CUBULATIONS OF SYMPLECTIC 4-MANIFOLDS

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Here is a construction of a simplicial decomposition of a symplectic 4-manifold into simplices which are of a standard form up to symplectomorphism. It is clear that much of the construction generalizes to higher dimensions.

A standard closed 4-simplex is a closed subset of $\mathbb{C}^2$ with coordinates z,w of the form

$$\Delta = \{(z,w)|z \in S_1, w \in S_2\}$$

where S_1 and S_2 are squares in the z and w planes. The simplex Δ inherits a symplectic form from the standard form $\frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ on $\mathbb{C}^2$. The areas of S_1 and S_2 are complete symplectic invariants of the closure of Δ. Lines parallel to the z and w coordinate planes give transverse symplectic foliations of Δ.

Definition. *A symplectic cubular decomposition (SC) of a symplectic 4-manifold M is a finite collection of embeddings of the closure of Δ into M which cover M. The embeddings should map leafs of the symplectic foliations of Δ into symplectic surfaces. Two different embeddings are either disjoint or intersect in the images of the boundaries of Δ. The intersection will be a boundary simplex of one of the Δ and along this simplex the images of the symplectic foliations should coincide.*

A topological cubular decomposition (TC) is defined similarly except that the embeddings are not required to be symplectic. However we do require that two distinct embeddings are either disjoint or the intersection is the image of a boundary simplex of each Δ.

Any M admits a TC, for example coming from a Morse decomposition.

Remark. *We observe that, at least if M is simply connected, then if M admits a SC its tangent bundle TM splits as a direct sum $TM = L_1 \oplus L_2$ of symplectic subbundles. At each point the splitting is given by the images of the coordinate directions in each simplex. This extends continuously across the boundaries of the simplices. If there is no continuous choice of an assignment of L_1 and L_2 to the components of the splitting, such an assignment does exist on a suitable double cover of M.*

Proposition 0.1. *After blowing up a finite number of points, any symplectic 4-manifold M admits a SC.*

Example If $M = S^2 \times S^2$ with its standard split symplectic form, we can take a TC of each factor. Here by a TC we mean a decomposition into images of squares in $\mathbb{C}$. The products of simplices in each factor give a SC of M.

Example Let $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ or the nontrivial S^2-bundle over S^2. We can write $M = D_1 \times S^2 \cup_h D_2 \times S^2$ where D_1 and D_2 are disks and the two components are glued using $h(e^{i\theta}, z) = (e^{i\theta}, e^{i\theta} z)$. Since h respects the symplectic structure on the

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S^2 fibers M is naturally a symplectic fibration π over the two-sphere $\Sigma = D_1 \cup D_2$. The symplectic structure on the fibers is given by the restriction of a global closed 2-form Ω and if σ is an area form on Σ then for K sufficiently large $\omega = \Omega + K\pi^*\sigma$ is a symplectic form on M. In this case we will see that M does not admit a TC which is also a SC.

Following [1] let B_+ and B_- denote the homology classes of the two sections of M corresponding to $\{0, \infty\} \subset S^2$, these have self-intersection ± 1. Denoting their Poincaré duals by $B_\pm^\vee$, it can be shown that M admits a symplectic form as above cohomologous to $\mu_+ B_+^\vee + \mu_- B_-^\vee$ if and only if $\mu_+ > \mu_- > 0$. Suppose that such a TC and SC structure exists on M, then we will find two associated cycles in M which can be represented by embedded symplectic surfaces. To do this, for each simplex in our decomposition we normalize so that the squares have center $\{0\} \in \mathbb{C}$ and look at the images of $\{0\} \times S_2$. We suppose that these images lie approximately tangent to the subbundle L_1 described above. They can be pieced together to form a symplectic surface with self-intersection 0.

Suppose that the surface is homologous to $S_1 = a_1 B_+ + b_1 B_-$. Since S_1 has positive symplectic area $a_1\mu_+ + b_1\mu_- > 0$ and the self-intersection number $S_1 \bullet S_1 = a_1^2 - b_1^2 = 0$. Another result of McDuff says that for any tame almost-complex structure on M, such as one making S_1 holomorphic, the class of B_- is represented by a holomorphic curve. Therefore $S_1 \bullet B_- = -b_1 \geq 0$. Hence $-b_1 = a_1 > 0$. Similarly we find another symplectic surface homologous to $S_2 = a_2 B_+ - a_2 B_-$ by looking at the other coordinates. But then $S_1 \bullet S_2 = a_1 a_2 - a_1 a_2 = 0$ which gives a contradiction since the surfaces intersect transversally in each simplex with positive intersection number. □

We now fix a compatible almost-complex structure J on (M, ω). After blowing up a perturbation of J extends to give a compatible almost-complex structure on the new symplectic manifold.

Lemma 0.1. *After blowing up M at a finite number of points the tangent bundle of the resulting symplectic manifold $\tilde{M}$ splits as a sum of complex line bundles. That is, there exist line bundles $\tilde{L_1}$ and $\tilde{L_2}$ over $\tilde{M}$ such that $(T\tilde{M}, J) = \tilde{L_1} \oplus \tilde{L_2}$.*

Proof Let T denote the tangent bundle of M and $\tilde{T}$ the tangent bundle of a blow up $\tilde{M}$ of M at m points. Let E_1,...,E_m be the exceptional divisors. Let L be a complex line bundle on M with $c_1(L)$ Poincaré dual to a divisor Σ (which may be assumed to be disjoint from the points to be blown up). Then let $\tilde{L}$ be a line bundle on $\tilde{M}$ with $c_1(\tilde{L})$ Poincaré dual to $\Sigma + k[E_1]$.

Now, $\tilde{T}$ will split with $\tilde{L}$ being a factor if and only if $\tilde{T} \otimes \tilde{L}^*$ admits a never vanishing section, that is, if $c_2(\tilde{T} \otimes \tilde{L}^*) = 0$.

We compute

$$c_2(T \otimes L^*) = c_1(L)^2 - c_1(L)c_1(T) + c_2(T)$$

and

$$c_2(\tilde{T} \otimes \tilde{L}^*) = c_1(\tilde{L})^2 - c_1(\tilde{L})c_1(\tilde{T}) + c_2(\tilde{T}).$$

Now,

$$c_1(\tilde{L}) = c_1(L) + k[E_1]^\vee,$$

$$c_1(\tilde{T}) = c_1(T) - \sum_{i=1}^{m} [E_i]^\vee$$

and

$$c_2(\tilde{T}) = c_2(T) + m.$$

Therefore

$$c_2(\tilde{T} \otimes \tilde{L}^*) = c_2(T \otimes L^*) - k^2 + k + m.$$

Hence $\tilde{T}$ splits provided we choose m and k such that $k^2 - k - m = c_2(T \otimes L^*)$. □

Example Let $M = \mathbb{C}P^2$ with its standard symplectic form ω. Then $T = T(M)$ does not split. Let L be the line bundle on $\mathbb{C}P^2$ with $c_1(L) = [\omega]$. Then $c_2(T \otimes L^*) = 1 - 3 + 3 = 1$. Therefore, setting $k = -1$, $m = 1$, we see that if we blow up M at a single point $T(\tilde{M})$ splits with one factor $\tilde{L}$ having first Chern class $[\omega] - [E_1]^\vee$. Here $\tilde{M}$ is just the nontrivial S^2 bundle over S^2 and $\tilde{L}$ is the vertical line bundle.

Proof of Proposition 0.1

We suppose from now that (M, ω) has been blown up sufficiently that with respect to a fixed compatible almost-complex structure J the complex vector bundle TM over M admits a splitting $TM = L_1 \oplus L_2$. Here L_2 can be taken to be the symplectic complement of L_1 in TM. We will measure distances and angles in M using the Riemannian metric $g(X, Y) = \omega(X, JY)$. We will construct an SC in which the tangent spaces to all leafs of the symplectic foliations of each cube are arbitrarily C^∞ close to one of the subbundles L_1 and L_2.

There exists an $\epsilon > 0$ such that at each point of $p \in M$ there exists a Darboux coordinate chart ϕ_p mapping the ball $B(p, \epsilon)$ to a neighborhood of 0 in $(\mathbb{C}^2, \omega_0)$. Furthermore $\phi_p(0)_*(J) = i$ and ϕ_p pushes forward the bundles L_1 and L_2 such that at 0 they coincide with the standard complex coordinate planes. Using these charts, on $B(p, \epsilon)$ we have natural projections onto $L_1(p)$ and $L_2(p)$.

We study the charts centered at a finite number of points such that the corresponding balls cover M. In fact, given a ball centered at a point p we may assume that all neighboring balls have centers lying in $B(p, \epsilon)$ and there exist C^∞ small (depending upon ϵ) symplectomorphisms mapping the foliations on $B(p, \epsilon)$ induced from the coordinate planes in $\mathbb{C}^2$ to the corresponding foliations of the $B(p_i, \epsilon)$, where the p_i are the centers of the neighboring balls.

We want to cover a neighborhood of each ball by a SC, ensuring compatibility between neighboring simplices.

Starting with the first ball, an SC structure on $\mathbb{C}^2$ pulls back under our chart to give a SC structure in a neighborhood of the ball.

We proceed to extend our SC over other balls by induction. Given a ball with center q, we will construct a similar product type SC on $B(q, \epsilon)$ and restrict to a neighborhood of the region which does not yet have a SC. We need to ensure that such a SC can be adjusted to extend our existing SC.

Part of the ball $B(q, \epsilon)$ may already be covered by an existing SC. By construction, the cells may be assumed to be of product type relative to foliations C^∞ close to the coordinate foliations on $B(q, \epsilon)$. Therefore the 3-cells on the boundary of this existing SC may be uniformly approximated by cells of the form $S \times I$ where S and I are a (solid) square and an interval respectively in one of the $L_i(q)$ and the product is defined with respect to the coordinate projections as before. Next we assume that the 3-cells on this boundary all intersect transversally (that is, 3 dimensional submanifolds slightly extending the cells intersect transversally). This can be easily arranged, at least in SCs constructed as we do below. Then if we choose the S and I sufficiently large our new cells will still intersect transversally and the relevant parts will form a new 3 dimensional subcomplex isomorphic to the original one, in the sense that there exists a C^∞ small ambient compactly supported isotopy of M mapping the boundary of the existing SC onto this new complex.

We replace the cells of our existing SC with their images under the ambient isotopy. So now the boundary of our existing SC intersects $B(q,\epsilon)$ in cells which are of product type with respect to the coordinate foliations of $B(q,\epsilon)$.

Using the canonical projections, we map the 1-skeleton of this boundary onto both $L_1(q)$ and $L_2(q)$. Generically the images of 1-cells will be intervals intersecting transversally which extend to give a decomposition of the planes into polygons. Taking a barycentric subdivision we get a decomposition of the planes into triangles, and dividing the triangles as in Figure 1 we will get TCs of the two planes. Taking a product type SC with respect to these two TCs now gives an SC of the remainder of $B(q,\epsilon)$ extending our existing SC as required.

□

References

[1] D. McDuff and D. Salamon, Introduction to symplectic topology. Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998