << N_2$ but both tend to infinity (since when N_2 is very large the holomorphic spheres in the $\mathcal{F}_i$ intersect the complement of U_i in a foliation arbitrarily close to the finite energy foliation of $S^2 \times S^2 \setminus U_i$ coming from letting $N_2 \to \infty$).

Lemma 28 *For a fixed singular J on M the foliations $F_0(J)$ and $F_1(J)$ formed by taking limits of a common convergent subsequence of $N_1 \to \infty$ coincide. Therefore we may denote the common foliation by $F(J)$.*

Proof

Suppose that a curve in the limiting $F_0(J)$ foliation intersects one in the limiting $F_1(J)$ foliation but that the curves do not have identical images. Then, by the positivity of intersections, any intersections of limiting finite energy curves must also be seen as intersections of holomorphic spheres in the foliations $\mathcal{F}_0$

and $\mathcal{F}_1$ of the complement of U_i when N_1 is large. By the positivity of intersections again, intersections are stable under perturbation and so we may assume that the curves have distinct asymptotic limits on ∂U_i. But this gives a contradiction since we can then topologically glue in planes in U_i homologous to the corresponding finite energy planes to produce spheres in the classes $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ with intersection number 2. □

Now we consider limits when the complex structure J_1 on M is nonsingular.

Lemma 29 *The foliation $F_i(J_1)$ of W coincides with $F(J)$, where J is the fixed singular structure on M above. Therefore we can denote the common foliation simply by F.*

Proof We study the limits of curves through a generic point $x \in \Delta$. By this we mean that the deformation index $I(C)$ of the finite energy curve C in $F(J)$ passing through the point x satisfies $I(C) = 2$. This is equivalent to saying that the constrained index (amongst curves passing through x) is 0. Suppose that the curve C' in $F_i(J_1)$ passing through x does not coincide with C. Note that if $F_i(J_1)$ differs from $F(J)$ then an open subset of the corresponding curves must be different, and in particular curves through a generic point will differ.

By considering a family of almost-complex structures J_t connecting $J_0 = J$ and J_1 and fixed on W (so that the almost-complex structures are degenerating along ∂U_i), we find corresponding families of J_t-holomorphic curves, say $C_{N_1,t}$, such that, as $N_1 \to \infty$, the curves $C_{N_1,0}$ have C as a limiting component in W and the curves $C_{N_1,1}$ have C' as a limiting component. Denote the intersection of a holomorphic curve D with a compact subset K of W by D^K. Then, given an $\epsilon > 0$, for N_1 sufficiently large, and provided the curves are suitable parameterized, $C^K_{N_1,0}$ is ϵ-close to C^K, and $C^K_{N_1,1}$ is ϵ-close to ${C'}^K$ in a fixed C^∞ topology. Now, as C and C' are distinct, the distance between C^K and ${C'}^K$ is bounded away from 0 in our C^∞ topology. Therefore, for any given small ϵ and N_1 sufficiently large we can find a J_t holomorphic sphere $C_{N_1,t}$ such that the distance between $C^K_{N_1,t}$ and C^K is exactly ϵ. Taking a limit, the same is true for the limiting finite energy curve, say C'' in W, that is, C^K

and C''^K are exactly ϵ apart. Then, if the almost-complex structure on W is regular and ϵ is chosen suitably, C'' must have positive constrained deformation index (as curves through x of constrained index 0 form a 0-dimensional set) or, equivalently, unconstrained index $I(C'') \geq 3$. But, following the analysis of M-L. Yau, see [36], for a suitable choice of contact form on Σ the Reeb orbits (of a bounded period) correspond either to Reeb orbits on a perturbed T^1S^2, or are multiple covers of orbits lying entirely in the 1-handles. In any case, they have Conley-Zehnder index at least 1 and therefore the components D of the limit of the $C_{N_1,t}$ in M all have nonnegative deformation index. Now $I(C'')+I(D) = 2$, the index of our original curves, and so it follows that $I(C'') \leq 2$ and we have a contradiction. □

Corollary 30 *Given a compact subset of singular and nonsingular almost-complex structures on M and a compact subset K of W, there exists an N_1 such that if the complex structure is stretched to length N_1 along Σ then (independently of the almost-complex structure on M) we may assume that the restriction to K of the curve in either $\mathcal{F}_0$ or $\mathcal{F}_1$ through a point $x \in \Delta$ lies C^∞ ϵ-close to the corresponding curve in F restricted to K.*

Otherwise, letting $N_1 \to \infty$, we reach a contradiction. In particular, if we stretch to length at least N_1 then curves in $\mathcal{F}_0$ and $\mathcal{F}_1$ through any pair of points $x, y \in \Delta$ which are distance order ϵ apart do not intersect in the fixed compact subset of W.

The following is the key proposition for the proof of Theorem 1.

Proposition 31 *There exists a family of almost-complex structures J_t and diffeomorphisms f_t of Δ such that $\mathcal{F}_0(t,x) \cap \mathcal{F}_1(t, f_t(x)) \in M$ for all $x \in \Delta$. Furthermore, we may assume that the sphere isotopy*

$$L_t = \{\mathcal{F}_0(t,x) \cap \mathcal{F}_1(t, f_t(x)) | x \in \Delta\}$$

satisfies $L_0 = \overline{\Delta}$ and $L_1 = L$.

Proof Letting $N_2 = \infty$, for $i = 0, 1$ we have two foliations of U_i and a single foliation of $S^2 \times S^2 \setminus U_i$ coming from limits of curves in $\mathcal{F}_0$ and $\mathcal{F}_1$. We can define diffeomorphisms f_i of Δ as follows. Each $p \in L_i$ lies on a unique plane P_0 in the foliation of the completion of U_i coming from limits of $\mathcal{F}_0$. Similarly, p lies on a unique plane P_1 in the completion of U_i coming from limits of $\mathcal{F}_1$. There is a unique plane Q_0 in $S^2 \times S^2 \setminus U_i$ whose negative asymptotic limit corresponds to the positive limit of P_0, and there is a unique plane Q_1 in $S^2 \times S^2 \setminus U_i$ whose negative asymptotic limit corresponds to the positive limit of P_1. If we denote the intersection of Q_0 with Δ by x then $f_i(x)$ can be defined to be the intersection with Δ of Q_1.

By construction the f_i are fixed-point free, therefore by Theorem 3 they can be connected by a family of fixed-point free diffeomorphisms f_t of Δ. Assume that for any $x \in \Delta$ and $t \in [0,1]$ the points x and $f_t(x)$ are at least ϵ' apart. Leaves of F which intersect Δ in points ϵ' apart we may assume to remain an ϵ apart on W (thought of as a compact subset of its completion). Therefore we can choose a corresponding N_1 as in Corollary 30 such that when we stretch to length N_1 along Σ the sphere in $\mathcal{F}_0$ through x does not intersect the sphere in $\mathcal{F}_1$ through $f_t(x)$ in W for any x, t. In other words the spheres intersect in M.

We can find a family of almost-complex structures J_t on $S^2 \times S^2$ such that if $t < \delta$ then J_t is stretched to length N_2 along ∂U_0; if $t > 1 - \delta$ then J_t is stretched to length N_2 along ∂U_1; if $\delta \leq t \leq 1 - \delta$ then J_t is stretched to length N_1 along Σ. Then we claim that if N_2 is chosen sufficiently large the spheres L_t are all disjoint from W as required. Given our choice of N_1 this is already established for the spheres L_t when $\delta \leq t \leq 1 - \delta$. For other t the claim follows by taking a limit as $N_2 \to \infty$. If the claim were false for $t < \delta$ then, taking the limit, we could find an $x \in \Delta$ such that the curve in the foliation of $S^2 \times S^2 \setminus U_0$ coming from $\mathcal{F}_0$ and passing through x intersects in the curve in the foliation of $S^2 \times S^2 \setminus U_0$ coming from $\mathcal{F}_1$ through $f_t(x)$. Indeed, the intersection point is a limit of intersection points of J_t-holomorphic spheres in W. But this is a contradiction as these two foliations coincide.

We remark that for a fixed large, but finite, N_2 this construction gives L_0 and L_1 only C^∞ close to $\overline{\Delta}$ and L respectively, but this can easily be corrected with a small adjustment of the f_t and Proposition 31 is established. $\square$

Using Proposition 31, we now complete the proof of Theorem 1. Following the method of section 3, we can find a family of symplectic forms ω_t on $S^2 \times S^2$ such that L_t is Lagrangian with respect to ω_t. The ω_t restrict to exact symplectic forms on M, say $\omega_t = d\alpha_t$, which tame J_t. In a tubular neighborhood $V = (-\epsilon, 0) \times \Sigma$ of the boundary $\Sigma = \{0\} \times \Sigma$ of M, define a function $\chi : V \to [0, 1)$ such that $\chi(r, y) = \chi(r)$ is an increasing function of r, $\chi(r, y) = 0$ for r close to $-\epsilon$ and $\chi(r, y) = 1$ for r close to 0. Then, first scaling α_t if necessary, we can replace it by $\beta_t = (1 - \chi)\alpha_t + \chi e^r \alpha$ in V. The new form $\omega_t = d\beta_t$ will still be symplectic and tamed by J_t (for α_t suitably scaled) but now agrees with ω near Σ. Assuming V to be disjoint from all L_t, the submanifolds L_t will still be Lagrangian with respect to ω_t.

We now apply Moser's method as in section 3 to find a symplectomorphism between (M, ω_t) and (M, ω) and thereby isotope the L_t into Lagrangian submanifolds of (M, ω). As before, this can be arranged to fix L_0 and L_1 and now also the neighborhood V. Thus it gives our Lagrangian isotopy as required.

5 Proof of Theorem 4

In this section we study the symplectic manifold W, which is a plumbing of two copies of T^*S^2. Namely we take two copies of T^*S^2 and identify the cotangent fibers projecting to a disk D in S^2 with a product $D \times E$ in each copy. We then identify the two copies of $D \times E$, preserving the product structure but reversing the factors. Alternatively W can be realized as a Stein manifold by adding a 2-handle to a disk bundle T^1S^2 along the boundary of one fiber, a Legendrian curve for the natural choice of contact structure.

In any case, W is naturally a symplectic manifold with symplectic form ω_0 and contains two Lagrangian spheres L_1 and L_2 corresponding to the two zero-sections. We will think of its non-compact end as a copy of $[0, \infty) \times M$ where M carries a contact structure with contact form α and the symplectic structure on the end is given by $\omega = d(e^t \alpha)$.

The manifold M is a lens space $L(3, 2)$. The contact form can be described as follows.

Let S be the 3-sphere given by

$$S = \{(z_1, z_2) \in \mathbb{C}^2 | H(x) = 1\}$$

where

$$H(x) = |z_1|^2 + \frac{1}{r^2}|z_2|^2$$

and equipped with the contact form $\lambda|_S$ where

$$\lambda = \frac{i}{4}\sum_{j=1}^{2}(z_j d\bar{z}_j - \bar{z}_j dz_j).$$

Let $r^2 > 1$ be an irrational number and let p_0 and p_1 denote the periodic orbits $\{z_2 = 0\} \cap S$ and $\{z_1 = 0\} \cap S$ respectively.

Lemma 32 (see [19] Lemma 1.6) *The associated Reeb vector field posesses precisely two periodic orbits p_0 and p_1. They are nondegenerate and have Conley-Zehnder indices $\mu(p_0) = 3$ and $\mu(p_1) = 2n + 1$ where $n < r^2 + 1 < n + 1$.*

Now we observe that S and $\lambda|_S$ are invariant under the map $\sigma : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{3}} z_1, e^{\frac{4\pi i}{3}} z_2)$ and so project to $L(3, 2)$ to give the contact form α. The orbits p_0 and p_1 triple cover periodic orbits x_0 and x_1 on our $L(3, 2)$. Let X be the corresponding Reeb vectorfield.

Our proof will proceed as follows. On $[0, \infty) \times M$ we choose a tame almost-complex structure J which is translation invariant, preserves the contact planes on M and satisfies $J(\frac{\partial}{\partial t}) = X$. Throughout the proof we will fix this almost-complex structure. It can be extended to a tame almost-complex structure J on W and for each extension we will describe a foliation of W by finite energy planes asymptotic to multiple covers of x_0. Let $L \subset W$ be a Lagrangian sphere homotopic to L_1. Then we pay specific attention to the pattern of the foliation relative to L when we change J by stretching the neck near L. This is all done in section 5.1.

In section 5.2, using our holomorphic foliations we can construct plurisubharmonic exhaustion functions on W. These functions will have exactly one

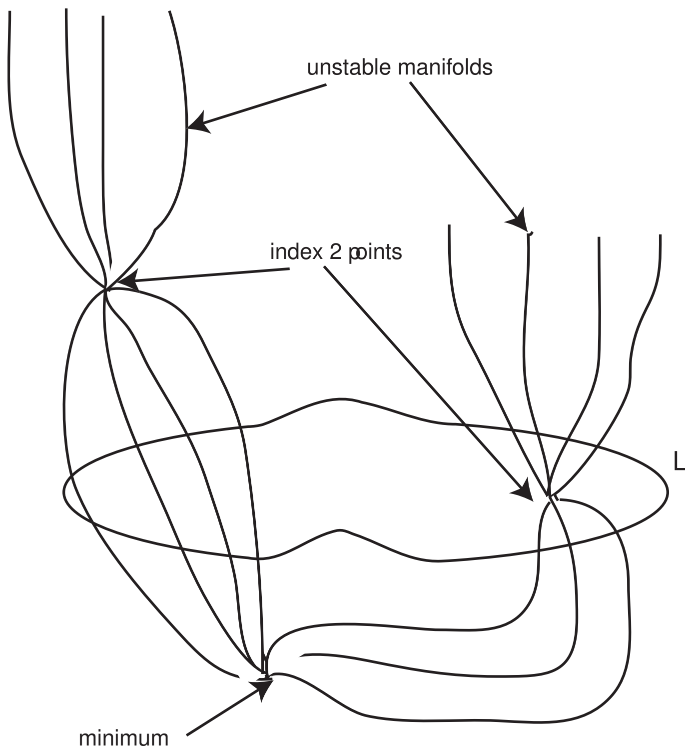

Figure 5: An unstable manifold disjoint from L

minimum and two critical points of index 2. It will turn out that after stretching the neck along L, the unstable manifold with respect to the upward gradient flow of one critical point will be disjoint from L. The arrangement we aim for is illustrated in Figure 5.

All such plurisubharmonic exhaustions give isotopic symplectic structures on W. The final part of the proof, in section 5.3, will use these isotopies to construct the symplectomorphism needed for our theorem. Of course Theorem 1 will also be used, in a form which says that a Lagrangian sphere disjoint from the unstable manifold of one critical point is Hamiltonian isotopic to the stable manifold of the other critical point.

5.1 Finite energy holomorphic curves in W

5.1.1 Finite energy foliations

As stated above, W admits a foliation by finite energy planes. More specifically the following is true.

Theorem 33 *For any tame extension J, the almost-complex manifold (W, J) can be foliated by finite energy planes. Exactly three planes in the foliation, E_0, E_1, E_2, are asymptotic to x_0. The other finite energy planes are all asymptotic to $3x_0$. After choosing orientations for L_1 and L_2 we may assume that $E_i \bullet L_j = -\delta_{ij}$ and $E_0 \bullet L_j = 1$ for $i, j = 1, 2$.*

Proof

This is very similar to the proof in [16], (which of course is heavily reliant on the series of papers [20], [21], [22]) but the arrangement of finite energy planes is different to the situation covered there. In fact, [16] described finite energy foliations of Stein manifolds diffeomorphic to disk bundles over S^2 whose boundaries are the Lens spaces $L(p, 1)$. The basic case of the foliation of T^*S^2 with boundary $\mathbb{R}P^3$ was worked out earlier in [15]. The proofs, and this one, follow the same path in that they start with the finite energy planes in $\mathbb{R} \times S^3$ constructed in [19] (using the method of filling by holomorphic disks) and project these to get finite energy planes in W which are topologically trivial relative to the boundary but appear in a 2-dimensional family. A process of elimination using index and area inequalities then determines the behaviour of the family of curves as they propagate into W.

More precisely, this reasoning, originating in the works of H. Hofer, K. Wysocki and E. Zehnder, [19], Theorem 5.1, implies that there is a 2-dimensional moduli space of unparameterized disjoint embedded finite energy planes asymptotic to $3x_0$. The planes lying in $[0, \infty) \times M$ are all disjoint from the cylinder $[0, \infty) \times x_0$ lying over the Reeb orbit and the natural S^1 action on $\mathbb{C}^2$ by rotation in the complex planes restricts to an action on our original 3-sphere which in turn descends to act on this subset of the moduli space. Then we may assume

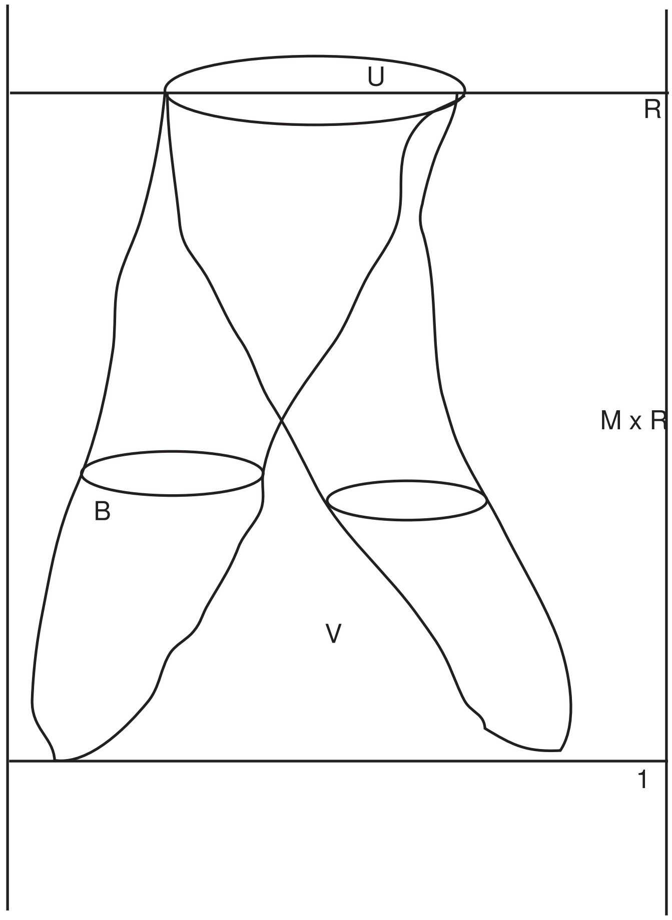

Figure 6: Finite energy planes making up B

that there exists an S^1 family of planes in $[0, \infty) \times M$ such that each plane in the family touches $\{1\} \times M$ in a single point (the family is given simply by the S^1 orbit of a plane whose $\mathbb{R}$ coordinate has a single absolute minimum). Choosing R large, this S^1 family will intersect $\{R\} \times M$ in an S^1 family of circles, a 2-torus, which bounds a solid torus U containing the periodic orbit x_0. Let B be the intersection of the S^1 family of finite energy planes with $[0, R] \times M$, see Figure 6.

Now, we may change our almost-complex structure near B such that it is biholomorphic to $\{|z_1| \leq 1, |z_2| = 1\} \subset \mathbb{C}^2$, where B itself is identified with

$\{|z_1| \leq 1, |z_2| = 1\}$ and a neighborhood U' of $\partial U \subset U$ with $\{1 - \epsilon < |z_2|^2 \leq 1, |z_2|^2 = 2 - |z_1|^2\}$. We may assume when perturbing the almost-complex structure that the planes intersecting B remain holomorphic, and also that the foliation of nearby planes coincides with the z_1-planes near $B \cup U$ and with the original foliation away from a small neighborhood. Let us replace $B \cup U'$ by the hypersurface $\{|z_2|^2 = h(|z_1|^2)\}$ where h is a concave decreasing function approximately equal to 1 when $|z_1|^2 < 1$ and equal to $2 - |z_1|^2$ when $|z_1|^2 > 1 + \epsilon$. Then the new hypersurface is strictly pseudoconvex and, together with $U \setminus U'$, bounds a domain V. We recall that (complete) plurisubharmonic exhaustions of Stein domains define a symplectic structure which depends only upon the underlying complex domain, see [10] Theorem 1.4.A, and in our case this structure is symplectomorphic to W (as it is isotopic to W through domains defined by $\mathbb{R}$ translates of B).

We are interested in an extension of our moduli space to a family of finite energy planes foliating V. After the perturbation of $B \cup U'$ we observe that there exists an S^1 family of finite energy planes (corresponding to $|z_2|^2 = h(0)$) which intersects B in a circle γ of complex tangencies. Other nearby finite energy planes in our moduli space intersect B in circles linking γ, see for example Figure 2 in [16]. Since M is an $L(3, 2)$, after choosing coordinates on U we may assume that the finite energy planes intersecting ∂U do so in $(3, 1)$ curves, where the first component represents the class of a longitude homotopic to x_0.

The planes in the moduli space intersecting U do not form a compact set. In fact, as in [16], Lemma 3.2, bubbling occurs and sequences of finite energy planes asymptotic to $3x_0$ will converge to three finite energy planes E_0, E_1, E_2 asymptotic to x_0. (The topology of V implies that we now get bubbling into three planes, energy considerations imply that they are all asymptotic to x_0.) We call these rigid planes since the moduli space of finite energy planes asymptotic to x_0 modulo reparameterization has dimension 0. Together with the finite energy planes asymptotic to $3x_0$ the rigid planes complete our foliation.

We notice as in [16] that V is homotopic to the intersections of the rigid planes with V, after identifying their boundaries in U. (This implies that there is no further bubbling.) To check the intersection numbers, we can choose a convenient almost-complex structure J since the numbers are independent of the choice. In fact there is an S^1 subgroup of symplectomorphisms of W which on each cotangent bundle corresponds to the extension via differentials of the rotations of L_j about the axis through the intersection point $q \in L_1 \cap L_2$. Let q_1 and q_2 be the antipodal point of q in L_1 and L_2 respectively. If the almost-complex structure is invariant under these symplectomorphisms, then so are the rigid planes (as they appear only in dimension 0). Stokes' Theorem implies that holomorphic planes cannot intersect our Lagrangians in circles (since they are symplectic and the symplectic form on W is exact) and so the rigid planes must intersect the two Lagrangians in fixed points of the S^1-action. A plane disjoint from the Lagrangians is homotopic to a plane in $[0, \infty) \times M$ where the asymptotic limit x_0 is not contractible. Therefore each rigid plane does indeed intersect a Lagrangian and we can order our planes so that $E_0 \cap L_j = \{q\}$, $E_1 \cap L_1 = \{q_1\}$ and $E_2 \cap L_2 = \{q_2\}$. Choosing orientations for L_1 and L_2 gives the theorem as required. □

Topologically the intersections of our finite energy planes with U can be visualized as follows. We note however that this is an idealized picture. In practice holomorphic curves can have quite complicated tangencies with pseudoconvex hypersurfaces. In the next section we will use the technique of filling by holomorphic disks to ensure that the pattern we describe here does indeed occur.

We look at a cross-section A of U. The interior of A has three special points corresponding to the intersection of A with the rigid planes. By taking R sufficiently large, the rigid planes can be assumed to intersect $U \subset \{R\} \times M$ transversally. A finite energy plane intersecting ∂U hits ∂A in three points. Choosing a path from one of these points to one of the special points determines a 1-parameter family of finite energy planes intersecting the path. The intersections of these planes with A generate two more paths from our points in ∂A to the remaining special points. Conversely a path in our moduli space

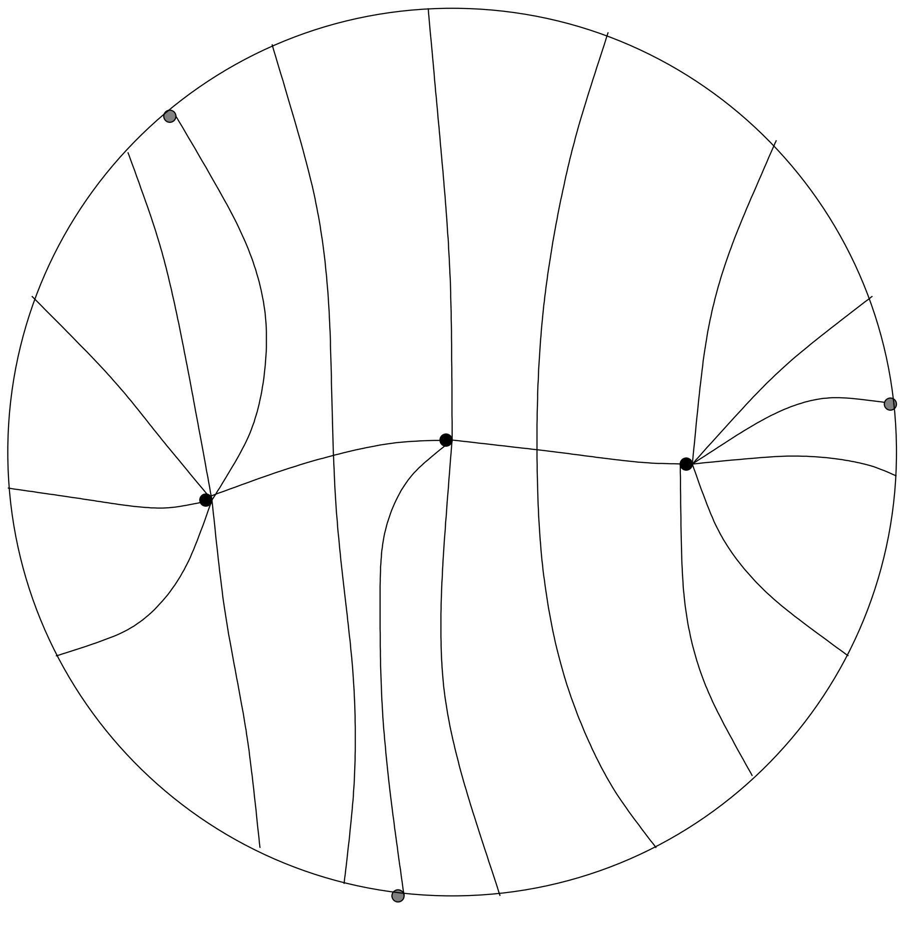

Figure 7: flowlines in the cross-section A

starting from a plane intersecting ∂U and converging to the bubbled planes generates three paths in A. Starting with other planes intersecting ∂U we can generate a vector field on A with elliptic points corresponding to the rigid planes. The vector field will necessarily have hyperbolic points corresponding to tangencies of finite energy planes with U. Assuming that there are no more elliptic points (which could occur if a finite energy plane became tangent to U from the outside) there must be two hyperbolic points and the various integral curves are illustrated in Figure 7. The three marked points on the boundary are the intersections of a typical finite energy plane with ∂A. With our choice of subscripts the central special elliptic point in Figure 7 corresponds to E_0. Notice that the same picture is obtained in each cross-section A_θ of U for $\theta \in S^1$ and we can continuously choose coordinates in each A_θ so that the elliptic points lie in the same position. But then the points on ∂A_θ corresponding to a fixed finite energy plane will rotate through $\frac{2\pi}{3}$ in these coordinates as θ moves once around. The integral curves leaving our points on A_θ can be chosen so that they correspond to the same family of finite energy planes for each θ. These integral curves will encounter a hyperbolic point for two values of θ, corresponding to a 1-parameter family in the moduli space becoming tangent to U twice before bubbling. Topologically this means that two curves in the plane must contract to the boundary and that the plane will bubble into three components.

5.1.2 Stretching the neck

In this subsection we consider which finite energy planes in the foliation will intersect L if we perform a stretching-the-neck operation to deform J along the boundary of a tubular neighborhood of L. The result is the following.

Proposition 34 *There exist tame extensions J on W such that the rigid planes E_0 and E_1 intersect L transversally in a single point each and E_2 is disjoint from L. The nonrigid planes intersecting L contain an S^1 family with the property that the planes in the family intersect U in two disjoint circles. The union of the first S^1 family of circles form a torus enclosing $E_0 \cap U$ and $E_1 \cap U$; the* *union of the second S^1 family of circles form a torus enclosing $E_2 \cap U$.*

Proof of Proposition 34 The almost-complex structure is replaced by other almost-complex structures J_N as in section 4 where Σ is now the boundary of a tubular neighborhood Z of our Lagrangian L, which of course is diffeomorphic to $\mathbb{R}P^3 = L(2,1)$. We fix a contact form on Σ as above, now quotienting S^3 by the map $\sigma : (z_1, z_2) \mapsto (-z_1, -z_2)$. Denote by y_0 and y_1 the corresponding Reeb periodic orbits on Σ. Note that this form and the corresponding Reeb vector field are nondegenerate, unlike the Morse-Bott type form on ∂U_i used in section 4.

The stretching-the-neck procedure in section 4 applies again here to produce a finite energy foliation of the completed tubular neighborhood of L which is now identified with $T^*L = T^*S^2$. Since y_1 has large Conley-Zehnder index the finite energy curves must be asymptotic to y_0. The resulting foliation was first described in [15], (see Theorem 2.1, or alternatively, for a description entirely in terms of finite energy planes, rather than disks, Theorem 2.3 in [16]). There are two finite energy planes asymptotic to y_0 and the remaining planes are asymptotic to $2y_0$. The planes asymptotic to y_0 have intersection number ± 1 with L. They are rigid in the sense that the corresponding moduli space has dimension 0.

Taking limits of finite energy planes in the holomorphic foliations of (W, J_N) also results in a collection of finite energy curves lying in a completion of $W \setminus Z$ and the symplectization of Σ equipped with suitable almost-complex structures. After taking subsequences and additional limits as in Lemma 27 we also obtain finite energy foliations of $W \setminus Z$.

Suppose that an embedded finite energy curve u in $W \setminus Z$ has one positive asymptotic limit mx_0 and k negative asymptotic limits asymptotic to $n_i y_0$, $1 \leq i \leq k$. The virtual dimension of the moduli space of finite energy curves containing u modulo reparameterization is given by

$$\text{index}(u) = -(2 - 1 - k) + \mu(mx_0) - \sum_{i=1}^{k} \mu(n_i y_0)$$

where the μ are Conley-Zehnder indices with respect to a suitable trivialization giving $c_1(TW) = 0$. For m, n_i not too large $\mu(mx_0) = 2\lfloor \frac{m}{3} \rfloor + 1$ and $\mu(n_i y_0) = 2\lfloor \frac{n_i}{2} \rfloor + 1$ where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. Hence

$$\text{index}(u) = 2(\lfloor \frac{m}{3} \rfloor - \sum_{i=1}^{k} \lfloor \frac{n_i}{2} \rfloor). \tag{1}$$

In particular all virtual indices are even. Note that by the compactness result of [3] such curves are the only ones which can appear as components in $W \setminus Z$ of limits of our curves in W. Recall also our assumption is that L is homotopic to L_1.

Lemma 35 *Limits of the rigid planes E_0 or E_1 as $N \to \infty$ contain a rigid plane in T^*L and a cylindrical component in $W \setminus Z$ with ends asymptotic to x_0 and y_0. Limits of the rigid plane E_2 have no components in T^*L.*

Proof

As their intersection number with L is ± 1, the limits of the J_N holomorphic rigid planes E_0 and E_1 must contain planes in T^*L, and as the indices of the limiting components of the rigid planes add to 0 we may assume that these planes are rigid. The corresponding components of the limits in $W \setminus Z$ have a single positive end asymptotic to x_0 (which implies that there is only one component in $W \setminus Z$ and it is not a multiple cover), and, from the above, at least one negative end asymptotic to a multiple of y_0. The index formula (1) implies that such a component has negative index unless it is a cylinder with ends asymptotic to x_0 and y_0. As the curve is not a multiple cover we may assume by regularity that it has nonnegative index.

Next, we note that for the component of the limit of the E_2 in $W \setminus Z$ to have nonnegative index, since $m = 1$ formula (1) implies that its negative asymptotic limit can cover y_0 at most once. Therefore if there is a component in T^*L it must be a single rigid curve. But as such a curve has intersection number ± 1 with L and as E_2 has intersection number 0 this is impossible. $\square$

Lemma 36 *The limits of E_0 and E_1 in $W \setminus Z$ coincide. Limits of nonrigid planes either have no component in T^*L or the components in $W \setminus Z$ consist of the limit of E_2 and a cylinder double covering the corresponding component of the limit of E_0.*

Figure 8 illustrates the arrangement of the various limits of rigid curves.

Proof

We first note that since all indices are even the limiting foliation of $W \setminus Z$ must consist of the images of a single moduli space of deformation index 2 curves together with isolated curves of index 0. Indeed, the image of the evaluation map applied to such a 2-dimensional moduli space will be both open and closed in $W \setminus Z$ with boundary consisting of images of curves of index 0, which have codimension 2. As we know the behaviour of curves on the cylindrical end this 2-dimensional moduli space consists of planes asymptotic to $3x_0$.

Suppose that a converging sequence of nonrigid planes has a limiting component in T^*L. As nonrigid planes have intersection number 0 with L this limiting component is not a single rigid plane and so the negative ends of the components in $W \setminus Z$ cover y_0 a total of at least two times. As the positive ends cover x_0 a total of three times we cannot have a plane asymptotic to $3x_0$ as part of the image (as this would be the only component, and we know there are negative ends) and so all components in $W \setminus Z$ have deformation index 0.

We next claim that for any converging sequence of nonrigid planes with a nontrivial component in T^*L the limiting component in $W \setminus Z$ has the same image. For suppose not. Then, arguing as in Lemma 29, we have sequences of planes C_N and C'_N whose limits have distinct images in $W \setminus Z$. For each large N we can find a family of J_N-holomorphic planes connecting C_N and C'_N which all intersect a compact subset of T^*L (for example the family of planes in the foliation which pass through a curve in T^*L between C_N and C'_N). There is a plane in this family, say D_N whose distance from C_N in a Hausdoff metric on $W \setminus Z$ is a fixed number independent of N and different from the distance of all index 0 curves in $W \setminus Z$ from the limit of the C_N. Taking a limit of a

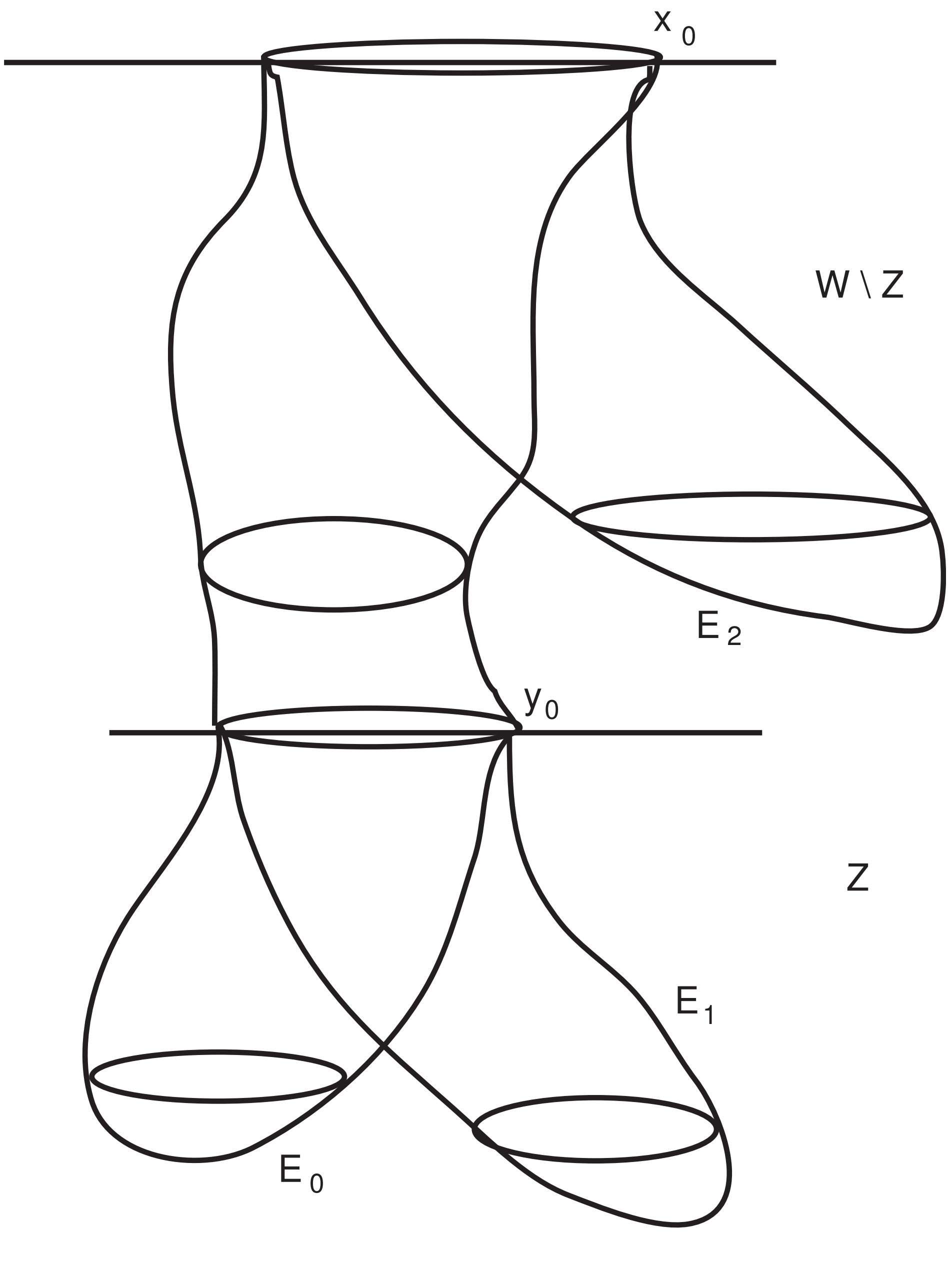

Figure 8: Limits of the rigid curves E_i

subsequence of the D_N then gives a contradiction.

Now if we take a limit of planes C_N which lie arbitrarily close to the rigid planes and converge to the rigid planes as $N \to \infty$ we see that these unique limiting components in $W \setminus Z$ have image coinciding with that of the rigid planes. If the component in T^*L is a plane asymptotic to $2y_0$ then the only possibility for the components in $W \setminus Z$ consist of a cylinder asymptotic to $2x_0$ and $2y_0$ and the plane asymptotic to x_0 equal to the limit of E_2. This cylinder must contain the limits of the E_0 and E_1 and so we conclude that these limits coincide and the map from the cylinder here is a double cover. □

Returning to the proof of Proposition 34, suppose that we fix a point $p \in T^*L$ disjoint from the rigid planes. Then by Lemma 36 the limits of the planes through p converge to a plane asymptotic to $2y_0$ in T^*L and the cylinder double covering the limits of the E_0 and E_1 and a plane equal to the limit of the E_2 in $W \setminus Z$. This implies by uniform convergence that for N sufficiently large the nonrigid planes through p will intersect U in two disjoint circles, one close to the intersection of U with E_0 and E_1 and homotopic to $2x_0$, the other close to the intersection with E_2. Looking at the points p lying in a small circle in L around the intersection with one of the rigid planes we find an S^1 family of curves satisfying the requirements of Proposition 34. □

5.2 Plurisubharmonic exhaustion functions

In this section we produce a filling (or foliation) of V by holomorphic disks with boundary on the perturbation of $B \cup U$ and use it to construct a plurisubharmonic exhaustion for V. The key property is that L will be disjoint from the unstable manifold of one of the two index 2 critical points.

Theorem 37 *For any extension J as in Lemma 34, the almost-complex manifold (V, J) admits a plurisubharmonic exhaustion function with three critical points, one a minimum and the others of index 2. The Lagrangian L is disjoint from the unstable manifold of one of the index 2 critical points.*

It would be convenient simply to use the intersections of V with finite energy planes as our filling. Unfortunately it seems hard to control the tangencies of such planes with U. Therefore we singularly foliate U with surfaces, each of which in turn can be singularly foliated by the boundaries of holomorphic disks. Together with the finite energy planes intersecting B these will complete the filling.

Proof

We have discussed an S^1 family of curves lying in $[R, \infty) \times M$ which intersect ∂U in the proof of Theorem 33 (they form the hypersurface B). As described at the end of the proof of Proposition 34 the J_N-holomorphic curves passing through a small circle in L around the intersection of L with E_0 give another S^1 family which forms the boundary of a tubular neighborhood of the intersection of the rigid planes with V. These two S^1 families bound a compact subset $\mathcal{N}$ of the moduli space of planes asymptotic to $3x_0$ and if $R' > R$ is sufficiently large we may assume that they all intersect $\{R'\} \times M$ transversally. Thus the curves in $\mathcal{N}$ will intersect a cross-section A of a neighborhood U' of x_0 in $\{R'\} \times M$ (as in Figure 7) in an annulus S with one boundary on ∂A, and each curve will intersect S in three points. The corresponding degree 3 cover from S to $\mathcal{N}$ shows that $\mathcal{N}$ itself is an annulus and will contain a circle (homotopic to a circumference) which consists of planes which miss L but still intersect a fixed compact subset of T^*L. Indeed, the curves which touch only the boundary of a small tubular neighborhood of L generically will correspond to a union of embedded circles in $\mathcal{N}$ and one of the circles must separate the two boundary components. As in Proposition 34, planes in these S^1 families converge, as we stretch along the boundary of a tubular neighborhood Z of L, to a nonrigid plane in T^*L and, in $W \setminus Z$, a cylinder covering the limits of E_0 and E_1 and a plane equal to the limit of E_2. Therefore for N sufficiently large we have that the curves in the family will intersect U transversally in two families of circles, one homotopic to $2x_0$ and the other to x_0. The first family will foliate a torus I enclosing $(E_0 \cup E_1) \cap U$ and the second will foliate a torus enclosing $E_2 \cap U$.

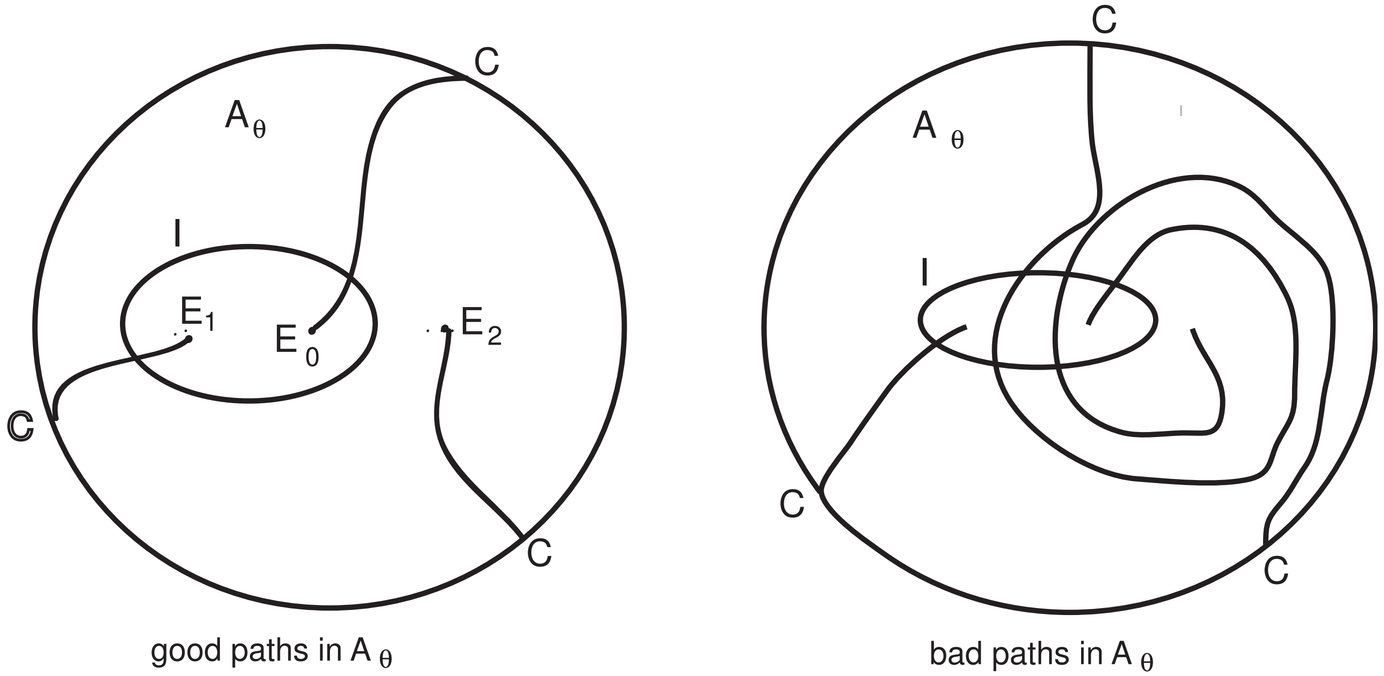

Figure 9: Different paths defining Σ_C

We define vector fields on each $A_\theta \subset U$ looking exactly as described in the previous section, but not necessarily corresponding to the intersections of the A_θ with finite energy planes. The integral curves of our vector field converging to the intersection of a particular curve C with ∂U will form a surface Σ_C diffeomorphic to a sphere with four disks removed. The four boundary components are the intersections of U with C, E_0, E_1 and E_2. Now, it is easy to adjust our vector field such that each of these surfaces intersect the torus I in the boundary of one of the finite energy planes in our S^1 family.

Remark 38 *There are different choices of these surfaces which are not homotopic relative to the $U \cap E_i$. Indeed, there are different homotopy classes of singular foliations in a cross-section A_θ. This will be important in the next section when we consider Lagrangian isotopies rather than global symplectomorphisms. The following arguments in this section apply to any choice of surfaces* *fulfilling our condition on their intersection with I, but we remark that this condition does impose a constraint on the homotopy class. To see this, observe that $I \cap A_\theta$ will be a circle enclosing $(E_0 \cup E_1) \cap A_\theta$ and a finite energy plane intersecting I will meet $I \cap A_\theta$ in two points. Looking at integral curves of our vector field converging to a $C \cap \partial A_\theta$ (which consists of three points) generically we will see three distinct curves with the other endpoints at the $E_i \cap A_\theta$. Our condition requires that exactly two of the curves intersect I, and do so in the intersection with a single finite energy plane. This property of the curves is not preserved, even up to homotopy, after, for instance, a Dehn twist along a circle enclosing $E_0 \cup E_2$. Figure 9 gives shows good and bad choices of paths in A_θ which define the surface Σ_C. The first path intersects $I \cap A_\theta$ correctly, the second, deformed by a Dehn twist, does not.*

Next we use the theory of filling by holomorphic disks, see [6], [2], [8], [14] (Theorem 1 in [14] unifies alot of the previous work) to singularly foliate each of the surfaces Σ_C above by boundaries of holomorphic disks in V. The theorem we need is the following. This is a simplified version of Theorem 1 in [14] valid only in the strictly pseudoconvex case. The theorem stated there accounts for more complicated behaviour when the boundary is only weakly pseudoconvex. We are free to assume that our complex structure is integrable near ∂V, and will make this assumption whenever convenient in the remainder of this section.

Recall that a surface Σ in ∂V has a characteristic foliation $\eta = T\Sigma \cap \xi$ where ξ is the 2-plane field $T(\partial V) \cap JT(\partial V)$ of complex tangencies. (In our situation ξ on U is a small perturbation of the tangents to the cross-sections.) On our Σ_C the characteristic foliation η is a line field away from two hyperbolic points (where Σ_C is tangent to ξ, which itself). For the proof of our theorem the intersections of the planes C and E_i with V, whose boundaries are the boundary components of Σ_C, act exactly like elliptic tangencies.

Theorem 39 *After perhaps a C^2 perturbation of the complex structure J near the families of hyperbolic points, each Σ_C has a singular foliation by circles. The foliation is smooth away from the hyperbolic points and includes the boundary*

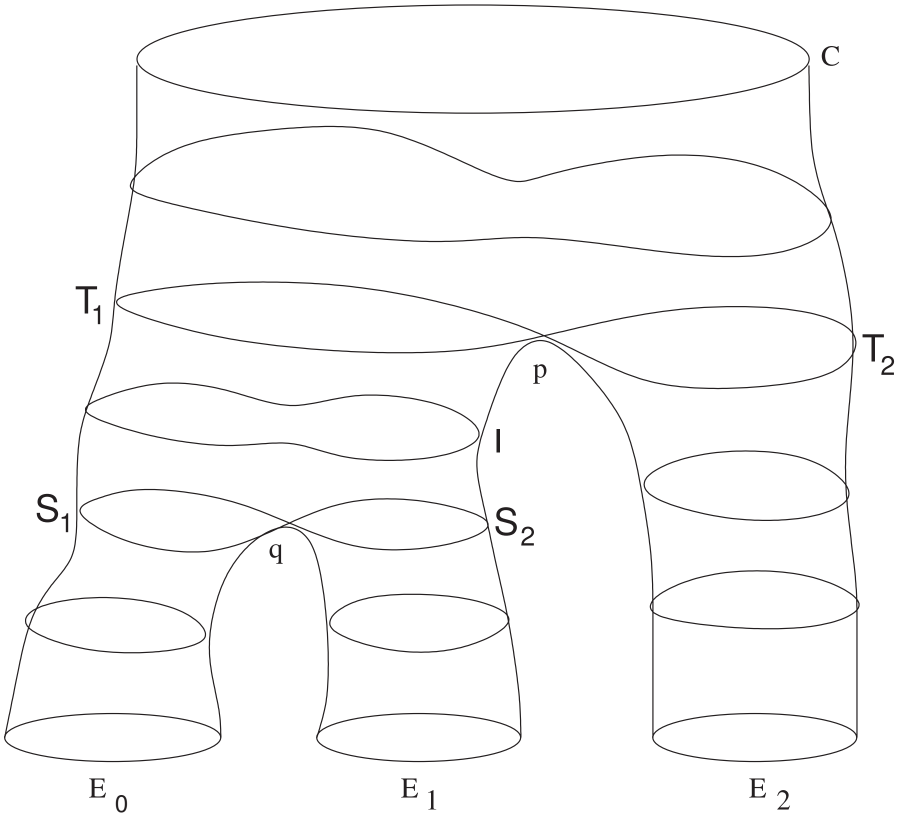

Figure 10: holomorphic disks filling a surface

circles. Leaves passing through the hyperbolic points are not smooth and exactly two different leaves intersect at each of these points. Each circle is the boundary of a holomorphic disk $u : (D, \partial D) \to (V, \Sigma_C)$, and the boundary circles are the intersections of the planes C and E_i with V. The images of the disks are disjoint apart from the boundary intersections at the hyperbolic points, and any holomorphic disk with boundary on Σ_C is one of the disks in our foliation (up to reparameterization and multiple covers).

Uniqueness implies that our fillings include the intersection of the surfaces with I. The filling looks as in Figure 10. In particular, since it includes the disk through I, the arrangement of the singular (hyperbolic) points p and q in relation to the E_i is as shown.

Disks in the fillings of different Σ_C are either disjoint or coincide with the rigid curves E_i. To see this, suppose by contradiction that a disk D in the filling of Σ_C and whose boundary is disjoint from the E_i intersects a disk D' in the filling of $\Sigma_{C'}$. Now, D and D' can each be added to at most two other such disks in their respective fillings to form a surface, say $\{D_i\}$ or $\{D'_j\}$, cobordant through a subset of the filling to $C \cap V$ or $C' \cap V$ respectively, see Figure 10. We know that C and C' are disjoint as they form part of our original foliation, and therefore the sum of the intersection numbers of the D_i and D'_j is zero. But then by positivity of intersection, since the boundaries of these disks are disjoint, this implies that the actual intersection must also be zero.

We construct a plurisubharmonic function by following [8], see also [15], [16]. We start by defining a function g which is constant on the holomorphic disks in our filling. We now fix $J = J_N$ for N suitably large. Recall that γ is the circle in B along which finite energy planes from our foliation are tangent to ∂V and let T_1, T_2 be tori in U formed by the boundaries of holomorphic disks passing through the hyperbolic points p and let S_1, S_2 be tori in U formed by the boundaries of holomorphic disks passing through the points q. We label things so that the inside of T_1 in U encloses S_1 and S_2, see again Figure 10. We define g to be a Morse function on γ with a single minimum at 0 and a single maximum at 1. As in [15] we define g to be constant on families of holomorphic disks converging to points on γ. These families of disks can be chosen to be parameterized either by an interval with the disks converging to the points $g^{-1}(t) \in \gamma$ for $t \leq \frac{1}{4}$ or $t \geq \frac{3}{4}$ or alternatively by an interval with one end converging to a point $g^{-1}(t) \in \gamma$ for $\frac{1}{4} < t < \frac{3}{4}$ and the other to a cusp-disk with boundary on $T_1 \cup T_2$. This defines g on the disks passing through the complement of the insides of T_1 and T_2.

Inside T_2 we simply define g to be constant on 1-parameter families of disks connecting the disks on which $g = t$. Inside T_1 we again define g to be constant on families of disks connecting the disks on which $g = t$ and $\frac{1}{4} < t < \frac{3}{8}$ or $\frac{5}{8} < t < \frac{3}{4}$. We also let $g = t$ on intervals of disks connecting disks with $g = t \in [\frac{3}{8}, \frac{5}{8}]$ on one side and cusp-disks with boundary on $S_1 \cup S_2$ on the other. Inside S_1 and S_2 we extend g to be constant on the 1-parameter families of disks connecting the disks on which $g = t$ as before. Altogether this defines a function g whose level-sets are foliated by holomorphic disks. Note that g has no critical points in the interior of V.

Now, as in [8], see also [15], [16], the level-sets of g are Levi flat (foliated by holomorphic curves) but we can perturb g such that they become pseudoconvex. One way to do this is to choose a function ψ on W which satisfies $dd^c\psi(X, JX) >> |d\psi|$ for any unit vector X tangent to the foliation (with respect to a fixed metric). Then we can replace g by $g + \psi$. Recall that the level-sets of a function f are pseudoconvex if $-dd^c f$ is positive on the complex tangencies $\ker(d^c f) \cap \ker(df)$. (Before the perturbation $-dd^c g$ vanishes on this subspace.) We then have that $dd^c(g + \psi)$ is positive on the complex tangencies of the level-sets of g, but the tangencies to the level-sets of $g + \psi$ differ only by order $\frac{|d\psi|}{|dg|}$ (which we can assume to be arbitrarily small) and so the same is true for these subspaces. Let us now denote the perturbation $g + \psi$ simply by g. Next, composing g with a sufficiently convex function ϕ on $\mathbb{R}$ with $\phi'' >> \phi' > 0$ it then becomes strictly plurisubharmonic. (To see this, we compute $-dd^c(\phi \circ g) = -d(\phi' d^c g) = -\phi' dd^c g + \phi'' d^c g \wedge dg$ and observe that the first term is positive on complex tangencies to the level sets of g while the second term vanishes, but on sufficiently transverse complex planes the second term is positive and overwhelms the first.) Finally set $f = \max(g, h)$ where h is a function increasing rapidly towards ∂V. The function f can be smoothed to give a plurisubharmonic exhaustion, see for example [13], Theorem 1.4.12. Investigating the pattern of holomorphic disks as in [15], section 3, we see that it has three critical points. There is an index 0 critical point near the minimum of g on γ and there are index 2 critical points near the maxima of g on $S_1 \cap S_2$ and $T_1 \cap T_2$. We call these points a and b respectively. The construction ensures that $f(b) > f(a)$. Furthermore $f < f(b)$ on all disks lying inside the hypersurface formed by the holomorphic disks intersecting I. Therefore L is disjoint from the unstable manifold of b as required, as it lies inside this hypersurface. □

5.3 Symplectomorphisms

The plurisubharmonic function f from the previous section gives a symplectic form $\omega = -dd^c f$ on W where $d^c f = df \circ J$ (as they are diffeomorphic we will now replace V by our original W). This in turn gives us a vector field $v = \text{grad} f$ defined by $v \rfloor \omega = d^c f$. By a suitable choice of $f = h$ near ∂V we may assume that v is complete in the sense that its positive integral flow exists for all time.

Further, as it is a plurisubharmonic exhaustion for an almost-complex structure Stein homotopic to a standard one on W, we can adjust f such that the stable manifolds of the two critical points are embedded Lagrangian spheres with respect to the form ω, which intersect transversally at the minimum. By Weinstein's Lagrangian neighborhood theorem applied to a pair of transversally intersecting Lagrangians, a neighborhood of these two stable manifolds is symplectomorphic to a neighborhood of $L_1 \cup L_2 \subset (W, \omega_0)$. We then use [10], see Proposition 1.8.4.A to imply the following.

Lemma 40 *(W, ω_0) and (W, ω) are symplectomorphic via a symplectomorphism ψ taking the stable manifolds of the critical points a and b of f onto L_1 and L_2 respectively.*

After perhaps adjusting f the following is also true. As above L denotes the Lagrangian sphere homotopic to L_1.

Lemma 41 *There exists a symplectomorphism ϕ from (W, ω_0) to (W, ω) taking L onto a Lagrangian sphere disjoint from the unstable manifold of the critical point b of f.*

Lemmas 40 and 41 together imply our Theorem 4. To see this, note that the one-parameter group of diffeomorphisms D_t generated by $-v = -\text{grad} f$ satisfy $D_t^* \omega = e^{-t}\omega$ so the spheres $D_t(\phi(L))$ are all Lagrangian. But a Lagrangian isotopy of spheres is also a Hamiltonian isotopy, and therefore we get a Hamiltonian isotopy of spheres from $\phi(L)$ to a Lagrangian sphere L' in a tubular neighborhood of the stable manifold of the critical point a. Since this tubular neighborhood can be taken to be symplectomorphic to a unit cotangent bundle of S^2, Theorem 1 implies that a further Hamiltonian diffeomorphism maps L' onto the stable manifold of a itself. We denote the Hamiltonian diffeomorphism mapping $\phi(L)$ onto the stable manifold of a by χ. Then, with ψ as in Lemma 40, $\psi^{-1} \circ \chi \circ \phi$ is the symplectomorphism required by Theorem 4.

Proof of Lemma 41

By choosing $f = h$ carefully near ∂V, now identified with the noncompact end of W, we may assume that $\omega = \omega_0$ outside of a compact subset of W. In fact, both forms are exact and we can write $\omega - \omega_0 = d\alpha$ where the 1-form α is identically zero outside of a compact set. Furthermore, since ω and ω_0 tame the same almost-complex structure, $\omega_t = (1-t)\omega_0 + t\omega$ is a symplectic form on W for all t.

Using Moser's method, we observe that the compactly supported time-dependent vector field X_t defined by $X_t \rfloor \omega_t = \alpha$ satisfies $\mathcal{L}_{X_t}\omega_t = \frac{d}{dt}\omega_t$ and so its flow generates a symplectomorphism from (W, ω_0) to (W, ω).

We are interested in the image of L under such a symplectomorphism, we recall that L is initially disjoint from the unstable manifold of b and we want to ensure that this remains the case under the flow of X_t. We will adjust f so that this will be the case.

Assume that a fixed tubular neighborhood Z of L is disjoint from the unstable manifold of b. Given the construction in Theorem 37 we may assume that $f \geq 0$ and $Z \subset f^{-1}([0,r])$ for some $r < f(b)$.

The composition of f with an increasing function $s : [0,\infty) \to [0,\infty)$ remains plurisubharmonic provided that $\frac{s''}{s'} >> 1$. We choose s (and its derivatives) to be very small on $[0,r]$ but then to increase rapidly on (r,∞). Thus we can replace f by another nonnegative plurisubharmonic exhaustion, still denoted by f, and having the property that $f|_Z < 1$. Further we arrange that $\omega(X,JX) << \omega_0(X,JX)$ on $f^{-1}([0,1])$ for all tangent vectors X, while $\omega(X,JX) >> \omega_0(X,JX)$ on $f^{-1}([2,3])$ for all X and now $f(b) > 3$. We observe that for reasonable choices of functions s the Moser flow will still exist for all time. Alternatively we can adjust ω_0 near ∂V also such that the flow still has compact support.

On the tubular neighborhood Z we have that ω and $d^c f$ are now uniformly small. Thus the length of X_t (relative to the Riemannian metric defined by ω_0 and J) remains bounded on this neighborhood for $t < \frac{1}{2}$ say. Therefore there exists a uniform ϵ (depending only upon ω_0, J and Z) such that the flow of L remains in Z for $t < \epsilon$. But for $t > \epsilon$ we can suppose that on $f^{-1}([2,3])$ the vector field X_t is closely approximated by $-\frac{1}{t}\text{grad}f$. Hence the flow of L remains in $f^{-1}([0,3])$ for all $0 \leq t \leq 1$ and so the symplectomorphism generated by X_t can indeed be arranged to leave L disjoint from the unstable manifold of b as required. □

6 Lagrangian isotopies and Dehn twists

In this section we use the analysis of section 5 to deduce Theorem 5.

First of all, by Weinstein's Theorem a Lagrangian 2-sphere has self-intersection -2, thus Lagrangian spheres in W are homologous to either L_1, L_2 or $L_1 \sharp L_2$. Up to Hamiltonian isotopy $L_1 \sharp L_2 = \tau_{L_2}(L_1)$ and so it suffices to prove the result assuming that L is homologous to L_1.

6.1 Section 5 revisited

Using the notation from the previous section, we recall that Theorem 33 constructed a finite energy foliation of (W, J) with respect to any tame almost-complex structure J which is standard outside of a compact set. In fact the finite energy foliation is described quite explicitly, in particular in terms of the intersection of the finite energy planes with a level $\{R\} \times M$, for R large. The rigid planes E_i intersect $\{R\} \times M$ transversally in a certain tubular neighborhood U of the Reeb orbit x_0. The boundary of U is foliated by circles in an S^1-family of finite energy planes and this family divides W into two pieces. We assume that the piece foliated by planes disjoint from U is disjoint from all of the Lagrangian spheres, and when we vary J it will always be fixed in this region.

In section 5.2, the finite energy foliation with respect to particular choices of J was used as the starting point to construct a plurisubharmonic exhaustion function f on a Stein domain $V \subset W$ with $\partial V = B \cup U$, see Theorem 37. The plurisubharmonic function has three critical points, one of index 0 and two of index 2. With respect to the Kähler structure associated to a well chosen plurisubharmonic function the two stable manifolds form Lagrangian spheres intersecting in a single point.

Such a plurisubharmonic exhaustion function can in fact be constructed for any of the almost-complex structures we consider, provided we neglect the requirement in Theorem 37 of unstable submanifolds avoiding a Lagrangian. Furthermore, the construction is essentially canonical given a choice of embedded curve in a cross-section A of U traveling from E_1 to E_2 through E_0. This is the content of the following lemma.

Lemma 42 *Given an almost-complex structure J on W and a path γ between the $E_i \cap A$, we can construct a plurisubharmonic exhaustion function f on W whose stable manifolds form two Lagrangian spheres intersecting transversally at a single point.*

The function is well defined given our data up to a homotopy through plurisub-

harmonic exhaustions with the same properties.

Proof

The rigid planes E_i and the finite energy foliation are determined by a given almost-complex structure J. However there are still ambiguities in the construction of the plurisubharmonic function in Theorem 37. First we must choose families of surfaces in U diffeomorphic to a sphere with four disks removed. One boundary of such a surface should coincide with the intersection of a finite energy plane with the boundary of U and the other three boundaries with the intersections of the E_i with U. It can be seen that U can be singularly foliated by such surfaces, the foliation being smooth away from the E_i. Now, the surfaces themselves can be chosen in an essentially canonical way (so that they intersect cross-sections as in Figure 7) given the position of the $E_i \cap U$ and a choice of embedded curve in a cross-section A traveling from E_1 to E_2 through E_0. To do this, we simply map the cross-section to the model picture in Figure 7, mapping rigid planes to the special points in the figure and the curve to the corresponding curve in the figure, and pull-back the foliation there. This is well-defined up to a homotopy fixing the rigid planes and the path (but not necessarily the boundary). It will be a consequence of the existence of different homotopy classes of such paths that there exist different Hamiltonian isotopy classes of Lagrangian spheres, see also Remark 38. Anyway, after the foliating surfaces are chosen we can construct a plurisubharmonic function with the required properties as follows, and do this canonically modulo a contractible set of choices.

This is done in a similar manner to Theorem 37. We first fill each of the surfaces by holomorphic disks, however for a general choice of almost-complex structure J we no longer have a torus I dividing the families of filling disks. Therefore the pattern of holomorphic disks in the filling is no longer necessarily that of Figure 10, that is, starting with a boundary in ∂U, the family may reach the hyperbolic complex tangency q before reaching p. Nevertheless we can construct a plurisubharmonic exhaustion in a canonical way by perturbing a function g constant on the holomorphic disks filling V. We define g as before on disks with boundary on B. To extend g to U we proceed as follows. Parameterize the boundaries of holomorphic disks on ∂U by $\psi \in S^1$ and thus the foliating surfaces starting from these boundaries. On each of the surfaces we can find a smooth function h_ψ which is constant on the boundaries of holomorphic disks, is equal to 1 on ∂U, and is equal to 0 on ∂E_i, $i = 0, 1, 2$. The h_ψ can be chosen to vary continuously with ψ and such that h_ψ has critical points only at the hyperbolic points of the surfaces. Then we can define a map $P : U \to D^2 = \{x^2 + y^2 \leq 1\}$ by assigning to a point in U the point in D^2 with polar coordinates (h_ψ, ψ). By adjusting the parameterization we may assume that $g|_{\partial U} = P^* L$ where $L = \frac{y+2}{4}$ and thus extend g to U by the same formula. After taking the maximum f of g and a function h increasing rapidly towards ∂V and smoothing appropriately, exactly as in Theorem 37, we see that as before f will have only three critical points, a minimum on the circle of complex tangencies in B and two index 2 critical points close to the hyperbolic points on the surface extending the maximum circle of g on ∂U. $\square$

6.2 Symplectomorphisms of (W, ω_0)

We will identify our symplectic structure ω_0 on W with the Kähler structure coming from a J_0-plurisubharmonic function, where J_0 is a fixed almost-complex structure and the plurisubharmonic function is constructed as above. Then the stable manifolds of the index 2 critical points correspond to L_1 and L_2. Suppose that J is another almost-complex structure tamed by ω_0. Let ω_1 be the symplectic form corresponding to a J-plurisubharmonic function constructed as above with 3 critical points. Then there are two natural symplectomorphisms from (W, ω_0) to (W, ω_1). Since both ω_0 and ω_1 tame the same almost-complex structure, convex linear combinations of the two forms are also symplectic and so by Moser's theorem we can generate a symplectomorphism ϕ between them. The flow used to define the symplectomorphism is well defined provided our exhaustion functions have sufficiently fast growth towards the boundary. On the other hand, as in Lemma 40, given a plurisubharmonic exhaustion its gradient flow is conformally symplectic with respect to the corresponding symplectic form. Therefore we get another symplectomorphism by first identifying neighborhoods of the stable manifolds using Weinstein's Theorem and extending this to a global symplectomorphism ψ using the gradient flows (see [10] section $1.8.4.A$ for these ideas). Composing the inverse of this symplectomorphism with the Moser diffeomorphism gives a symplectomorphism $\Phi = \psi^{-1} \circ \phi$ of (W, ω_0) determined by a tame almost-complex structure J and corresponding plurisubharmonic function (that is, up to isotopy by J and the path γ of Lemma 42). If $J = J_0$ and γ is chosen as for the definition of ω_0 (with the identification above) then this map is the identity. In summary we have the following.

Lemma 43 *An almost-complex structure J on W and a path γ between the intersections of the rigid planes with a cross-section A of U determine up to isotopy a symplectomorphism Φ of (W, ω_0). If $J = J_0$ and γ is a particular path σ then $\Phi = \Phi_0 = \mathrm{id}$.*

Now let L be a Lagrangian sphere in W homologous to L_1. It was shown in Lemma 37 that there exists an almost-complex structure J and plurisubharmonic function constructed as above generating a symplectic form ω_1 such that the corresponding unstable manifold of one of the index 2 critical points is disjoint from L. Furthermore, under the Moser map ϕ from $(W, \omega_0) \to (W, \omega_1)$, Lemma 41 implies that the Lagrangian L can be arranged to stay disjoint from this unstable manifold. Therefore by Theorem 1, composing with a Hamiltonian diffeomorphism we may assume that the Moser map takes L to one of the stable manifolds. Thus the symplectomorphism $\Phi = \Phi_1$ of (W, ω_0) maps L onto L_1.

This all gives a method of constructing a Lagrangian isotopy of L_1. Namely we start with the almost-complex structure J_1 and path γ_1 generating the map Φ_1 above and look at a family of almost-complex structures J_t connecting J_0 and J_1. The rigid planes vary continuously with t and so we can find a corresponding family of paths γ_t and hence symplectomorphisms Φ_t. Things are chosen such that $\Phi_1(L) = L_1$ and so $\Phi_t(L)$ gives a Lagrangian isotopy starting from L_1.

However γ_t is determined up to homotopy by γ_1 and so may not equal σ up to a homotopy fixing the rigid planes $E_i \cap A$. In particular we cannot guarantee that γ_0 together with J_0 generate the identity symplectomorphism, and thus an isotopy from L_1 to L. In the next section we examine the effect of carrying out exactly the same construction starting with $\tau(L)$ rather than L, where τ is a symplectic Dehn twist.

6.3 Symplectic Dehn twists

In section 6.2 we described how a Lagrangian sphere L homologous to L_1 enables us to construct a Lagrangian isotopy from L_1 to a Lagrangian sphere $\Phi(L)$ where Φ is the symplectomorphism from Lemma 43 determined by J_0 and a path γ_0 between the rigid planes. To find γ_0 one starts with an almost-complex structure J_1 and path γ_1 generating a plurisubharmonic function with the properties of Lemma 37. Then we choose a family of almost-complex structures J_t connecting J_0 and J_1, these define a family of rigid planes E_i and we can then define, uniquely up to homotopy, a continuous family of compatible paths γ_t. So γ_0 depends not just on the J_0-holomorphic rigid planes but on the path of J_t-holomorphic rigid planes, in particular their intersection with the cross-section A. Here we examine how those intersections change if we repeat the whole construction with $\tau(L)$ instead of L, where τ is an even power of the symplectic Dehn twist about L_1 or L_2. Since τ is an even power it is smoothly isotopic to the identity, in particular $\tau(L)$ is still homologous to L_1.

First note that as τ has compact support the almost-complex structure $\tau(J)$ is a compatible almost-complex structure with a cylindrical end on (W, ω_0) whenever J is. Furthermore, the new foliation is the image of the J-holomorphic one under τ and in particular the intersection of the rigid planes with our cross-section A will be identical. Thus if J_1 and γ_1 generate a plurisubharmonic function f with an unstable manifold disjoint from L then $\tau(J_1)$ and γ_1 generate a plurisubharmonic function $\tau(f)$ with an unstable manifold disjoint from $\tau(L)$.

If J_t is a family of almost-complex structures interpolating between J_0 and

J_1 then $\tau(J_t)$ interpolates between $\tau(J_0)$ and $\tau(J_1)$. We observe that the intersection of $\tau(J_t)$-holomorphic finite energy planes with U are exactly the same as the intersections of the J_t-holomorphic finite energy planes. Therefore to understand the new family of intersections $E_i \cap U$ it suffices to understand the intersections $E_i \cap U$ for a family of tame almost-complex structures connecting $\tau(J_0)$ and J_0.

Case of T^*S^2

Before this, we first consider T^*S^2 with its standard symplectic form. This again can be thought of as a Stein manifold with open end symplectomorphic to $[0, \infty) \times N$ where $N = \mathbb{R}P^3$ with its standard contact form. The Reeb flow here can be identified with the geodesic flow on S^2 with a fixed round metric. We also fix a tame almost-complex structure J_0 invariant under the natural action of $\mathrm{Isom}(S^2)$. Then as described in [15], and used in [16] and [18], T^*S^2 admits a finite energy foliation with all planes asymptotic to multiples of a Reeb orbit y_0 corresponding to, say, the equator on S^2. The foliation now contains two rigid planes F_0 and F_1 asymptotic to the single orbit y_0 and all other finite energy planes in the foliation are asymptotic to $2y_0$. Let τ_{S^2} denote the square of the symplectic Dehn twist about the zero section in T^*S^2, which we may assume is supported in a neighborhood of the zero section. Denote by T^rS^2 the cotangent vectors of length r in the round metric.

Lemma 44 *Let J_t be a family of almost-complex structures interpolating between J_0 and $\tau_{S^2}(J_0)$. Let R be very large, V a neighborhood of y_0 in T^RS^2 and B a cross-section of V transverse to the Reeb flow. Then the intersections of the J_t-holomorphic rigid planes F_i with B rotate their positions exactly once in the interval $0 \le t \le 1$.*

We recall that the J_1-holomorphic planes are just the images of the J_0-holomorphic planes under τ_{S^2}, and so their intersections with B are identical.

Proof

We begin with a few remarks. The rigid planes project to opposite hemispheres on the S^2. Now rotation about the axis perpendicular to the equator preserves y_0 and J_0 and so also the rigid finite energy planes. It follows that each intersects the zero-section at either the north or south pole and intersects the tubes of radius r, denoted $T^r S^2$, in circles projecting to parallels on S^2. The square τ_{S^2} of the symplectic Dehn twist about the zero-section can be thought of as the Hamiltonian flow of $H = \frac{1}{2}|p|^2$ if the cotangent vector has length $|p| \leq 2\pi$ and the identity if $|p| \geq 2\pi$. (In other words, the tubes are preserved and for $r < 2\pi$ the diffeomorphism of $T^r S^2$ is the time-r geodesic flow.) This map τ_{S^2} is isotopic to the identity through (non-compactly supported) symplectomorphisms τ_t, where τ_t is equal to the Hamiltonian flow of $H(tp)$ for $|p| \leq \frac{2\pi}{t}$ and the identity for $|p| \geq \frac{2\pi}{t}$. We observe that $\tau_t(J_0)$ for $0 < t \leq 1$ give a family of tame almost-complex structures converging to J_0 as $t \to 0$. In fact, for R sufficiently large, $\tau_t(J_0)|_{T^{\geq R}S^2}$ is approximately equal to J_0 for all t since τ_t acts as the geodesic flow on a fixed level (which we can assume to preserve the relevent CR structure) and is approximately translation invariant for R large. Therefore after a small adjustment we will think of $\tau_t(J_0)$ as a compactly supported variation of J_0. In a level $T^R S^2$ let us choose coordinates (x, y) in the cross-section B transverse to our Reeb orbit at $(0, 0)$ such that our rigid J_0-holomorphic planes intersect in points $(\pm\epsilon, 0)$. Then we observe that for $0 < t \leq 1$ the positions of $\tau_t(F_i) \cap B$ perform one complete rotation. Since the space of almost-complex structures is contractible, any family connecting $\tau_{S^2}(J_0)$ and J_0 will have the same effect on the intersections. □

Case of planes in W

Returning to our original situation, consider first the case when τ is the square of the symplectic Dehn twist about L_1. Then a family of almost-complex structures J_t on W connecting $\tau(J_0)$ and J_0 can be chosen to be fixed away from a neighborhood of L_1, and we can derive the following from Lemma 44.

Corollary 45 *Up to isotopy, the intersections of the J_t-holomorphic rigid planes E_0 and E_1 with A rotate their position once over the interval $0 \leq t \leq 1$ while the intersection of E_2 with A remains fixed.*

Proof We will again follow the methods of section 5. Let M be a tubular neighborhood of L_1, symplectomorphic to a tubular neighborhood $T^{\leq r}S^2$ of the zero-section in T^*S^2. We suppose that the $J_t = J_0$ outside of M for all t. We denote by J_t^T the result of stretching J_t to length T along the boundary of M. Then, rather than studying J_t-holomorphic planes for $0 \leq t \leq 1$, it is enough to show that for some N the intersections of the J_t^N-holomorphic rigid planes E_0 and E_1 with A rotate their positions once. Indeed, as τ has compact support, the intersections of J_t^T-holomorphic and $\tau(J_t^T)$-holomorphic rigid planes with A coincide on the interval $0 \leq T \leq N$. Therefore, in a path from J_0 to J_1 which first connects J_0 to J_0^N, then connects J_0^N to $\tau(J_0^N)$, and finally connects $\tau(J_0^N)$ to $\tau(J_0) = J_1$, any rotations on the first and last segments will cancel. By Lemma 35, for N sufficiently large E_2 is disjoint from M and so remains fixed with respect to almost-complex structures in the path from J_0^N to $\tau(J_0^N)$.

We stretch the neck a length $N \to \infty$ along the boundary of M. In the limit as $N \to \infty$ we have complex structures $J_{t,\infty}$ on T^*S^2 and our J_t^N-holomorphic finite energy foliations of W converge to $J_{t,\infty}$-holomorphic finite energy foliations of T^*S^2. These foliations may be taken to be exactly those described in the model case of Lemma 44 (we get uniqueness near the boundary from [19] and then globally by regularity), in particular the limits of the rigid planes rotate positions once for $0 \leq t \leq 1$. We recall also that the limits of the rigid planes E_0 and E_1 in the completion of $W \setminus M$ converge to the same finite energy cylinder, see Lemmas 35 and 36. We look at the intersections of our E_i with a 1-parameter family of surfaces intersecting this cylinder transversally. The surfaces can be chosen to be tangent to a cross-section A in U at one end and tangent to a tube T^rS^2 at the other. Then for N sufficiently large our finite energy planes E_i will intersect these surfaces transversally and so their relative rotation will be the same in each. But by uniform convergence away from the punctures of J_t^N-holomorphic spheres to their limiting finite energy curves, see [3], the rotation of the J_t^N-holomorphic E_i in a T^rS^2 will be the same as that of the limits with respect to the $J_{t,\infty}$, in other words they rotate once. Our corollary follows. $\square$

Exactly the same argument applies for a symplectic Dehn twist about L_2.

Corollary 46 *Let J_t be a family of almost-complex structures connecting J_0 and $J_1 = \tau_2(J_0)$ where τ_2 is the square of the symplectic Dehn twist about L_2. Then, up to isotopy, the intersections of the J_t-holomorphic rigid planes E_0 and E_2 with A rotate their position once over the interval $0 \leq t \leq 1$, while the intersection of the rigid plane E_1 remains fixed.*

6.4 Conclusion of the proof of Theorem 5

We have seen in Corollaries 45 and 46 that for a suitable composition of Dehn twists τ, a family of almost-complex structures connecting $\tau(J_0)$ and J_0 can produce any relative movement of the $E_i \cap U$ up to homotopy. So given a J_1, we can find a τ such that a family of almost-complex structures J_t connecting $\tau(J_1)$ and J_0 produces any relative movement of the $E_i \cap U$. Suppose that J_1 and γ_1 generate a symplectomorphism Φ from Lemma 43 mapping L onto L_1. Then $\tau(J_1)$ and γ_1 generate a symplectomorphism mapping $\tau(L)$ onto L_1. But τ can be chosen such that given a family of almost-complex structures interpolating between $\tau(J_1)$ and J_0, the intersections of the rigid planes with A are such that the corresponding family of paths γ_t between the E_i end with a γ_0 which is homotopic relative to the E_i to σ, the path generating the identity map in Lemma 43. Thus the Lagrangian isotopy $\Phi_t(\tau(L))$ is an isotopy between L_1 and $\tau(L)$ and our theorem is proved as required.

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