Inner Products, Parseval and Power Calculations

A fundamental property of the Fourier series expansion of a time function on the interval $(0, T)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}, 0 \leq t \leq T$$

is that the functions $e^{jk2\pi t/T}$ are mutually orthogonal on the interval $(0, T)$. Orthogonality of two functions $f(t)$ and $g(t)$ is defined as

$$\int_0^T f(t) g^*(t) dt = 0,$$

where the integral of one function with the conjugate of the other is called the *inner product* of f and g.

Consider now the scaled inner product of any two functions $x(t)$ and $y(t)$, whose Fourier series coefficients are the sets $\{a_k\}$ and $\{b_k\}$, respectively. Then

$$\frac{1}{T}\int_0^T x(t) y^*(t) dt = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T} \sum_{m=-\infty}^{\infty} b_m^* e^{-jm2\pi t/T} dt,$$

We can interchange the well-behaved integration and summations (assume conditions for convergence of Fourier series are met) and rearrange the above as

$$\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k b_m^* \frac{1}{T}\int_0^T e^{jk2\pi t/T} e^{-jm2\pi t/T} dt.$$

By the orthogonality of these functions, this scaled integral is zero any time k and m are not equal, and equals 1 when $k = m$. We could also directly demonstrate this, since the integral of the complex exponential rotating around the complex plane an integer number of times is always zero (recall our "Zero Sum Theorem," extended to integrals). So the double summation becomes a single summation after we throw out all the zero integrals for $k \neq m$, and we get

$$\frac{1}{T}\int_0^T x(t) y^*(t) dt = \sum_{k=-\infty}^{\infty} a_k b_k^*. \tag{1}$$

The left-hand side is the inner product of the continuous-time functions, and the right-hand side is the inner product of the two series $\{a_k\}$ and $\{b_k\}$.

If we let $x(t)$ and $y(t)$ be the same real-valued function, (1) becomes Parseval's Theorem for the total energy on $(0, T)$ or, with averaging by dividing by T, the average power in $x(t)$. All this also applies to an inner product between voltage and current, so all the above may be applied in electric power problems. A special case of this is our standard formula for complex power with a single frequency voltage and current, $\mathbf{S} = \mathbf{V}\mathbf{I}^*$. Here, through our use of rms values for both, we have effectively condensed each function into a single Fourier series coefficient, rather than using the two conjugate symmetric values.