ON THE VARIETY OF LAGRANGIAN SUBALGEBRAS, II

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Abstract. VERSION FRANÇAISE: Motivé par le théorème de Drinfeld sur les espaces de Poisson homogènes, nous étudions la variété $\mathcal{L}$ des sous-algèbres de Lie Lagrangiennes de $\mathfrak{g} \oplus \mathfrak{g}$ pour $\mathfrak{g}$, une algèbre de Lie complexe semisimple. Soit G le groupe adjointe de $\mathfrak{g}$. Nous montrons que les adhérences des $(G \times G)$-orbites dans $\mathcal{L}$ sont les variétés sphériques et lisses. Aussi, nous classifions les composantes irréductibles de $\mathcal{L}$ et nous montrons qu'elles sont lisses. Nous employons quelques méthodes de M. Yakimov pour donner une nouvelle description et une nouvelle preuve de la classification de Karolinsky des orbites diagonales de G dans $\mathcal{L}$, quel, comme cas spécial, donne la classification de Belavin-Drinfeld des $\mathfrak{r}$-matrices quasitriangulaires de $\mathfrak{g}$. En outre, $\mathcal{L}$ possède une structure de Poisson canonique, et nous calculons son rang à chaque point and nous décrivons sa décomposition en feuilles symplectiques en termes des intersections des orbites des deux sous-groupes de $G \times G$.

ENGLISH VERSION: Motivated by Drinfeld's theorem on Poisson homogeneous spaces, we study the variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ for a complex semi-simple Lie algebra $\mathfrak{g}$. Let G be the adjoint group of $\mathfrak{g}$. We show that the $(G \times G)$-orbit closures in $\mathcal{L}$ are smooth spherical varieties. We also classify the irreducible components of $\mathcal{L}$ and show that they are smooth. Using some methods of M. Yakimov, we give a new description and proof of Karolinsky's classification of the diagonal G-orbits in $\mathcal{L}$, which, as a special case, recovers the Belavin-Drinfeld classification of quasi-triangular r-matrices on $\mathfrak{g}$. Furthermore, $\mathcal{L}$ has a canonical Poisson structure, and we compute its rank at each point and describe its symplectic leaf decomposition in terms of intersections of orbits of two subgroups of $G \times G$.

Contents

1. Introduction

Let $\mathfrak{d}$ be a $2n$-dimensional Lie algebra over $k = \mathbb{R}$ or $\mathbb{C}$, together with a symmetric, non-degenerate, and ad-invariant bilinear form $\langle\, , \rangle$. A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ is said to be *Lagrangian* if $\mathfrak{l}$ is maximal isotropic with respect to $\langle\, , \rangle$, i.e., if $\dim_k \mathfrak{l} = n$ and if $\langle x, y \rangle = 0$ for all $x, y \in \mathfrak{l}$. By a *Lagrangian splitting* of $\mathfrak{d}$ we mean a direct sum decomposition $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Lagrangian subalgebras of $\mathfrak{d}$. Denote by $\mathcal{L}(\mathfrak{d})$ the set of all Lagrangian subalgebras of $\mathfrak{d}$. It is an algebraic subvariety of the Grassmannian $\mathrm{Gr}(n, \mathfrak{d})$ of n-dimensional subspaces of $\mathfrak{d}$, and every connected Lie group D with Lie algebra $\mathfrak{d}$ acts on $\mathcal{L}(\mathfrak{d})$ via the adjoint action of D on $\mathfrak{d}$. We proved in [E-L2] that each Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ gives rise to a Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$, making $\mathcal{L}(\mathfrak{d})$ into a Poisson variety. Moreover, if L_1 and L_2 are the connected subgroups of D with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, all the L_1 and L_2-orbits in $\mathcal{L}(\mathfrak{d})$ are Poisson submanifolds of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$.

The above construction in [E-L2] was motivated by the work of Drinfeld [Dr] on Poisson homogeneous spaces. Indeed, a Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ of $\mathfrak{d}$ gives rise to the Manin triple $(\mathfrak{d}, \mathfrak{l}_1, \mathfrak{l}_2)$, which in turn defines Poisson structures π_1 and π_2 on L_1 and L_2 respectively, making them into Poisson Lie groups [K-S]. A Poisson space (M, π) is said to be (L_1, π_1)-homogeneous if L_1 acts on M transitively and if the action map $L_1 \times M \to M$ is Poisson. In [Dr], Drinfeld constructed an L_1-equivariant map $M \to \mathcal{L}(\mathfrak{d})$ for every (L_1, π_1)-homogeneous Poisson space (M, π) and proved that (L_1, π_1)-homogeneous Poisson spaces correspond to L_1-orbits in $\mathcal{L}(\mathfrak{d})$ in this way. The Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$ is constructed in such a way that the Drinfeld map $M \to \mathcal{L}(\mathfrak{d})$ is a Poisson map. In many cases, the Drinfeld map $M \to \mathcal{L}(\mathfrak{d})$ is a local diffeomorphism onto its image. Thus we can think of L_1-orbits in $\mathcal{L}(\mathfrak{d})$ as as models for (L_1, π_1)-homogeneous Poisson spaces. For this reason, it is interesting to study the geometry of the variety $\mathcal{L}(\mathfrak{d})$, the L_1 and L_2-orbits in $\mathcal{L}(\mathfrak{d})$, and the Poisson structures $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$.

There are many examples of Lie algebras $\mathfrak{d}$ with symmetric, non-degenerate, and ad-invariant bilinear forms. The geometry of $\mathcal{L}(\mathfrak{d})$ is different from case to case. Moreover, there can be many Lagrangian splittings for a given $\mathfrak{d}$, resulting in many Poisson structures on $\mathcal{L}(\mathfrak{d})$.

Example 1.1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and regard it as a real Lie algebra, and let $\langle \, , \, \rangle$ be the imaginary part of the Killing form of $\mathfrak{g}$. The geometry of $\mathcal{L}(\mathfrak{g})$ was studied in [E-L2]. In particular, we determined the irreducible components of $\mathcal{L}(\mathfrak{g})$ and classified the G-orbits in $\mathcal{L}(\mathfrak{g})$, where G is the adjoint group of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. Then both $\mathfrak{k}$ and $\mathfrak{a} + \mathfrak{n}$ are Lagrangian, so $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$ is a Lagrangian splitting, resulting in a Poisson structure on $\mathcal{L}(\mathfrak{g})$ which we denote by π_0. Many interesting Poisson manifolds appear as G or K-orbits inside $\mathcal{L}(\mathfrak{g})$, where K is the connected subgroup of G with Lie algebra $\mathfrak{k}$. Among such Poisson manifolds are the flag manifolds of G and the compact symmetric spaces associated to real forms of G. Detailed studies of the Poisson geometry of these Poisson structures and some applications to Lie theory have been given in [Lu1], [Lu2], [E-L1], and [Ft-L]. For example, a flag manifold X of G can be identified with a certain K-orbit in $\mathcal{L}(\mathfrak{g})$. The resulting Poisson structure π_0 on X is called the *Bruhat-Poisson structure* because its symplectic leaves are Bruhat cells in X. In [Lu1] and [E-L1], we established connections between the Poisson geometry of π_0 and the harmonic forms on X constructed by Kostant [Ko] in 1963, and we gave a Poisson geometric interpretation of the Kostant-Kumar approach [K-K] to Schubert calculus on X.

Example 1.2. Let $\mathfrak{g}$ be any n-dimensional Lie algebra, and let $\mathfrak{d}$ be the semi-direct product of $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$. The symmetric bilinear form $\langle x+\xi, \, y+\eta\rangle = \langle x, \eta\rangle + \langle y, \xi\rangle$ for $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ is non-degenerate and ad-invariant. When $\mathfrak{g}$ is semi-simple, Lagrangian subalgebras of $\mathfrak{d}$ are not easy to classify (except for low dimensional cases), for, as a sub-problem, one needs to classify all abelian subalgebras of $\mathfrak{g}$. See [K-S], [Ka-St], and their references for more detail. The description of the geometry of $\mathcal{L}(\mathfrak{d})$ in this case is an open problem.

In this paper, we consider the complexification of Example 1.1. Namely, let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ the direct sum Lie algebra with the bilinear form

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \qquad x_1, x_2, y_1, y_2 \in \mathfrak{g},$$

where $\ll \, , \, \gg$ is a fixed symmetric, non-degenerate, and ad-invariant bilinear form on $\mathfrak{g}$. The variety of Lagrangian subalgebras of $\mathfrak{d}$ with respect to $\langle \, , \, \rangle$ will be denoted by $\mathcal{L}$.

The classification of Lagrangian subalgebras of $\mathfrak{d}$ has been given by Karolinsky [Ka], and Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$ have been classified by Delorme [De]. In this paper, we establish the first few steps in the study of the Poisson structures on $\mathcal{L}$ defined by Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$. Namely, we will first describe the geometry of $\mathcal{L}$ in the following terms:

1) the $(G \times G)$-orbits in $\mathcal{L}$ and their closures, where G is the adjoint group of $\mathfrak{g}$;

2) the irreducible components of $\mathcal{L}$;

We then look at the Poisson structure Π_0 on $\mathcal{L}$ defined by the so-called *standard* Lagrangian splitting $\mathfrak{d} = \mathfrak{g}_\Delta + \mathfrak{g}^*_{\mathrm{st}}$, where $\mathfrak{g}_\Delta = \{(x, x) : x \in \mathfrak{g}\}$ and $\mathfrak{g}^*_{\mathrm{st}} \subset \mathfrak{b} \oplus \mathfrak{b}^-$, where $\mathfrak{b}$ and $\mathfrak{b}^-$ are two opposite Borel subalgebras of $\mathfrak{g}$. Let $G_\Delta = \{(g, g) : g \in G\}$, and let B and B^- be the Borel subgroups of G with Lie algebras $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. We will compute the rank of Π_0 in $\mathcal{L}$ and study the symplectic leaf decomposition of $\mathcal{L}$ with respect to Π_0 in terms of the intersections of G_Δ and $(B \times B^-)$-orbits in $\mathcal{L}$.

We regard the Poisson structure Π_0 as the most important Poisson structure on $\mathcal{L}$, especially since it is more closely related to applications to other areas of mathematics. The study of the symplectic leaf decomposition of the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ defined by an arbitrary Lagrangian splitting $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ will be carried out in [Lu-Y2] using results of [Lu-Y1]. This analysis is much more technically involved, so we think it is worthwhile to present the important special case of Π_0 separately. In particular, in computing the rank of Π_0 in $\mathcal{L}$, we need to classify G_Δ-orbits in $\mathcal{L}$ and compute the normalizer subalgebra of $\mathfrak{g}_\Delta$ for every $\mathfrak{l} \in \mathcal{L}$. The classification of G_Δ-orbits in $\mathcal{L}$ follows directly from a special case of Theorem 2.2 in [Lu-Y1]. However, the inductive procedure used in the proof of Theorem 2.2 in [Lu-Y1] is rather involved. To make the paper self-contained and more comprehensible, we present a shorter inductive proof for the classification of G_Δ-orbits in $\mathcal{L}$. Our proof and the one in [Lu-Y1] are both adapted from [Y]. The description of the normalizer subalgebra of $\mathfrak{g}_\Delta$ at $\mathfrak{l} \in \mathcal{L}$ can be derived from Theorem 2.5 in [Lu-Y1]. Again for the purpose of completeness and comprehensibility, we give a simple and more direct computation of these subalgebras.

We point out that E. Karolinsky has in [Ka] given a classification of G_Δ-orbits in $\mathcal{L}$ in different terms. Our classification is more in line with that of Lagrangian splittings in [De], and in particular, the Belavin-Drinfeld theorem [B-Dr] on Lagrangian splittings of the form $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}$ follows easily from our classification, but does not seem to be easily derived from Karolinsky's classification. A special case of our classification is the classification of G_Δ-orbits on the wonderful compactification of G, and this special case was first determined by Lusztig in [Lusz1] and [Lusz2], where it is used to develop a theory of character sheaves for the compactifications. Our arguments are somewhat different from those in [Lusz1] and [Lusz2]. It would be very interesting to find connections between Poisson geometry and character sheaves.

We now give more details of the main results in this paper:

Following O. Schiffmann [Sch], we define a *generalized Belavin-Drinfeld triple* (generalized BD-triple) to be a triple (S, T, d), where S and T are two subsets of the set Γ of vertices of the Dynkin diagram of $\mathfrak{g}$, and $d : S \to T$ is an isometry with respect to $\ll\, , \gg$. For a generalized BD-triple (S, T, d), let P_S and P_T^- be respectively the standard parabolic subgroups of G of type S and opposite type T (see Notation 2.11) and Levi decompositions $P_S = M_S N_S$ and $P_T^- = M_T N_T^-$. Let G_S and G_T be the quotients of M_S and M_T by their centers respectively, and let $\chi_S : M_S \to G_S$ and $\chi_T : M_T \to G_T$ be the natural projections. Let $\gamma_d : G_S \to G_T$ be the group isomorphism induced by d, and define the subgroup $R_{S,T,d}$ of $P_S \times P_T^-$ by

$$R_{S,T,d} = \{(m_S, m_T) \in M_S \times M_T :\ \gamma_d(\chi_S(m_S)) = \chi_T(m_T)\}(N_S \times N_T^-).$$

In Section 2, we establish the following facts on $(G\times G)$-orbits and their closures in $\mathcal{L}$ (Proposition 2.17, Corollary 2.22, and Proposition 2.25):

1) Every $(G \times G)$-orbit in $\mathcal{L}$ is isomorphic to $(G \times G)/R_{S,T,d}$ for a generalized BD-triple (S, T, d), so the $(G \times G)$-orbit types in $\mathcal{L}$ correspond bijectively to generalized BD-triples for G; Every $(G \times G)$-orbit in $\mathcal{L}$ is a $(G \times G)$-spherical homogeneous space.

2) For a generalized BD-triple (S, T, d), the closure of a $(G\times G)$-orbit of type (S, T, d) is a fiber bundle over the flag manifold $G/P_S \times G/P_T^-$ whose fiber is isomorphic to a De Concini-Procesi compactification of G_S.

We also study in Section 2 the irreducible components of $\mathcal{L}$. We prove (Corollary 2.27, Theorem 2.29, and Theorem 2.32):

1) The irreducible components of $\mathcal{L}$ are roughly (see Theorem 2.32 for detail) labeled by quadruples (S, T, d, ϵ), where (S, T, d) are generalized BD-triples and $\epsilon \in \{0, 1\}$;

2) The irreducible component corresponding to (S, T, d, ϵ) is a fiber bundle over the flag manifold $G/P_S \times G/P_T^-$ with fiber isomorphic to the product of a De Concini-Procesi compactification of G_S and a homogeneous space of a special orthogonal group. In particular, all the irreducible components of $\mathcal{L}$ are smooth;

3) $\mathcal{L}$ has two connected components.

In Section 3, we give the classification of G_Δ-orbits in $\mathcal{L}$, and we compute the normalizer subgalgebra of $\mathfrak{g}_\Delta$ at every $\mathfrak{l} \in \mathcal{L}$. In Section 4, we compute the rank of the Poisson structure Π_0. In particular, we prove (Theorem 4.5 and Theorem 4.14):

Every non-empty intersection of a G_Δ-orbit $\mathcal{O}$ and a $(B \times B^-)$-orbit $\mathcal{O}'$ in $\mathcal{L}$ is a regular Poisson variety with respect to Π_0; The Cartan subgroup $H_\Delta = \{(h, h) : h \in H\}$ of G_Δ, where $H = B \cap B^-$, acts transitively on the set of symplectic leaves in $\mathcal{O} \cap \mathcal{O}'$.

An interesting Poisson subvariety of $(\mathcal{L}, \Pi_0)$ is the De Concini-Procesi compactification $Z_1(G)$ of G as the closure of the $(G \times G)$-orbit through $\mathfrak{g}_\Delta \in \mathcal{L}$. Conjugacy classes in G and their closures in $Z_1(G)$ are all Poisson subvarieties of $(Z_1(G), \Pi_0)$. In particular, Π_0 restricted to a conjugacy class C in G is non-degenerate precisely on the intersection of C with the open Bruhat cell B^-B (see Corollary 4.7). It will be particularly interesting to compare the Poisson structure Π_0 on the unipotent variety in G with the Kirillov-Kostant structure on the nilpotent cone in $\mathfrak{g}^*$. Another interesting Poisson subvariety of $(\mathcal{L}, \Pi_0)$ is the De Concini-Procesi compactification X_σ of a complex symmetric space G/G^σ for an involutive automorphism σ of G (Proposition 3.21). Intersections of G_Δ-orbits and $(B \times B^-)$-orbits inside the closed $(G \times G)$-orbits in $\mathcal{L}$ are related to double Bruhat cells in G (see Example 4.9), and Kogan and Zelevinsky [K-Z] have constructed toric charts on some of the symplectic leaves in these closed orbits. It would be interesting to see how their methods can be applied to other symplectic leaves of Π_0.

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2. The variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$

Throughout this paper, $\mathfrak{g}$ will be a complex semi-simple Lie algebra, and $\ll\, ,\, \gg$ will be a fixed symmetric and non-degenerate ad-invariant bilinear form on $\mathfrak{g}$. Equip the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the symmetric non-degenerate ad-invariant bilinear form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \qquad x_1, x_2, y_1, y_2 \in \mathfrak{g}. \tag{2.1}$$

A Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ is by definition an n-dimensional complex Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ that is isotropic with respect to $\langle\, ,\, \rangle$.

Notation 2.1. We will denote by $\mathcal{L}$ the variety of all Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ and by $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ the variety of all n-dimensional isotropic *subspaces* of $\mathfrak{g} \oplus \mathfrak{g}$.

Let G be the adjoint group of $\mathfrak{g}$. In this section, we will use the classification of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ by Karolinsky [Ka] to study the $(G \times G)$-orbits and their closures in $\mathcal{L}$, and we will determine the irreducible components of $\mathcal{L}$.

2.1. **Lagrangian subspaces.** Let U be a finite-dimensional complex vector space with a symmetric and non-degenerate bilinear form $\langle \, , \, \rangle$. A subspace V of U is said to be Lagrangian if V is maximal isotropic with respect to $\langle \, , \, \rangle$. If $\dim U = 2n$ or $2n+1$, Witt's theorem says that the dimension of a Lagrangian subspace of U is n. Denote by $\mathcal{L}_{\mathrm{space}}(U)$ the set of Lagrangian subspaces of U. It is a closed algebraic subvariety of $\mathrm{Gr}(n, U)$, the Grassmannian of n-dimensional subspaces of U.

Proposition 2.2 ([A-C-G-H], pp. 102-103)**.** *Let* $\dim U = 2n$ *(resp.* $2n+1$*) with* $n > 0$*. Then* $\mathcal{L}_{\mathrm{space}}(U)$ *is a smooth algebraic subvariety of* $\mathrm{Gr}(n, U)$ *with two (resp. one) connected components, each of which is isomorphic to* $SO(2n, \mathbb{C})/P$ *(resp.* $SO(2n+1, \mathbb{C})/P$*)) where* P *has Levi factor isomorphic to* $GL(n, \mathbb{C})$*. Moreover,* $\mathcal{L}_{\mathrm{space}}(U)$ *has complex dimension* $\frac{n(n-1)}{2}$ *(resp.* $\frac{n(n+1)}{2}$*). When* $\dim U = 2n$*, two Lagrangian subspaces* V_1 *and* V_2 *are in the same connected component of* $\mathcal{L}_{\mathrm{space}}(U)$ *if and only if* $\dim(V_1) - \dim(V_1 \cap V_2)$ *is even.*

Notation 2.3. For $U = \mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form $\langle \, , \, \rangle$ in (2.1), let $\mathcal{L}^0$ be the intersection of $\mathcal{L}$ with the connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ containing the diagonal of $\mathfrak{g} \oplus \mathfrak{g}$. The intersection of $\mathcal{L}$ with the other connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ will be denoted by $\mathcal{L}^1$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{n}$ and $\mathfrak{n}^-$ be the opposite nilpotent subalgebras of $\mathfrak{g}$ corresponding to a choice of positive roots for $(\mathfrak{g}, \mathfrak{h})$. For a Lagrangian subspace V of $\mathfrak{h} \oplus \mathfrak{h}$ with respect to $\langle \, , \, \rangle$, let $\mathfrak{l}_V = V + \{(x, y) : x \in \mathfrak{n}, y \in \mathfrak{n}^-\}$. Then $\mathfrak{l}_V \in \mathcal{L}$. It is easy to see from Proposition 2.2 that $\mathfrak{l}_{V_1}$ and $\mathfrak{l}_{V_2}$ are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ if and only if V_1 and V_2 are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{h} \oplus \mathfrak{h})$. In particular, $\mathcal{L}^1$ is non-empty.

2.2. **Isometries.** We collect some results on automorphisms that will be used in later sections.

Notation 2.4. Throughout this paper, we fix a Cartan subalgebra $\mathfrak{h}$ and a choice Σ^+ of positive roots in the set Σ of all roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, and let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ be the root decomposition. Let Γ be the set of simple roots in Σ^+. For $\alpha \in \Sigma$, let $H_\alpha \in \mathfrak{h}$ be such that $\ll H_\alpha, H \gg = \alpha(H)$ for all $H \in \mathfrak{h}$. For $\alpha \in \Sigma^+$, fix root vectors $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $\ll E_\alpha, E_{-\alpha} \gg = 1$. For a subset S of Γ, let $[S]$ be the set of roots in the linear span of S, and let G_S be the adjoint group of the semisimple Lie algebra $\mathfrak{g}_S$ given by

$$\mathfrak{g}_S = \mathrm{span}_{\mathbb{C}}\{H_\alpha : \alpha \in S\} + \sum_{\alpha \in [S]} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

For $S, T \subset \Gamma$, we are interested in Lie algebra isomorphisms $\mathfrak{g}_S \to \mathfrak{g}_T$ that preserve the restrictions of $\ll \, , \, \gg$ to $\mathfrak{g}_S$ and $\mathfrak{g}_T$. We will simply refer to this property as preserving $\ll \, , \, \gg$.

Definition 2.5. Let $S, T \subset \Gamma$. By an isometry from S to T we mean a bijection $d : S \to T$ such that $\ll d\alpha, d\beta \gg = \ll \alpha, \beta \gg$ for all $\alpha, \beta \in S$, where $\ll \alpha, \beta \gg = \ll H_\alpha, H_\beta \gg$. Let $I(S, T)$

be the set of all isometries from S to T. Following [Sch], a triple (S, T, d), where $S, T \subset \Gamma$ and $d \in I(S, T)$, will also be called a *generalized Belavin-Drinfeld (generalized BD-)triple for G.*

Lemma 2.6. *For any $S, T \subset \Gamma$ and $d \in I(S, T)$, there is a unique isomorphism $\gamma_d : \mathfrak{g}_S \to \mathfrak{g}_T$ such that*

$$\gamma_d(E_\alpha) = E_{d(\alpha)}, \quad \gamma_d(H_\alpha) = H_{d(\alpha)}, \quad \forall \alpha \in S. \tag{2.2}$$

Moreover, γ_d preserves $\ll\, , \gg$, and for every Lie algebra isomorphism $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ preserving $\ll\, , \gg$, there is a unique $d \in I(S, T)$ and a unique $g \in G_S$ such that $\mu = \gamma_d \mathrm{Ad}_g$.

Proof. Existence and uniqueness of γ_d is by Theorem 2.108 in [Kn]. For $\alpha \in \Sigma^+$, let $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ be such that $\gamma_d(E_\alpha) = \lambda_\alpha E_{d\alpha}$ and $\gamma_d(E_{-\alpha}) = \mu_\alpha E_{-d\alpha}$. By applying γ_d to the identity $[E_\alpha, E_{-\alpha}] = H_\alpha$ we get $\lambda_\alpha \mu_\alpha = 1$ for every $\alpha \in \Sigma^+$. It follows that γ_d preserves $\ll\, , \gg$. Suppose that $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ is a Lie algebra isomorphism preserving $\ll\, , \gg$. Let d_1 be any isomorphism from the Dynkin diagram of $\mathfrak{g}_S$ to the Dynkin diagram of $\mathfrak{g}_T$. Let $\gamma_{d_1} : \mathfrak{g}_S \to \mathfrak{g}_T$ be defined as in (2.2). Then $\nu := \gamma_{d_1}^{-1} \mu$ is an automorphism of $\mathfrak{g}_S$. Recall that there is a short exact sequence

$$1 \longrightarrow G_S \longrightarrow \mathrm{Aut}_{\mathfrak{g}_S} \longrightarrow \mathrm{Aut}_S \longrightarrow 1,$$

where $\mathrm{Aut}_{\mathfrak{g}_S}$ is the group of automorphisms of $\mathfrak{g}_S$, and Aut_S is the group of automorphisms of the Dynkin diagram of $\mathfrak{g}_S$. Let $d_2 \in \mathrm{Aut}_S$ be the image of ν under the map $\mathrm{Aut}_{\mathfrak{g}_S} \to \mathrm{Aut}_S$ and write $\nu = \gamma_{d_2} \mathrm{Ad}_g$ for some $g \in G_S$. Thus $\mu = \gamma_{d_1} \gamma_{d_2} \mathrm{Ad}_g = \gamma_{d_1 d_2} \mathrm{Ad}_g$. Since μ and Ad_g are isometries of $\ll\, , \gg$, so is $\gamma_{d_1 d_2}$. Thus $d := d_1 d_2 \in I(S, T)$ is an isometry, and $\mu = \gamma_d \mathrm{Ad}_g$.

Uniqueness of d and g follows from the fact that if $g_0 \in G_S$ preserves a Cartan subalgebra and acts as the identity on all simple root spaces, then g_0 is the identity element.

Q.E.D.

Definition 2.7. A $\ll\, , \gg$-preserving Lie algebra isomorphism $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ is said to be of *type* d for $d \in I(S, T)$ if d is the unique element in $I(S, T)$ such that $\mu = \gamma_d \mathrm{Ad}_g$ for some $g \in G_S$.

2.3. **Karolinsky's classification.** Karolinsky [Ka] has classified the Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ with respect to the bilinear form $\langle\, , \rangle$ given in (2.1). We recall his results now.

Notation 2.8. For a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, let $\mathfrak{n}$ be its nilradical and $\mathfrak{m} := \mathfrak{p}/\mathfrak{n}$ its Levi factor. Let $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{z}$ be the decomposition of $\mathfrak{m}$ into the direct sum of its derived algebra $[\mathfrak{m}, \mathfrak{m}]$ and its center $\mathfrak{z}$. Recall that $[\mathfrak{m}, \mathfrak{m}]$ is semisimple and that the restrictions of the bilinear form $\ll\, , \gg$ to $\mathfrak{m}$ and $\mathfrak{z}$ are both non-degenerate. If $\mathfrak{p}'$ is another parabolic subalgebra, denote its nilradical, Levi factor, and center of Levi factor, etc. by $\mathfrak{n}'$, $\mathfrak{m}'$, and $\mathfrak{z}'$, etc.. When speaking of Lagrangian subspaces of $\mathfrak{z} \oplus \mathfrak{z}'$, we mean with respect to the restriction of $\langle\, , \rangle$ to $\mathfrak{z} \oplus \mathfrak{z}'$.

Definition 2.9. A quadruple $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is called admissible if $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$, $\mu : [\mathfrak{m}, \mathfrak{m}] \to [\mathfrak{m}', \mathfrak{m}']$ is a Lie algebra isomorphism preserving $\ll\, , \gg$, and V is a Lagrangian subspace of $\mathfrak{z} \oplus \mathfrak{z}'$.

If $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is admissible, set

$$\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V) := \{(x, x') : x \in \mathfrak{p}, x' \in \mathfrak{p}', \mu(x_{[\mathfrak{m},\mathfrak{m}]}) = x'_{[\mathfrak{m}',\mathfrak{m}']}, (x_{\mathfrak{z}}, x'_{\mathfrak{z}'}) \in V\} \subset \mathfrak{g} \oplus \mathfrak{g},$$

where for $x \in \mathfrak{p}$, $x_{[\mathfrak{m},\mathfrak{m}]} \in [\mathfrak{m}, \mathfrak{m}]$ and $x_{\mathfrak{z}} \in \mathfrak{z}$ are respectively the $[\mathfrak{m}, \mathfrak{m}]$- and $\mathfrak{z}$-components of $x + \mathfrak{n} \in \mathfrak{p}/\mathfrak{n} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{z}$. We use similar notation for $x' \in \mathfrak{p}'$.

Theorem 2.10 ([Ka]). *$\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is a Lagrangian subalgebra if $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is admissible, and every Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ is of the form $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ for some admissible quadruple.*

2.4. **Partition of $\mathcal{L}$.**

Notation 2.11. Recall the fixed choice Σ^+ of positive roots from Notation 2.4. Set

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}.$$

A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is said to be *standard* if it contains the Borel subalgebra $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}$. Recall also that, for $S \subset \Gamma$, $[S]$ denotes the set of roots in the linear span of S. Let

$$\mathfrak{m}_S = \mathfrak{h} + \sum\nolimits_{\alpha \in [S]} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S = \sum\nolimits_{\alpha \in \Sigma^+ - [S]} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S^- = \sum_{\alpha \in \Sigma^+ - [S]} \mathfrak{g}_{-\alpha}$$

and $\mathfrak{p}_S = \mathfrak{m}_S + \mathfrak{n}_S$, $\mathfrak{p}_S^- = \mathfrak{m}_S + \mathfrak{n}_S^-$. We will refer to $\mathfrak{p}_S$ as the standard parabolic subalgebra of $\mathfrak{g}$ defined by S and $\mathfrak{p}_S^-$ the *opposite* of $\mathfrak{p}_S$. A parabolic subalgebra of $\mathfrak{p}$ of $\mathfrak{g}$ is said to be of *type* S if $\mathfrak{p}$ is conjugate to $\mathfrak{p}_S$ and of the *opposite-type* S if $\mathfrak{p}$ is conjugate to $\mathfrak{p}_S^-$. Let $\mathfrak{g}_S = [\mathfrak{m}_S, \mathfrak{m}_S]$ as in Notation 2.4 and

$$\mathfrak{h}_S = \mathfrak{h} \cap \mathfrak{g}_S = \mathrm{span}_{\mathbb{C}}\{H_\alpha : \alpha \in [S]\}, \qquad \mathfrak{z}_S = \{x \in \mathfrak{h} : \alpha(x) = 0, \forall \alpha \in S\}.$$

Then $\mathfrak{p}_S = \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S$ and $\mathfrak{p}_S^- = \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S^-$. The connected subgroups of G with Lie algebras $\mathfrak{p}_S, \mathfrak{p}_S^-, \mathfrak{m}_S, \mathfrak{n}_S$ and $\mathfrak{n}_S^-$ will be respectively denoted by P_S, P_S^-, M_S, N_S and N_S^-. Correspondingly we have $P_S = M_S N_S$ and $P_S^- = M_S N_S^-$, where $M_S \cap N_S = \{e\} = M_S \cap N_S^-$. Recall that G_S denotes the adjoint group of $\mathfrak{g}_S$. Let $\chi_S : P_S \to G_S$ be the composition of the projection from $P_S = M_S N_S$ to M_S along N_S and the projection $M_S \to G_S$. The similarly defined projection from P_S^- to G_S will also be denoted by χ_S.

Returning to the notation in Notation 2.8, we have

Lemma-Definition 2.12. *Let $(\mathfrak{p}, \mathfrak{p}', \mu)$ be a triple, where $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$, and $\mu : [\mathfrak{m}, \mathfrak{m}] \to [\mathfrak{m}', \mathfrak{m}']$ is a Lie algebra isomorphism preserving $\ll, \gg$. Assume that $\mathfrak{p}$ is of type S and $\mathfrak{p}'$ is of opposite-type T. Let $g_1, g_2 \in G$ be such that $\mathrm{Ad}_{g_1}\mathfrak{p} = \mathfrak{p}_S$ and $\mathrm{Ad}_{g_2}\mathfrak{p}' = \mathfrak{p}_T^-$. Let $\overline{\mathrm{Ad}_{g_1}}$ and $\overline{\mathrm{Ad}_{g_2}}$ be the induced Lie algebra isomorphisms*

$$\overline{\mathrm{Ad}_{g_1}} : [\mathfrak{m}, \mathfrak{m}] \longrightarrow \mathfrak{g}_S, \qquad \overline{\mathrm{Ad}_{g_2}} : [\mathfrak{m}', \mathfrak{m}'] \longrightarrow \mathfrak{g}_T.$$

If $\mu' := \overline{\mathrm{Ad}_{g_2}} \circ \mu \circ (\overline{\mathrm{Ad}_{g_1}})^{-1} : \mathfrak{g}_S \to \mathfrak{g}_T$ is of type $d \in I(S,T)$ as in Definition 2.7, we say that $(\mathfrak{p}, \mathfrak{p}', \mu)$ is of type (S, T, d). The type of $(\mathfrak{p}, \mathfrak{p}', \mu)$ is independent of the choice of g_1 and g_2.

Proof. If h_1 and h_2 in G are such that $\mathrm{Ad}_{h_1}\mathfrak{p} = \mathfrak{p}_S$ and $\mathrm{Ad}_{h_2}\mathfrak{p}' = \mathfrak{p}_T^-$, then there exist $p_S \in P_S$ and $p_T^- \in P_T^-$ such that $h_1 = p_S g_1$ and $h_2 = p_T^- g_2$. Thus

$$\mu'' := \overline{\mathrm{Ad}_{h_2}} \circ \mu \circ (\overline{\mathrm{Ad}_{h_1}})^{-1} = \overline{\mathrm{Ad}_{p_T^-}} \circ \mu' \circ (\overline{\mathrm{Ad}_{p_S}})^{-1}.$$

The action of $\overline{\mathrm{Ad}_{p_S}}$ on $\mathfrak{g}_S$ is by definition the adjoint action of $\chi_S(p_S) \in G_S$ on $\mathfrak{g}_S$. Similarly for the action of $\overline{\mathrm{Ad}_{p_T^-}}$ on $\mathfrak{g}_T$. Thus by Definition 2.7, the two maps μ' and μ'' have the same type.

Q.E.D.

We are now ready to partition $\mathcal{L}$. Recall the definitions of $\mathcal{L}^0$ and $\mathcal{L}^1$ in Notation 2.3.

Definition 2.13. Let $S, T \subset \Gamma, d \in I(S,T)$, and $\epsilon \in \{0,1\}$. Define $\mathcal{L}^\epsilon(S,T,d)$ to be the set of all Lagrangian subalgebras $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ such that

1) $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V) \in \mathcal{L}^\epsilon$;
2) $(\mathfrak{p}, \mathfrak{p}', \mu)$ is of type (S, T, d).

We say that $\mathfrak{l} \in \mathcal{L}$ is of *type* (ϵ, S, T, d) if $\mathfrak{l} \in \mathcal{L}^\epsilon(S,T,d)$.

It is clear that

$$\mathcal{L} = \bigcup_{\epsilon \in \{0,1\}} \bigcup_{S,T \subset \Gamma, d \in I(S,T)} \mathcal{L}^\epsilon(S,T,d) \tag{2.3}$$

is a disjoint union, and each $\mathcal{L}^\epsilon(S,T,d)$ is invariant under $G \times G$. Set

$$\mathfrak{n}_S \oplus \mathfrak{n}_T^- = \{(x,y) : x \in \mathfrak{n}_S, y \in \mathfrak{n}_T^-\} \subset \mathfrak{g} \oplus \mathfrak{g},$$

and for each $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, set

$$\mathfrak{l}_{S,T,d,V} = V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\} \in \mathcal{L}. \tag{2.4}$$

Note that $\dim \mathfrak{z}_S = \dim \mathfrak{z}_T$ because $\mathfrak{h}_S \cong \mathfrak{h}_T$ and $\mathfrak{h} = \mathfrak{z}_S + \mathfrak{h}_S = \mathfrak{z}_T + \mathfrak{h}_T$ are direct sums.

Lemma-Definition 2.14. *For $V_1, V_2 \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, $\mathfrak{l}_{S,T,d,V_1}$ and $\mathfrak{l}_{S,T,d,V_2}$ are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ if and only if V_1 and V_2 are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. For $\epsilon = \{0,1\}$, let*

$$\mathcal{L}^\epsilon_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T) = \{V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T) : \mathfrak{l}_{S,T,d,V} \in \mathcal{L}^\epsilon\}.$$

Proof. The statement follows from Proposition 2.2 and the fact

$$\dim(\mathfrak{l}_{S,T,d,V_1}) - \dim(\mathfrak{l}_{S,T,d,V_1} \cap \mathfrak{l}_{S,T,d,V_2}) = \dim(V_1) - \dim(V_1 \cap V_2).$$

Q.E.D.

Proposition 2.15. *For any generalized BD-triple (S,T,d) and $\epsilon \in \{0,1\}$,*

$$\mathcal{L}^\epsilon(S,T,d) = \bigcup_{V \in \mathcal{L}^\epsilon_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)} (G \times G) \cdot \mathfrak{l}_{S,T,d,V} \qquad (\textit{disjoint union}). \tag{2.5}$$

Proof. By Definition 2.13, every $(G \times G)$-orbit in $\mathcal{L}^\epsilon(S,T,d)$ passes through an $\mathfrak{l}_{S,T,d,V}$ for some $V \in \mathcal{L}^\epsilon_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. If $V_1, V \in \mathcal{L}^\epsilon_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$ are such that $\mathfrak{l}_{S,T,d,V_1} = \mathrm{Ad}_{(g_1,g_2)} \mathfrak{l}_{S,T,d,V}$, then (g_1, g_2) normalizes $\mathfrak{n}_S \oplus \mathfrak{n}_T^-$, so $(g_1,g_2) \in P_S \times P_T^-$, and it follows that $V_1 = V$.

Q.E.D.

2.5. $(G \times G)$**-orbits in** $\mathcal{L}$**.** The following theorem follows immediately from Proposition 2.15 and the decomposition of $\mathcal{L}$ in (2.3).

Theorem 2.16. *Every $(G \times G)$-orbit in $\mathcal{L}$ passes through an $\mathfrak{l}_{S,T,d,V}$ for a unique quadruple (S,T,d,V), where $S,T \subset \Gamma, d \in I(S,T)$ and $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$.*

For each $S, T \subset \Gamma$ and $d \in I(S,T)$, define the group (see Notation 2.11)

$$R_{S,T,d} := \{(p_S, p_T^-) \in P_S \times P_T^- : \gamma_d(\chi_S(p_S)) = \chi_T(p_T^-)\} \subset P_S \times P_T^-. \tag{2.6}$$

Proposition 2.17. *Let $S, T \subset \Gamma, d \in I(S,T)$, and $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$.*

1) The $(G\times G)$-orbit in $\mathcal{L}$ through $\mathfrak{l}_{S,T,d,V}$ is isomorphic to $(G\times G)/R_{S,T,d}$ and it has dimension $n-z$, where $n = \dim \mathfrak{g}$ and $z = \dim \mathfrak{z}_S$.

2) $(G\times G)\cdot \mathfrak{l}_{S,T,d,V}$ fibers over $G/P_S \times G/P_T^-$ with fiber isomorphic to G_S.

Proof. It is routine to check that the stabilizer of $\mathfrak{l}_{S,T,d,V}$ is $R_{S,T,d}$, and the dimensional formula follows. The fiber in 2) may be identified with $(P_S \times P_T^-)/R_{S,T,d}$, which may be identified with G_S via the map

$$(P_S \times P_T^-)/R_{S,T,d} \longrightarrow G_S : \quad (p_S, p_T^-) \mapsto \gamma_d^{-1}(\chi_T(p_T^-))(\chi_S(p_S))^{-1}$$

Q.E.D.

Remark 2.18. It follows that the number of orbit types for $G\times G$ in $\mathcal{L}$ is exactly the number of generalized BD-triples for G.

Lemma 2.19. *$R_{S,T,d}$ is connected.*

Proof. The projection map $p : R_{S,T,d} \to P_S$, $(p_S, p_T^-) \mapsto p_S$ is surjective and has fiber $N_T^- \times Z_T$, where Z_T is the center of the group M_T. Clearly N_T^- is connected, and Z_T is connected by Proposition 8.1.4 of [C]. Since P_S is connected, it follows that $R_{S,T,d}$ is also connected.

Q.E.D.

2.6. **$(B\times B^-)$-orbits in $\mathcal{L}$.** Let B and B^- be the Borel subgroups of G with Lie algebras $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$ respectively. By Proposition 2.17, to classify $(B\times B^-)$-orbits in $\mathcal{L}$, it suffices to consider $(B\times B^-)$-orbits in $(G\times G)/R_{S,T,d}$ for any generalized BD-triple (S,T,d).

Let W be the Weyl group of Σ. For $S\subset\Gamma$, let W_S be the subgroup of W generated by elements in S, and let W^S be the set of minimal length representatives of elements in the cosets in W/W_S. Fix a representative $\dot{w}$ in G for each $w\in W$. The following assertion can be proved in the same way as Lemma 1.3 in [Sp]. It also follows directly from Proposition 8.1 of [Lu-Y1].

Proposition 2.20. *Let (S,T,d) be an generalized BD-triple for G. Then every $(B\times B^-)$-orbit in $(G\times G)/R_{S,T,d}$ is of the form $(B\times B^-)(\dot{w},\dot{v})R_{S,T,d}$ for a unique pair $(w,v)\in W\times W^T$.*

Corollary 2.21. *Every $(B\times B^-)$-orbit in $\mathcal{L}$ goes through exactly one point in $\mathcal{L}$ of the form $\mathrm{Ad}_{(\dot{w},\dot{v})}\mathfrak{l}_{S,T,d,V}$, where (S,T,d) is a generalized BD-triple, $V\in\mathcal{L}_{space}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$, and $(w,v)\in W\times W^T$.*

Since each $(G\times G)$-orbit in $\mathcal{L}$ has finitely many $(B\times B^-)$-orbits, at least one of them is open.

Corollary 2.22. *All $(G\times G)$-orbits in $\mathcal{L}$ are $(G\times G)$-spherical homogeneous spaces.*

2.7. **The De Concini-Procesi compactifications $Z_d(G)$ of G.** In this section, we will consider the closure in $\mathcal{L}$ of some special $(G\times G)$-orbits. Namely, when $S=T=\Gamma$ and $d\in I(\Gamma,\Gamma)$, we have the graph $\mathfrak{l}_{\gamma_d}$ of $\gamma_d : \mathfrak{g}\to\mathfrak{g}$ as a point in $\mathcal{L}$:

$$\mathfrak{l}_{\gamma_d} = \{(x, \gamma_d(x)) : x\in\mathfrak{g}\}. \tag{2.7}$$

The $(G\times G)$-orbit in $\mathcal{L}$ through $\mathfrak{l}_{\gamma_d}$ can be identified with G by the map

$$I_d : G \longrightarrow (G\times G)\cdot\mathfrak{l}_{\gamma_d} : \; g\longmapsto \{(x,\, \gamma_d\mathrm{Ad}_g(x)) : x\in\mathfrak{g}\}. \tag{2.8}$$

The identification I_d is $(G \times G)$-equivariant if we equip G with the action of $G \times G$ given by

$$(g_1, g_2) \cdot g = \gamma_d^{-1}(g_2) g g_1^{-1}. \tag{2.9}$$

Since an orbit of an algebraic group on a variety is locally closed (Section 8.3 in [Hu]), the orbit $(G\times G)\cdot \mathfrak{l}_{\gamma_d}$ has the same closure in the Zariski topology and in the classical topology. The closure $\overline{(G \times G) \cdot \mathfrak{l}_{\gamma_d}}$, called a *De Concini-Procesi compactification of* G, is a smooth projective variety of dimension $n = \dim G$ (see [D-P, §6]). We denote this closure by $Z_d(G)$.

It is known [D-P] that $G \times G$ has finitely many orbits in $Z_d(G)$ indexed by subsets of Γ. Indeed, for each $S \subset \Gamma$, let $\mathfrak{l}_{S,d} \in \mathcal{L}$ be given by

$$\mathfrak{l}_{S,d} = \mathfrak{n}_S \oplus \mathfrak{n}_{d(S)}^- + \{(x, \gamma_d(x)) : x \in \mathfrak{m}_S\}. \tag{2.10}$$

Choose $\lambda \in \mathfrak{h}$ such that $\alpha(\lambda) = 0$ for $\alpha \in S$ and $\alpha(\lambda) > 0$ for $\alpha \in \Gamma - S$, and let $e^\lambda : \mathbb{C}^* \to H$ be the one parameter subgroup of H corresponding to λ. Then it is easy to see that

$$\lim_{t\to+\infty} \mathrm{Ad}_{(e^\lambda(t),e)} \mathfrak{l}_{\gamma_d} = \mathfrak{l}_{S,d} \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}).$$

Thus $\mathfrak{l}_{S,d} \in Z_d(G)$. It is easy to see that $\mathfrak{l}_{\gamma_d} \in \mathcal{L}^\epsilon(\Gamma, \Gamma, d)$ for $\epsilon = (\dim \mathfrak{h} - \dim \mathfrak{h}^{\gamma_d}) \mod 2$. Thus $\mathfrak{l}_{S,d} \in \mathcal{L}^\epsilon(S, d(S), d|_S)$ for the same value of ϵ.

Theorem 2.23. [D-P] *For every* $d \in I(\Gamma, \Gamma)$, $Z_d(G) = \bigcup_{S\subset\Gamma}(G \times G) \cdot \mathfrak{l}_{S,d}$.

2.8. **Closures of** $(G \times G)$**-orbits in** $\mathcal{L}$**.** Let (S, T, d, V) be a quadruple where (S, T, d) is a generalized BD-triple and $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. For $\mathfrak{l}_{S,T,d,V}$ given in (2.4), we will now study the closure of the $(G \times G)$-orbit through $\mathfrak{l}_{S,T,d,V}$ in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. To this end, let $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$ be the Grassmannian of m-dimensional subspaces in $\mathfrak{g}_S \oplus \mathfrak{g}_T$, where $m = \dim \mathfrak{g}_S$. For the Lie algebra isomorphism $\gamma_d : \mathfrak{g}_S \to \mathfrak{g}_T$ given in (2.2), let

$$\mathfrak{l}_{\gamma_d} = \{(x,\, \gamma_d(x)) : x \in \mathfrak{g}_S\} \in \mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T).$$

Notation 2.24. Let $Z_d(G_S)$ be the closure of $(G_S \times G_T) \cdot \mathfrak{l}_{\gamma_d}$ in $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$. Under the identification

$$G_S \longrightarrow (G_S \times G_T) \cdot \mathfrak{l}_{\gamma_d} : \ g \longmapsto \{(x, \gamma_d \mathrm{Ad}_g x) : x \in \mathfrak{g}_S\}, \tag{2.11}$$

$Z_d(G_S)$ can be regarded as a $(P_S \times P_T^-)$-equivariant compactification of G_S, where $P_S \times P_T^-$ act on G_S by

$$(p_S, p_T^-) \cdot g_S = \gamma_d^{-1}(\chi_T(p_T^-))\, g_S\, (\chi_S(p_S))^{-1}, \quad (p_S, p_T^-) \in P_S \times P_T^-,\ g_S \in G_S \tag{2.12}$$

(see Notation 2.11), and on $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$ via the group homomorphism $\chi_S \times \chi_T : P_S \times P_T^- \to G_S \times G_T$.

Proposition 2.25. *For every generalized BD-triple* (S, T, d) *and every* $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$*,*

1) the closure $\overline{(G \times G) \cdot \mathfrak{l}_{S,T,d,V}}$ *in* $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ *is a smooth subvariety of* $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ *of dimension* $n - z$*, where* $n = \dim \mathfrak{g}$ *and* $z = \dim \mathfrak{z}_S$*, and the map*

$$\mathbf{a} :\ (G\times G) \times_{(P_S\times P_T^-)} Z_d(G_S) \longrightarrow \overline{(G \times G) \cdot \mathfrak{l}_{S,T,d,V}} :\ [(g_1, g_2), \mathfrak{l}] \longmapsto \mathrm{Ad}_{(g_1,g_2)}(V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \mathfrak{l})$$

is a $(G \times G)$*-equivariant isomorphism;*

2) $\overline{(G\times G)\cdot \mathfrak{l}_{S,T,d,V}}$ *is the finite disjoint union*

$$\overline{(G\times G)\cdot \mathfrak{l}_{S,T,d,V}} = \bigcup_{S_1\subset S} (G\times G)\cdot \mathfrak{l}_{S_1,d(S_1),d_1,V_1},$$

where for $S_1\subset S$*, we set* $d_1 = d|_{S_1}$*, and* $V_1 = V + \{(x,\gamma_d(x)) : x\in \mathfrak{h}_S\cap \mathfrak{z}_{S_1}\}\subset \mathfrak{z}_{S_1}\oplus\mathfrak{z}_{T_1}$.

Proof. Since $G/P_S\times G/P_T^-$ is complete, the image of $\mathbf{a}$ is closed. Since $\mathbf{a}[(e,e),\mathfrak{l}_{\gamma_d}] = \mathfrak{l}_{S,T,d,V}$ and $\mathbf{a}$ is $(G\times G)$-equivariant, $\mathbf{a}$ has dense image. Hence $\mathbf{a}$ is onto. 2) follows easily from the fact that $\mathbf{a}$ is onto and the description of orbits in $Z_d(G_S)$ in Theorem 2.23. Further, by this description of orbits, for $\mathfrak{l}\in Z_d(G_S)$, the nilradical of $V + (\mathfrak{n}_S\oplus\mathfrak{n}_T^-)+\mathfrak{l}$ contains $\mathfrak{n}_S\oplus\mathfrak{n}_T^-$. It follows that if $\mathbf{a}[(g_1,g_2),\mathfrak{l}]\in \mathbf{a}[(e,e),Z_d(G_S)]$, then (g_1,g_2) is in the normalizer of the nilradical of $V+(\mathfrak{n}_S\oplus\mathfrak{n}_T^-)+\mathfrak{l}$. This normalizer, and hence also (g_1,g_2), is contained in $P_S\times P_T^-$. Now suppose $\mathbf{a}[(g_1,g_2),\mathfrak{l}] = \mathbf{a}[(x_1,x_2),\mathfrak{l}_1]$ for $x_1,x_2\in G$ and $\mathfrak{l}_1\in Z_d(G_S)$. Then $\mathbf{a}[(x_1^{-1}g_1,x_2^{-1}g_2),\mathfrak{l}] = \mathbf{a}[(e,e),\mathfrak{l}_1]$, so by the above $(x_1^{-1}g_1,x_2^{-1}g_2)\in P_S\times P_T^-$. It follows easily that $\mathbf{a}$ is injective.

To show that the differential $\mathbf{a}_*$ of $\mathbf{a}$ is injective everywhere, it suffices to show $\mathbf{a}_*$ is injective at all points $[(e,e),\mathfrak{l}]$ by $(G\times G)$-equivariance. For $X\in\mathfrak{g}\oplus\mathfrak{g}$, let ξ_X be the vector field on $(G\times G)\times_{(P_S\times P_T^-)} Z_d(G_S)$ tangent to the $(G\times G)$-action. Recall that the tangent space at a plane U to the Grassmannian $\mathrm{Gr}(n,Y)$ may be identified with $\mathrm{Hom}(U,Y/U)$. Since $Z_d(G_S)\subset \mathrm{Gr}(\dim(\mathfrak{g}_S),\mathfrak{g}_S+\mathfrak{g}_T)$, it follows that the tangent space to $(G\times G)\times_{(P_S\times P_T^-)} Z_d(G_S)$ at $[(e,e),\mathfrak{l}]$ is spanned by the set of ξ_X for $X\in\mathfrak{n}_S^-\oplus\mathfrak{n}_T$ and the set of ϕ in $\mathrm{Hom}(\mathfrak{l},(\mathfrak{g}_S\oplus\mathfrak{g}_T)/\mathfrak{l})$ tangent to $Z_d(G_S)$. We identify the tangent space to $\mathrm{Gr}(n,\mathfrak{g}\oplus\mathfrak{g})$ at $\mathbf{a}[(e,e),\mathfrak{l}]$ with $\mathrm{Hom}(V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-+\mathfrak{l},(\mathfrak{g}\oplus\mathfrak{g})/(V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-+\mathfrak{l}))$. Then $\mathbf{a}_*(\xi_X)=\eta_X$, where for $Y\in V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-+\mathfrak{l}$, $\eta_X(Y) = \mathrm{ad}_X(Y) \mod (V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-+\mathfrak{l})$. Moreover, $\mathbf{a}_*(\phi)=\tilde{\phi}$, where $\tilde{\phi}(x)=\phi(x)$ for $x\in\mathfrak{l}$, and $\tilde{\phi}(y)=0$ if $y\in V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-$. We further identify $(\mathfrak{g}\oplus\mathfrak{g})/(V+\mathfrak{n}_S\oplus\mathfrak{n}_T^-+\mathfrak{l})$ with $(\mathfrak{z}_S\oplus\mathfrak{z}_T)/V+\mathfrak{n}_S^-\oplus\mathfrak{n}_T+(\mathfrak{g}_S\oplus\mathfrak{g}_T)/\mathfrak{l}$. Now suppose that $\mathbf{a}_*(\xi_X+\phi)=0$. Since $\mathrm{ad}_X(\mathfrak{l})\subset\mathfrak{n}_S^-\oplus\mathfrak{n}_T$ and $\tilde{\phi}(\mathfrak{l})\subset(\mathfrak{g}_S\oplus\mathfrak{g}_T)/\mathfrak{l}$, it follows that $\tilde{\phi}=0$. Hence, $\phi=0$, and $\xi_X=0$ since $\mathbf{a}_*$ is injective on $\mathfrak{n}_S^-\oplus\mathfrak{n}_T$. By Corollary 14.10 of [Harr], $\mathbf{a}$ is an isomorphism.

Q.E.D.

Consider now the case when S and T are the empty set $\emptyset$, so $d=1$. By Theorem 2.16, every $(G\times G)$-orbit in $\mathcal{L}^0(\emptyset,\emptyset,1)\cup\mathcal{L}^1(\emptyset,\emptyset,1)$ goes through a unique Lagrangian subalgebra of the form

$$\mathfrak{l}_V = V + (\mathfrak{n}\oplus\mathfrak{n}^-), \tag{2.13}$$

where $V\in\mathcal{L}_{\mathrm{space}}(\mathfrak{h}\oplus\mathfrak{h})$. The following fact follows immediately from Proposition 2.25.

Corollary 2.26. *For every* $V\in\mathcal{L}_{\mathrm{space}}(\mathfrak{h}\oplus\mathfrak{h})$*, the* $(G\times G)$*-orbit through* $\mathfrak{l}_V$ *is isomorphic to* $G/B\times G/B^-$*. These are the only closed* $(G\times G)$*-orbits in* $\mathcal{L}$*.*

Corollary 2.27. *$\mathcal{L}$ has two connected components.*

Proof. In Section 2.1, we observed that $\mathcal{L}$ has at least two connected components, namely $\mathcal{L}^0$ and $\mathcal{L}^1$. Since every orbit of an algebraic group on a variety has a closed orbit in its boundary (see Section 8.3 in [Hu]), every point in $\mathcal{L}$ is in the same connected component as $\mathfrak{l}_V$ for some $V\in\mathcal{L}_{\mathrm{space}}(\mathfrak{h}\oplus\mathfrak{h})$. Thus $\mathcal{L}$ has at most two connected components.

Q.E.D.

2.9. **The geometry of the strata $\mathcal{L}^\epsilon(S,T,d)$.** Fix a generalized BD-triple (S,T,d) and an $\epsilon \in \{0,1\}$. Let $P_S \times P_T^-$ act on G_S by (2.12) and trivially on $\mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. Consider the associated bundle $(G \times G) \times_{(P_S \times P_T^-)} (G_S \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T))$ over $G/P_S \times G/P_T^-$ and the map

$$\mathbf{a}: \quad (G \times G) \times_{(P_S \times P_T^-)} (G_S \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)) \longrightarrow \mathcal{L}^\epsilon(S,T,d) \tag{2.14}$$

$$[(g_1,g_2),(g,V)] \longmapsto \mathrm{Ad}_{(g_1,g_2)}\mathfrak{l}_{g,V}, \tag{2.15}$$

where $\mathfrak{l}_{g,V} = V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d \mathrm{Ad}_g(x)) : x \in \mathfrak{g}_S\}$ for $g \in G_S$.

Proposition 2.28. *For every $S,T \subset \Gamma, d \in I(S,T)$, and $\epsilon \in \{0,1\}$, $\mathcal{L}^\epsilon(S,T,d)$ is a smooth connected subvariety of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ of dimension $n + \frac{z(z-3)}{2}$, where $n = \dim \mathfrak{g}$ and $z = \dim \mathfrak{z}_S$, and the map $\mathbf{a}$ in (2.14) is a $(G \times G)$-equivariant isomorphism.*

Proof. Consider the $(G \times G)$-equivariant projection

$$J: \ \mathcal{L}^\epsilon(S,T,d) \longrightarrow G/P_S \times G/P_T^- : \ \mathfrak{l}(\mathfrak{p},\mathfrak{p}',\mu,V) \longmapsto (\mathfrak{p},\mathfrak{p}'). \tag{2.16}$$

Let $\mathcal{F}^\epsilon(S,T,d)$ be the fibre of J over the point $(\mathfrak{p}_S, \mathfrak{p}_T^-) \in G/P_S \times G/P_T^-$. By Lemma 4, p. 26 of [Sl], the map

$$(G \times G) \times_{(P_S \times P_T^-)} \mathcal{F}^\epsilon(S,T,d) \longrightarrow \mathcal{L}^\epsilon(S,T,d): \ [(g_1,g_2),\mathfrak{l}] \longmapsto \mathrm{Ad}_{(g_1,g_2)}\mathfrak{l}$$

is a $(G \times G)$-equivariant isomorphism. By Lemma 2.6,

$$\mathcal{F}^\epsilon(S,T,d) = \{\mathfrak{l}_{g,V}: \ g \in G_S, \ V \in \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)\}.$$

The identification $G_S \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T) \to \mathcal{F}^\epsilon(S,T,d) : (g,V) \mapsto \mathfrak{l}_{g,V}$ is $(P_S \times P_T^-)$-equivariant. It follows that $\mathbf{a}$ is a $(G \times G)$-equivariant isomorphism. The dimension claim follows from Propositions 2.2 and 2.17. Smoothness and connectedness of $\mathcal{L}^\epsilon(S,T,d)$ follow easily.

Q.E.D.

2.10. **The geometry of the closure of $\overline{\mathcal{L}^\epsilon(S,T,d)}$.** For a generalized BD-triple (S,T,d) and an $\epsilon \in \{0,1\}$, recall that $Z_d(G_S)$ is an $(P_S \times P_T^-)$-equivariant compactification of G_S. Let again $P_S \times P_T^-$ act trivially on $\mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. We omit the proof of the following Theorem since it is similar to the proof of Theorem 2.25.

Theorem 2.29. *For every generalized BD-triple (S,T,d) and $\epsilon \in \{0,1\}$, the closure $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is a smooth algebraic variety of dimension $n + \frac{z(z-3)}{2}$, where $n = \dim(\mathfrak{g})$, $z = \dim \mathfrak{z}_S$, and*

$$\mathbf{a}: \quad (G \times G) \times_{(P_S \times P_T^-)} (Z_d(G_S) \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)) \longrightarrow \overline{\mathcal{L}^\epsilon(S,T,d)} \tag{2.17}$$

$$[(g_1,g_2),(\mathfrak{l},V)] \longmapsto \mathrm{Ad}_{(g_1,g_2)}(V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \mathfrak{l}) \tag{2.18}$$

is a $(G \times G)$-equivariant isomorphism.

Corollary 2.30. *For every generalized BD-triple (S,T,d) and $\epsilon \in \{0,1\}$,*

$$\overline{\mathcal{L}^\epsilon(S,T,d)} = \bigcup_{V \in \mathcal{L}^\epsilon_{space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)} \ \bigcup_{S_1 \subset S} (G \times G) \cdot \mathfrak{l}_{S_1, d(S_1), d, V_1(V,S_1)} \tag{2.19}$$

is a disjoint union, where for $S_1 \subset S$ and $V \in \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$,

$$V_1(V,S_1) = V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S \cap \mathfrak{z}_{S_1}\} \subset \mathfrak{z}_{S_1} \oplus \mathfrak{z}_{d(S_1)}.$$

Remark 2.31. 1). Since $Z_d(G_S)$ is also the closure in the classical topology of $(G_S \times G_T) \cdot \mathfrak{l}_{\gamma_d}$ inside $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$, $\mathcal{L}^\epsilon(S,T,d)$ also has the same closure in the two topologies of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$.

2). Since $Z_d(G_S) - G_S$ has dimension strictly lower than $m = \dim G_S$, it follows from the proof of Theorem 2.29 that $\overline{\mathcal{L}^\epsilon(S,T,d)} - \mathcal{L}^\epsilon(S,T,d)$ is of strictly lower dimension than $\overline{\mathcal{L}^\epsilon(S,T,d)}$.

2.11. **Irreducible components of $\mathcal{L}$.** Since each $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is smooth and connected, it is a closed irreducible subvariety of $\mathcal{L}$. Since

$$\mathcal{L} = \bigcup_{\epsilon \in \{0,1\}} \bigcup_{S,T \subset \Gamma, d \in I(S,T)} \overline{\mathcal{L}^\epsilon(S,T,d)}$$

is a finite union, the irreducible components of $\mathcal{L}$ are those $\overline{\mathcal{L}^\epsilon(S,T,d)}$ that are not properly contained in some other such set.

Theorem 2.32. *$\overline{\mathcal{L}^\epsilon(S,T,d)}$ is an irreducible component of $\mathcal{L}$ unless $|\Gamma - S| = 1$, $T = d_1(S)$ for some $d_1 \in I(\Gamma,\Gamma)$, $d = d_1|_S$, and $\epsilon = (\dim \mathfrak{h} - \dim \mathfrak{h}^{\gamma_{d_1}}) \mod 2$.*

Proof. When (S,T,d,ϵ) are as described in the proposition, $\dim \mathfrak{z}_S = 1$, so $\mathcal{L}^\epsilon(S,T,d)$ consists of a single $(G \times G)$-orbit which lies in $Z_{d_1}(G)$ by Theorem 2.23. We need to show that this is the only nontrivial case when the closure $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is contained in another $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)}$.

Assume that $\mathcal{L}^\epsilon(S,T,d)$ is in the boundary of $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)}$. Then by Corollary 2.30, $S \subset S_1$ and $T \subset T_1$. By Remark 2.31, $\dim \mathcal{L}^\epsilon(S,T,d) < \dim \mathcal{L}^\epsilon(S_1,T_1,d_1)$, and thus

$$\frac{1}{2}\dim(\mathfrak{z}_S)(\dim(\mathfrak{z}_S) - 3) < \frac{1}{2}\dim(\mathfrak{z}_{S_1})(\dim(\mathfrak{z}_{S_1}) - 3)$$

by the dimension formula in Proposition 2.28. Since $S \subset S_1$, so $\dim(\mathfrak{z}_S) \geq \dim(\mathfrak{z}_{S_1})$, these two inequalities imply that $\dim(\mathfrak{z}_{S_1}) = 0$ and $\dim(\mathfrak{z}_S) = 1$ or 2. In particular, $S_1 = T_1 = \Gamma$, so $\epsilon = (\dim \mathfrak{h} - \dim \mathfrak{h}^{\gamma_{d_1}}) \mod 2$, and $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)} = Z_{d_1}(G)$.

If $\dim(\mathfrak{z}_S) = 2$, $\mathcal{L}^\epsilon(S,T,d)$ contains infinitely many $(G \times G)$-orbits by Theorem 2.16 and Proposition 2.2. Since $Z_{d_1}(G)$ has only finitely many $(G \times G)$-orbits, $\mathcal{L}^\epsilon(S,T,d)$ can not be contained in $Z_{d_1}(G)$. Assume that $\dim(\mathfrak{z}_S) = 1$. Then by Proposition 2.28, $\mathcal{L}^\epsilon(S,T,d)$ is a single $(G \times G)$-orbit. By the description of the $(G \times G)$-orbits in $Z_{d_1}(G)$ in Theorem 2.23, T and d must be as described in the proposition.

Q.E.D.

Example 2.33. For $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $\mathcal{L}$ has two irreducible components. One is the De Concini-Procesi compactification $Z_{\mathrm{id}}(G)$ of $G = PSL(2,C)$ which is isomorphic to $\mathbb{C}P^3$ (see [D-P]), and the other is the closed $(G \times G)$-orbit through $\mathfrak{h}_\Delta + (\mathfrak{n} \oplus \mathfrak{n}^-)$, and is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

For $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, there are four irreducible components $Z_{\mathrm{id}}(G), Z_{d_1}(G), C_1$ and C_2, where $Z_{\mathrm{id}}(G)$ and $Z_{d_1}(G)$ are the two De Concini-Procesi compactifications of $G = PSL(3,\mathbb{C})$ corresponding to the identity and the non-trivial automorphism of the Dynkin diagram of $\mathfrak{sl}(3,\mathbb{C})$, and C_1 and C_2 are the two components $\mathcal{L}_0(\emptyset,\emptyset,d)$ and $\mathcal{L}_1(\emptyset,\emptyset,d)$. Both C_1 and C_2 have dimension 7. Moreover, $Z_{\mathrm{id}}(G) \cap C_1$ is a 6-dimensional closed $(G \times G)$-orbit, and so is $Z_{d_1}(G) \cap C_2$.

3. Classification of G_Δ-orbits in $\mathcal{L}$

By Theorem 2.16, to describe the G_Δ-orbits in $\mathcal{L}$, it suffices to describe $(G_\Delta, R_{S,T,d})$-double cosets in $G \times G$ for all generalized BD-triples (S,T,d) for Γ, where $R_{S,T,d}$ is given by (2.6). A general double coset theorem in [Lu-Y1] classifies $(R_{S',T',d'}, R_{S,T,d})$-double cosets in $G \times G$ for two arbitrary generalized BD-triples (S', T', d') and (S, T, d). However, the proof in [Lu-Y1] is rather technically involved. In this section, we present a simplified proof for the special case when $R_{S',T',d'} = G_\Delta$. Our method and the one used in [Lu-Y1] are both adapted from [Y]. The G_Δ-orbits in the wonderful compactifications were also studied by Lusztig in [Lusz1] and [Lusz2] using a somewhat different method.

3.1. **Some results on Weyl groups and generalized BD-triples.** The results in this section, while different in presentation, are closely related to some combinatorial results in [Be], which were used and extended in [Lusz1] and [Lusz2]. In particular, the limit of a sequence studied in [Be] is closely related to the set $S(v, d)$ in Proposition 3.5.

Notation 3.1. Let W be the Weyl group of Γ. For $F \subset \Gamma$, let W_F be the subgroup of W generated by elements in F. If $E, F \subset \Gamma$, let ${}^EW^F$ be the set of minimal length representatives for double cosets from $W_E \backslash W / W_F$, and set $W^F = {}^{\emptyset}W^F$. If $E_1, E_2 \subset F$, the set of minimal length representatives in W_F for the double cosets from $W_{E_1} \backslash W_F / W_{E_2}$ will be denoted by ${}^{E_1}(W_F)^{E_2}$. If $u \in {}^{E_1}(W_F)^{E_2}$ and $v \in {}^{E_1'}(W_{F'})^{E_2'}$, we can regard both $u \in W_F$ and $v \in W_{F'}$ as elements in W, and by uv we will mean their product in W.

Definition 3.2. Let (S, T, d) be a generalized BD-triple in Γ. For $v \in W^T$, regarding vd as a map $S \to \Delta$, we define $S(v, d) \subset S$ to be the largest subset in S that is invariant under vd. In other words,

$$S(v, d) = \{\alpha \in S : (vd)^n \alpha \in S, \forall \text{ integer } n \geq 1\}. \tag{3.1}$$

Parts 1) and 2) in the following Lemma 3.3 follow directly from Proposition 2.7.5 of [C] or Lemma 4.3 of [Y], and Part 3) is a special case of Lemma 5.3 in [Lu-Y1].

Lemma 3.3. *1) If $w \in {}^SW^T$ and $u \in (W_S)^{S \cap w(T)}$, then $uw \in W^T$;*

2) Every $v \in W^T$ has a unique decomposition $v = uw$, where $w \in {}^SW^T$ and $u \in (W_S)^{S \cap w(T)}$. Moreover, $l(v) = l(u) + l(w)$;

3) For $w \in {}^SW^T$, set $T_w = S \cap w(T)$ and $S_w = d^{-1}(T \cap w^{-1}(S))$ and regard (S_w, T_w, wd) as a generalized BD-triple in S. Then for any $u \in (W_S)^{T_w}$, one has $S_w(u, wd) = S(uw, d)$, where $S_w(u, wd)$ is the largest subset of S_w that is invariant under uwd.

Notation 3.4. Fix a generalized BD-triple (S, T, d). Let $\mathcal{Q}_{S,T,d}$ denote the set of all sequences $\mathbf{q} = \{\mathbf{q}_i\}_{i \geq 0}$ of quadruples

$$\mathbf{q}_i = (S_i, T_i, d_i, w_i), \quad i \geq 0,$$

where, for each $i \geq 0$,

1) (S_i, T_i, d_i) is a generalized BD-triple and $(S_0, T_0, d_0) = (S, T, d)$;

2) $w_i \in {}^{S_i}(W_{S_{i-1}})^{T_i}$ (we set $S_{-1} = \Gamma$);

3) the triple $(S_{i+1}, T_{i+1}, d_{i+1})$ is obtained from $\mathbf{q}_i$ as follows:

$$T_{i+1} = S_i \cap w_i(T_i), \quad d_{i+1} = w_i d_i, \quad S_{i+1} = d_{i+1}^{-1}(T_{i+1}).$$

Note that $S_{i+1}, T_{i+1} \subset S_i$ for all i. For $\mathbf{q} = \{\mathbf{q}_i\}_{i \geq 0} \in \mathcal{Q}_{S,T,d}$, let i_0 be the smallest integer such that $S_{i_0+1} = S_{i_0}$. Then it is easy to see that

$$\mathbf{q}_i = \mathbf{q}_{i_0+1} = (S_{i_0}, w_{i_0}(T_{i_0}), w_{i_0} d_{i_0}, 1)$$

for all $i \geq i_0 + 1$, where 1 is the identity element of W. Set

$$v_\infty(\mathbf{q}) = w_{i_0} w_{i_0-1} \cdots w_0 \quad \text{and} \quad S_\infty(\mathbf{q}) = S_{i_0}.$$

Proposition 3.5 is a direct consequence of Lemma 3.3. See also Proposition 2.5 of [Lusz1].

Proposition 3.5. *Let (S, T, d) be a generalized BD-triple for Γ. Then for any $\mathbf{q} \in \mathcal{Q}_{S,T,d}$, $v := v_\infty(\mathbf{q}) \in W^T$, and $S_\infty(\mathbf{q}) = S(v, vd)$. Moreover, the map $\mathcal{Q}_{S,T,d} \to W^T : \mathbf{q} \mapsto v_\infty(\mathbf{q})$ is bijective.*

3.2. **A double coset theorem.** For this section, G will be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$, not necessarily of adjoint type. We use the same notation as in Notation 2.11 and Notation 3.1 for various subalgebras of $\mathfrak{g}$ and subgroups of G. We will define a class of subgroups R of $G \times G$ that are slightly more general than the groups $R_{S,T,d}$, and we will prove a theorem on (G_Δ, R)-double cosets in $G \times G$ for such an R.

Definition 3.6. Let (S, T, d) be a generalized BD-triple in Γ. Let C_S (resp. C_T) be a subgroup of the center Z_S (resp. Z_T) of M_S (resp. M_T), and let $\theta_d : M_S/C_S \to M_T/C_T$ be a group isomorphism that maps the one-dimensional unipotent subgroup of M_S/C_S defined by α to the corresponding subgroup of M_T/C_T defined by $d\alpha$ for each $\alpha \in [S]$. By a (S, T, d)-admissible subgroup of $G \times G$ we mean a subgroup $R = R(C_S, C_T, \theta_d)$ of $P_S \times P_T^-$ of the form

$$R(C_S, C_T, \theta_d) = \{(m, m') \in M_S \times M_T : \theta_d(mC_S) = m'C_T\}(N_S \times N_T^-). \tag{3.2}$$

Clearly $R(Z_S, Z_T, \gamma_d) = R_{S,T,d}$. Let R be any (S, T, d)-admissible subgroup of $G \times G$. Recall that the subset $S(v, d)$ of S for $v \in W^T$ is defined in (3.1). If $\dot{v}$ is a representative of v in G, set

$$R_{\dot{v}} = (M_{S(v,d)} \times M_{S(v,d)}) \cap ((\mathrm{id} \times \mathrm{Ad}_{\dot{v}})R),$$

where $\mathrm{Ad}_{\dot{v}} : G \to G : g \mapsto \dot{v} g \dot{v}^{-1}$. Let $R_{\dot{v}}$ act on $M_{S(v,d)}$ (from the right) by

$$m \cdot (m_1, m_1') = (m_1')^{-1} m m_1, \quad m \in M_{S(v,d)}, \ (m_1, m_1') \in R_{\dot{v}}. \tag{3.3}$$

For $(g_1, g_2) \in G \times G$, let $[g_1, g_2]$ be the double coset $G_\Delta(g_1, g_2)R$ in $G \times G$.

Theorem 3.7. *Let (S, T, d) be a generalized BD-triple, and let $R = R(C_S, C_T, \theta_d)$ be an (S, T, d)-admissible subgroup of $G \times G$ as given in (3.2). For $v \in W^T$, let $S(v, d) \subset S$ be given in (3.1), and let $\dot{v}$ be a fixed representative of v in G. Then*

1) every (G_Δ, R)-double coset in $G \times G$ is of the form $[m, \dot{v}]$ for some $v \in W^T$ and $m \in M_{S(v,d)}$;

2) Two double cosets $[m_1, \dot{v}_1]$ and $[m_2, \dot{v}_2]$ in 1) coincide if and only if $v_1 = v_2 = v$ and m_1 and m_2 are in the same $R_{\dot{v}}$-orbit in $M_{S(v,d)}$ for the $R_{\dot{v}}$ action on $M_{S(v,d)}$ given in (3.3).

We present the main induction step in the proof of Theorem 3.7 in a lemma. Recall that each $w \in {}^S W^T$ gives rise to the generalized BD-triple (S_w, T_w, wd) in S as in Lemma 3.3. Set

$$N_{S_w}^S = N_{S_w} \cap M_S, \quad \text{and} \quad N_{T_w}^{S,-} = N_{T_w}^- \cap M_S.$$

Fix a representative $\dot{w}$ in G, and define

$$R_{\dot{w}}^S = ((M_{S_w} \times M_{T_w}) \cap ((\mathrm{id} \times \mathrm{Ad}_{\dot{w}})R)) \left(N_{S_w}^S \times N_{T_w}^{S,-}\right). \tag{3.4}$$

Then $R_{\dot{w}}^S$ is an (S_w, T_w, wd)-admissible subgroup of $M_S \times M_S$ defined by the subgroup C_S of Z_{S_w}, the subgroup $w(C_T)$ of Z_{T_w} and the group isomorphism $\mathrm{Ad}_{\dot{w}} \theta_d : M_{S_w}/C_S \to M_{T_w}/w(C_T)$.

Lemma 3.8. *1) Every (G_Δ, R)-double coset in $(G\times G)$ is of the form $[m, m'\dot{w}]$ for a unique $w\in {}^SW^T$ and some $m\in M_S$.*

2) $[m_1, m_1'\dot{w}] = [m_2, m_2'\dot{w}]$, where $w\in {}^SW^T$ and $(m_1, m_1'), (m_2, m_2')\in M_S\times M_S$, if and only if (m_1, m_1') and (m_2, m_2') are in the same $((M_S)_\Delta, R^S_{\dot{w}})$-double coset in $M_S\times M_S$.

Proof. Consider the right action of $P_S\times P_T^-$ on $G_\Delta\backslash(G\times G)$ by right translations. By the Bruhat decomposition $G=\bigcup_{w\in {}^SW^T} P_S w P_T^-$, the set of $(P_S\times P_T^-)$-orbits is parameterized by the set $\{G_\Delta(e,\dot{w}) : w\in {}^SW^T\}$. Let $w\in {}^SW^T$. The stabilizer subgroup of $P_S\times P_T^-$ at $G_\Delta(e,\dot{w})$ is $P_S\cap(\dot{w}P_T^-\dot{w}^{-1})$ considered as a subgroup of $P_S\times P_T^-$ via the embedding

$$P_S\cap(\dot{w}P_T^-\dot{w}^{-1})\longrightarrow P_S\times P_T^- :\ p_S\longmapsto (p_S,\ \dot{w}^{-1}p_S\dot{w}). \tag{3.5}$$

Thus the set of R-orbits in $G_\Delta\backslash(G\times G)$ can be identified with the disjoint union over $w\in {}^SW^T$ of the spaces of R-orbits in $P_S\cap(\dot{w}P_T^-\dot{w}^{-1})\backslash(P_S\times P_T^-)$. Thus, for every $w\in {}^SW^T$, we have an injective map

$$(P_S\cap \dot{w}P_T^-\dot{w}^{-1})\backslash P_S\times P_T^-/R\longrightarrow G_\Delta\backslash G\times G/R \tag{3.6}$$

given by $(P_S\cap \dot{w}P_T^-\dot{w}^{-1})(p_S, p_T^-)R\to [p_S, \dot{w}p_T^-]$. We will complete the proof by identifying

$$(P_S\cap(\dot{w}P_T^-\dot{w}^{-1})\backslash P_S\times P_T^-/R\ \cong\ (M_S)_\Delta\backslash M_S\times M_S/R^S_{\dot{w}} \tag{3.7}$$

through a series of steps. Let $\pi_S : P_S\to M_S$ and $\pi_T : P_T^-\to M_T$ be the projections with respect to the decompositions $P_S = M_S N_S$ and $P_T^- = M_T N_T^-$. Then $\pi_S\times\pi_T : P_S\times P_T^-\to M_S\times M_T$ gives an identification

$$P_S\cap(\dot{w}P_T^-\dot{w}^{-1})\backslash P_S\times P_T^-/R\longrightarrow R_1\backslash M_S\times M_T/R_2, \tag{3.8}$$

where $R_1 = (\pi_S\times\pi_T)(P_S\cap(\dot{w}P_T^-\dot{w}^{-1}))$ and $R_2 = (M_S\times M_T)\cap R$. Since the projection from $(M_S\times M_T)\cap R$ to M_T is onto with kernel $(C_S\times\{e\})$, the map

$$\phi_w :\ (M_S\times M_T)/R_2\longrightarrow (M_S\times M_S)/(M_S)_\Delta(C_S\times\{e\})$$

that maps $(m_S, m_T)R_2$ to $(m_S', m_S)((M_S)_\Delta(C_S\times\{e\}))$ is a well-defined bijection, where for $m_T\in M_T$, m_S' is any element in M_S such that $(m_S', m_T)\in R_2$. Thus ϕ_w induces an identification

$$\psi_w :\ R_1\backslash M_S\times M_T/R_2\longrightarrow R_3\backslash M_S\times M_S/((M_S)_\Delta(C_S\times\{e\})), \tag{3.9}$$

where

$$R_3 \stackrel{\text{def}}{=} \{(m_S', m_S)\in M_S\times M_S :\ \exists m_T\in M_T \,\text{such that}\, (m_S, m_T)\in R_1, (m_S', m_T)\in R_2\}.$$

By Theorem 2.8.7 of [C],

$$P_S\cap(\dot{w}P_T^-\dot{w}^{-1}) = (M_S\cap \mathrm{Ad}_{\dot{w}}(M_T))(M_S\cap\mathrm{Ad}_{\dot{w}}(N_T^-))(N_S\cap\mathrm{Ad}_{\dot{w}}(M_T))(N_S\cap\mathrm{Ad}_{\dot{w}}(N_T^-)). \tag{3.10}$$

Note that $M_S\cap\mathrm{Ad}_{\dot{w}}(M_T) = M_{S\cap w(T)}$, $M_S\cap\mathrm{Ad}_{\dot{w}}(N_T^-) = N^{S,-}_{S\cap w(T)} = M_S\cap N^-_{S\cap w(T)}$, and $N_S\cap\mathrm{Ad}_{\dot{w}}(M_T) = N^T_{T\cap w^{-1}(S)} = M_T\cap N_{T\cap w^{-1}(S)}$. Thus

$$R_1 = \left\{(m, \mathrm{Ad}_{\dot{w}^{-1}}(m)) : m\in M_{S\cap w(T)}\right\}\left(N^{S,-}_{S\cap w(T)}\times N^T_{T\cap w^{-1}(S)}\right).$$

Therefore $(m_S', m_S)\in R_3$ if and only if there exist $n\in N^{S,-}_{S\cap w(T)}$, $n_1\in N^T_{T\cap w^{-1}(S)}$, and $m\in M_{S\cap w(T)}$ such that $m_S = mn$ and $(m_S', \mathrm{Ad}_{\dot{w}^{-1}}(m)n_1)\in R_2$. It follows from the definition of R that $(m_S', m_S)\in R_3$ if and only if there exist $m'\in M_{S_w}, m\in M_{T_w}$, $n\in N^{S,-}_{T_w}$, and

$n' \in N^S_{S_w} = M_S \cap N_{S_w}$ such that $m_S = mn, m'_S = m'n'$ and $(m', \mathrm{Ad}_{\dot{w}}^{-1}(m)) \in R$. Thus $R_3 = R^S_{\dot{w}}$. Since $C_S \times \{e\} \subset R^S_{\dot{w}}$, the (right) action of $C_S \times \{e\}$ on $R^S_{\dot{w}} \backslash (M_S \times M_S)$ is trivial. Thus we have

$$\begin{aligned} R_3 \backslash M_S \times M_S/((M_S)_\Delta(C_S \times \{e\})) &\cong R^S_{\dot{w}} \backslash M_S \times M_S/((M_S)_\Delta(C_S \times \{e\})) \\ &\cong R^S_{\dot{w}} \backslash M_S \times M_S/(M_S)_\Delta \\ &\cong (M_S)_\Delta \backslash M_S \times M_S/R^S_{\dot{w}}, \end{aligned}$$

where the last identification is induced by the inverse map of $M_S \times M_S$.

Combining the above identification with the identifications in (3.8)-(3.9) and the inclusion of (3.6), we get a well-defined injective map $(M_S)_\Delta \backslash M_S \times M_S/R^S_{\dot{w}} \to G_\Delta \backslash G \times G/R$ given by

$$(M_S)_\Delta(m, m')R^S_{\dot{w}} \to [((m')^{-1}, \dot{w}\theta_d(m^{-1}))] = [(m')^{-1}m, \dot{w}] = [m, m'\dot{w}].$$

This finishes the proof of Lemma 3.8.

Q.E.D.

Proof of Theorem 3.7. By Lemma 3.8, each (G_Δ, R) double coset in $G \times G$ determines a unique $w \in {}^SW^T$ and a unique double coset $[m, m']_1 \in (M_S)_\Delta \backslash M_S \times M_S/R^S_{\dot{w}}$. Let $(S_0, T_0, d_0, w_0) = (S, T, d, w)$. By successively applying Lemma 3.8 to a sequence of smaller subgroups, we obtain a sequence $\mathbf{q}$ of quadruples $\mathbf{q}_i = (S_i, T_i, d_i, w_i)$ as in Notation 3.4, as well as a double coset in $(M_{S_i})_\Delta \backslash M_{S_i} \times M_{S_i}/R_i$, where R_i is the subgroup of $M_{S_i} \times M_{S_i}$ defined analogously to $R^S_{\dot{w}}$.

As in Notation 3.4, let i_0 be the smallest integer such that $S_{i_0+1} = S_{i_0}$ and let $v = v_\infty(\mathbf{q}) = w_{i_0}w_{i_0-1}\cdots w_0$. Then each (G_Δ, R)-double coset in $G \times G$ is of the form $[m, m'\dot{v}]$ for $m \in M_{S_{i_0+1}}$. By Proposition 3.5, $v \in W^T$, and $S_{i_0+1} = S(v, d)$. Moreover, $R_{i_0+1} = R_{\dot{v}}$ by definition. Thus double cosets in $(M_{S_{i_0+1}})_\Delta \backslash M_{S_{i_0+1}} \times M_{S_{i_0+1}}/R_{i_0+1}$ coincide with double cosets in $(M_{S(v,d)})_\Delta \backslash M_{S(v,d)} \times M_{S(v,d)}/R_{\dot{v}}$. It is easy to see that the map

$$(M_{S(v,d)})_\Delta \backslash M_{S(v,d)} \times M_{S(v,d)}/R_{\dot{v}} \longrightarrow M_{S(v,d)}/R_{\dot{v}} : \ [m, m'] \longmapsto [{m'}^{-1}m]$$

is a bijection. This proves Theorem 3.7.

Q.E.D.

3.3. G_Δ-orbits in $\mathcal{L}$.

Notation 3.9. For a generalized BD-triple (S, T, d), $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, $m \in M_{S(v,d)}$, $v \in W^T$, and $\dot{v} \in G$ a fixed representative of v in G, set

$$\mathfrak{l}_{S,T,d,V,\dot{v},m} = \mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}, \tag{3.11}$$

where $\mathfrak{l}_{S,T,d,V}$ is given in (2.4). Define

$$R_{\dot{v}} = \{(m_1, m_1') \in M_{S(v,d)} \times M_{S(v,d)} : \gamma_d(\chi_S(m_1)) = \chi_T(\dot{v}^{-1}m_1'\dot{v})\}, \tag{3.12}$$

and let $R_{\dot{v}}$ act on $M_{S(v,d)}$ (from the right) by

$$m \cdot (m_1, m_1') = (m_1')^{-1}mm_1, \quad m \in M_{S(v,d)}, (m_1, m_1') \in R_{\dot{v}}. \tag{3.13}$$

As an immediate corollary of Theorem 3.7, we have

Corollary 3.10. *Every G_Δ-orbit in $\mathcal{L}$ passes through an $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ for a unique generalized BD-triple (S,T,d), a unique $V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, a unique $v \in W^T$, and some $m \in M_{S(v,d)}$; Two such Lagrangian subalgebras $\mathfrak{l}_{S,T,d,V,v,m_1}$ and $\mathfrak{l}_{S,T,d,V,v,m_2}$ are in the same G_Δ-orbit if and only if m_1 and m_2 are in the same $R_{\dot{v}}$-orbit in $M_{S(v,d)}$.*

3.4. **Normalizer subalgebras of $\mathfrak{g}_\Delta$ at $\mathfrak{l} \in \mathcal{L}$.** For $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ as in Corollary 3.10, we now compute its normalizer subalgebra $\mathfrak{n}(\mathfrak{l})$ in $\mathfrak{g} \cong \mathfrak{g}_\Delta = \{(x,x) : x \in \mathfrak{g}\}$. Introduce

$$\phi := \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S\mathrm{Ad}_m^{-1} : \ \mathfrak{p}_S \longrightarrow \mathfrak{g}.$$

Consider the standard parabolic subalgebra $\mathfrak{p}_{S(v,d)}$ and its decomposition $\mathfrak{p}_{S(v,d)} = \mathfrak{z}_{S(v,d)} + \mathfrak{g}_{S(v,d)} + \mathfrak{n}_{S(v,d)}$ (see Notation 2.11).

Lemma 3.11. *The map $\phi = \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S\mathrm{Ad}_m^{-1}$ leaves each of $\mathfrak{z}_{S(v,d)}$, $\mathfrak{g}_{S(v,d)}$, and $\mathfrak{n}_{S(v,d)}$ invariant. Moreover, $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent.*

Proof. Let $x \in \mathfrak{z}_{S(v,d)}$. Then $\phi(x) = \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(x) \in \mathfrak{h}$. For $\alpha \in S(v,d)$, since $(vd)^{-1}\alpha \in S(v,d)$,

$$\alpha(\phi(x)) = ((vd)^{-1}\alpha)(\chi_S(x)) = ((vd)^{-1}\alpha)(x) = 0.$$

Thus $\phi(x) \in \mathfrak{z}_{S(v,d)}$, so $\mathfrak{z}_{S(v,d)}$ is ϕ-invariant. Since both $\mathrm{Ad}_{\dot{v}}\gamma_d$ and Ad_m^{-1} leave $\mathfrak{g}_{S(v,d)}$ invariant, we see that $\phi|_{\mathfrak{g}_{S(v,d)}} = \mathrm{Ad}_{\dot{v}}\gamma_d\mathrm{Ad}_m^{-1}$ leaves $\mathfrak{g}_{S(v,d)}$ invariant.

To show that $\mathfrak{n}_{S(v,d)}$ is ϕ-invariant and that $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent, set $\Sigma_0^+ = \Sigma^+ - [S]$, and for $j \geq 1$, set

$$\Sigma_j^+ = \{\alpha \in \Sigma^+ : \alpha \in [S], vd\alpha \in [S], \cdots, (vd)^{j-1}(\alpha) \in [S], (vd)^j(\alpha) \notin [S]\}. \tag{3.14}$$

Then $\Sigma^+ - [S(v,d)] = \cup_{j\geq 0}\Sigma_j^+$ and $vd(\Sigma_j^+) \subset \Sigma_{j-1}^+$ for $j \geq 1$. For $j \geq 0$, set $\mathfrak{n}_j = \oplus_{\alpha\in\Sigma_j^+}\mathfrak{g}_\alpha$. Then $\mathfrak{n}_0 = \mathfrak{n}_S$, and $\mathfrak{n}_{S(v,d)} = \sum_{j\geq 0}\mathfrak{n}_j$ is a finite direct sum. It is easy to prove by induction on j that

$$\alpha \in \Sigma_j^+, \beta \in [S(v,d)], \alpha+\beta \in \Sigma \implies \alpha+\beta \in \Sigma_j^+, \quad \forall j \geq 0. \tag{3.15}$$

It follows that $[\mathfrak{m}_{S(v,d)}, \mathfrak{n}_j] \subset \mathfrak{n}_j$ for each $j \geq 0$. Thus $\mathrm{Ad}_m\mathfrak{n}_j = \mathfrak{n}_j, \forall j \geq 0$. By setting $\mathfrak{n}_{-1} = 0$, we then have $\mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(\mathfrak{n}_j) \subset \mathfrak{n}_{j-1}, \forall j \geq 0$. Thus $\phi(\mathfrak{n}_j) \subset \mathfrak{n}_{j-1}$ for all $j \geq 0$, and $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent.

Q.E.D.

Remark 3.12. Similar arguments imply the same statements for $\mathfrak{n}^-_{S(v,d)}$. In particular, for $j \geq 0$, set $\mathfrak{n}_j^- = \oplus_{\alpha\in\Sigma_j^+}\mathfrak{g}_{-\alpha}$ and set $\mathfrak{n}_{-1}^- = 0$. Then $\phi(\mathfrak{n}_j^-) \subset \mathfrak{n}_{j-1}^-$ for $j \geq 0$, $\mathfrak{n}_S^- = \mathfrak{n}_0^-$, and $\mathfrak{n}^-_{S(v,d)} = \sum_{j\geq 0}\mathfrak{n}_j^-$ is a direct sum.

Since $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent, we can define

$$\psi := (1-\phi)^{-1} = 1 + \phi + \phi^2 + \phi^3 + \cdots : \ \mathfrak{n}_{S(v,d)} \longrightarrow \mathfrak{n}_{S(v,d)}.$$

Let $\Sigma_v^+ = \{\alpha \in \Sigma^+ : v^{-1}\alpha \in \Sigma^-\}$. Since $v([T] \cap \Sigma^+) \subset \Sigma^+$, we have $\Sigma_v^+ \subset \Sigma^+ - [S(v,d)]$. Let

$$\mathfrak{n}_v = \oplus_{\alpha\in\Sigma_v^+}\mathfrak{g}_\alpha = \mathfrak{n} \cap \mathrm{Ad}_{\dot{v}}(\mathfrak{n}^-).$$

Then $\mathfrak{n}_v \subset \mathfrak{n}_{S(v,d)}$.

Theorem 3.13. *The normalizer subalgebra* $\mathfrak{n}(\mathfrak{l})$ *in* $\mathfrak{g}_\Delta \cong \mathfrak{g}$ *of* $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ *in (3.11) is*

$$\mathfrak{n}(\mathfrak{l}) = \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v),$$

where $\mathfrak{g}^{\phi}_{S(v,d)}$ *is the fixed point set of* $\phi|_{\mathfrak{g}_{S(v,d)}} = \mathrm{Ad}_{\dot{v}}\gamma_d\mathrm{Ad}_m^{-1}$ *in* $\mathfrak{g}_{S(v,d)}$*, and*

$$\mathfrak{z}'_{S(v,d)} = \{z \in \mathfrak{z}_{S(v,d)} : z - \phi(z) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T\} = \{z \in \mathfrak{z}_{S(v,d)} : \gamma_d\chi_S(z) = \chi_T(\mathrm{Ad}_{\dot{v}}^{-1}z)\}.$$

Proof. The normalizer subgroup $R_{S,T,d}$ of $\mathfrak{l}_{S,T,d,V}$ in $G \times G$ has Lie algebra

$$\mathfrak{r}_{S,T,d} = (\mathfrak{z}_S \oplus \mathfrak{z}_T) + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\} = \{(x,y) \in \mathfrak{p}_S \oplus \mathfrak{p}_T^- : \gamma_d\chi_S(x) = \chi_T(y)\}.$$

Since $\mathfrak{l} = \mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}$, it follows that $\mathfrak{n}(\mathfrak{l}) = \{x \in \mathfrak{g} :\ (\mathrm{Ad}_m^{-1}x,\ \mathrm{Ad}_{\dot{v}}^{-1}x) \in \mathfrak{r}_{S,T,d}\}$. Thus $x \in \mathfrak{n}(\mathfrak{l})$ if and only if $x \in \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$ and $\gamma_d\chi_S(\mathrm{Ad}_m^{-1}(x)) = \chi_T(\mathrm{Ad}_{\dot{v}}^{-1}x)$, which is equivalent to

$$x - \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(\mathrm{Ad}_m^{-1}(x)) \in \mathrm{Ad}_{\dot{v}}(\mathfrak{z}_T + \mathfrak{n}_T^-). \tag{3.16}$$

Let χ_S also denote the projection $\mathfrak{g} \to \mathfrak{g}_S$ with respect to the decomposition $\mathfrak{g} = \mathfrak{n}_S^- + \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S$, so $\phi : x \mapsto \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(\mathrm{Ad}_m^{-1}x)$ is defined for all $x \in \mathfrak{g}$. Let $\mathfrak{c}$ be the set of all $x \in \mathfrak{g}$ satisfying (3.16). Then $\mathfrak{n}(\mathfrak{l}) = \mathfrak{c} \cap \left(\mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-\right)$.

Consider the decomposition $\mathfrak{g} = \mathfrak{n}^-_{S(v,d)} + \mathfrak{m}_{S(v,d)} + \mathfrak{n}_{S(v,d)}$. Since $\alpha \notin v([T])$ implies that $\alpha \notin [S(v,d)]$, we have

$$\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- = \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}^- + \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n} \ \subset\ \mathfrak{n}^-_{S(v,d)} + \mathfrak{n}_{S(v,d)}.$$

Moreover, it is easy to see that $\left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n} = \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}_{S(v,d)} = \mathfrak{n}_v$, so

$$\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- = \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}^-_{S(v,d)} + \mathfrak{n}_v. \tag{3.17}$$

Now let $x \in \mathfrak{g}$ and write $x = x_- + x_0 + x_+$, where $x_- \in \mathfrak{n}^-_{S(v,d)}, x_0 \in \mathfrak{m}_{S(v,d)}$, and $x_+ \in \mathfrak{n}_{S(v,d)}$. It follows from Lemma 3.11 and (3.17) that $x \in \mathfrak{c}$, i.e., x satisfies (3.16), if and only if

$$\begin{cases} x_0 - \phi(x_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T \\ x_+ - \phi(x_+) \in \mathfrak{n}_v \\ x_- - \phi(x_-) \in \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}^-_{S(v,d)}. \end{cases} \tag{3.18}$$

Write $x_0 = z_0 + y_0$, where $z_0 \in \mathfrak{z}_{S(v,d)}$ and $y_0 \in \mathfrak{g}_{S(v,d)}$. Since $\mathrm{Ad}_{\dot{v}}\mathfrak{z}_T \subset \mathfrak{z}_{S(v,d)}$ and since both $\mathfrak{z}_{S(v,d)}$ and $\mathfrak{g}_{S(v,d)}$ are ϕ-invariant, $x_0 - \phi(x_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T$ if and only if $z_0 - \phi(z_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T$ and $y_0 - \phi(y_0) = 0$, which is the same as $x_0 \in \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)}$. Recall that $\psi = (1-\phi)^{-1}$ on $\mathfrak{n}_{S(v,d)}$. Thus, $x_+ - \phi(x_+) \in \mathfrak{n}_v$ if and only if $x_+ \in \psi(\mathfrak{n}_v)$. Since $\mathfrak{n}_v \subset \mathfrak{n}_{S(v,d)}$,

$$\psi(\mathfrak{n}_v) \subset \mathfrak{p}_S \cap (\mathfrak{n}_v + \phi(\mathfrak{n}_{S(v,d)})) \subset \mathfrak{p}_S \cap (\mathrm{Ad}_{\dot{v}}(\mathfrak{n}^- + \mathfrak{m}_T)) \subset \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-.$$

Note that $\mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} \subset \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$. Thus $\mathfrak{c} \cap (\mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-) = \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v) + \mathfrak{c}'$, where $\mathfrak{c}'$ consists of all

$$x_- \in \mathfrak{n}^-_{S(v,d)} \cap \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^- \subset \mathfrak{n}^-_{S(v,d)} \cap \mathfrak{m}_S$$

satisfying the third condition in (3.18). It suffices to show that $\mathfrak{c}' = 0$.

We regard the direct sum decomposition $\mathfrak{n}^-_{S(v,d)} = \sum_{j\geq 0}\mathfrak{n}_j^-$ from Remark 3.12 as a grading of $\mathfrak{n}^-_{S(v,d)}$. Let $U = \mathfrak{n}^-_{S(v,d)} \cap \mathfrak{m}_S = \sum_{j>0}\mathfrak{n}_j^-$ and let $Y = \mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- \cap \mathfrak{n}^-_{S(v,d)}$. Clearly, U and Y are graded subspaces of $\mathfrak{n}^-_{S(v,d)}$, since they are sums of root spaces. Since $U \subset \mathfrak{g}_S$, ϕ is injective on

U. Moreover, the image of ϕ is in $\mathrm{Ad}_{\dot{v}}(\mathfrak{g}_T)$ so Y has zero intersection with the image of ϕ. The fact that $\mathfrak{c}' = 0$ now follows from the following simple linear algebra fact in Lemma 3.14.

Q.E.D.

Lemma 3.14. *Let $V = \oplus V_i$ be a graded vector space with graded subspaces U and Y. Let ϕ be an endomorphism of V such that 1) $\phi(V_i) \subset V_{i-1}$ for all i, 2) $Y \cap \mathrm{Im}(\phi) = 0$, and 3) $U \cap \mathrm{Ker}(\phi) = 0$. Then $\{v \in U : v - \phi(v) \in Y\} = 0$.*

Remark 3.15. Theorem 3.13 implies that

$$\mathfrak{n}(\mathfrak{l}) \subset \mathfrak{p}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^- = \mathfrak{p}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^- = \mathfrak{m}_{S(v,d)} + \mathfrak{n}_v + \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T$$

and that $\mathfrak{n}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l}) \cap \mathfrak{m}_{S(v,d)} + \mathfrak{n}(\mathfrak{l}) \cap (\mathfrak{n}_v + \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T)$, where

$$(\mathfrak{n}(\mathfrak{l}) \cap \mathfrak{m}_{S(v,d)})_\Delta = (\mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)})_\Delta = (\mathfrak{m}_{S(v,d)})_\Delta \cap \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}_{S,T,d},$$

and $\mathfrak{n}(\mathfrak{l}) \cap (\mathfrak{n}_v + \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T)$ is the graph of the map

$$\psi - 1 = \phi\psi = \phi + \phi^2 + \cdots : \mathfrak{n}_v \to \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T.$$

In [Lu-Y1], the map $\psi - 1$ is shown to be related to some set-theoretical solutions to the Quantum Yang-Baxter Equation.

3.5. **Intersections of $\mathfrak{g}_\Delta$ with $\mathfrak{l} \in \mathcal{L}$.** By Corollary 3.10, to compute $\mathfrak{g}_\Delta \cap \mathfrak{l}$ for any $\mathfrak{l} \in \mathcal{L}$, we may assume that $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ as given in (3.11).

Proposition 3.16. *For the Lagrangian subalgebra $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ as given in (3.11), let the notation be as in Theorem 3.13. Then*

$$\mathfrak{g}_\Delta \cap \mathfrak{l}_{S,T,d,V,\dot{v},m} = \mathrm{Ad}_{(m,\dot{v})}V' + \left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta,$$

where $V' = \{(z, v^{-1}z) : z \in \mathfrak{z}'_{S(v,d)}\} \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\})$.

Proof. Set $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$. By Theorem 3.13,

$$\mathfrak{g}_\Delta \cap \mathfrak{l} \subset \mathfrak{n}(\mathfrak{l}) = \left(\mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta.$$

Since $\left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta \subset \mathfrak{l}$, we see that

$$\mathfrak{g}_\Delta \cap \mathfrak{l} = \left((\mathfrak{z}'_{S(v,d)})_\Delta \cap \mathfrak{l}\right) + \left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta,$$

and

$$\left(\mathfrak{z}'_{S(v,d)}\right)_\Delta \cap \mathfrak{l} = \left(\mathfrak{z}'_{S(v,d)}\right)_\Delta \cap \mathfrak{l} \cap (\mathfrak{h} \oplus \mathfrak{h}) = \mathrm{Ad}_{(m,\dot{v})}V'.$$

Q.E.D.

Recall that a Belavin-Drinfeld triple [B-Dr] for $\mathfrak{g}$ is a generalized Belavin-Drinfeld triple (S, T, d) with the nilpotency condition: for every $\alpha \in S$, there exists an integer $n \geq 1$ such that $\alpha, d\alpha, \ldots, d^{n-1}\alpha \in S$ but $d^n\alpha \notin S$. The nilpotency condition is equivalent to $S(1, d) = \emptyset$, where 1 is the identity element in the Weyl group W.

Definition 3.17. A *Belavin-Drinfeld system* is a quadruple (S, T, d, V), where (S, T, d) is a Belavin-Drinfeld triple, and V is a Lagrangian subspace of $\mathfrak{z}_S \oplus \mathfrak{z}_T$ such that

$$\mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}) = 0.$$

We now derive a theorem of Belavin and Drinfeld [B-Dr] from Proposition 3.16.

Corollary 3.18. *[Belavin-Drinfeld] A Lagrangian subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$ has trivial intersection with $\mathfrak{g}_\Delta$ if and only if $\mathfrak{l}$ is G_Δ-conjugate to a Lagrangian subalgebra of the form $\mathfrak{l}_{S,T,d,V}$, where (S, T, d, V) is a Belavin-Drinfeld system.*

Proof. It is clear from Proposition 3.16 that $\mathfrak{g}_\Delta \cap \mathfrak{l}_{S,T,d,V} = 0$ if (S, T, d, V) is a Belavin-Drinfeld system. Suppose that $\mathfrak{g}_\Delta \cap \mathfrak{l}_{S,T,d,V,\dot{v},m} = 0$, where $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ is as in (3.11). Since $\dim \psi(\mathfrak{n}_v) = l(v)$, the length of v, and since every automorphism of a semi-simple Lie algebra has fixed point set of dimension at least one [Wi], $v = 1$ and $S(1, d) = \emptyset$. In this case, V' as in Proposition 3.16 is given by

$$V' = \mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}),$$

so $\mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}) = 0$, and we have $\mathfrak{l}_{S,T,d,V,\dot{v},m} = \mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}$ for some $m \in H$ and $\dot{v} \in H$. Note that in this case

$$R_{\dot{v}} = \{(h_1, h_2) \in H \times H : \gamma_d(\chi_S(h_1)) = \chi_T(h_2)\}$$

and $R_{\dot{v}}$ acts on H from the right by $h \cdot (h_1, h_2) = hh_1h_2^{-1}$, where $h \in H$ and $(h_1, h_2) \in R_{\dot{v}}$. Consider $\mathbf{m} : R_{\dot{v}} \to H : (h_1, h_2) \mapsto h_1h_2^{-1}$. The assumption $V' = 0$ implies that the dimension of the kernel of the differential of $\mathbf{m}$ is less than or equal to $\dim(\mathfrak{z}_T)$. It follows that the differential of $\mathbf{m}$ is onto, so $\mathbf{m}$ is onto. By Corollary 3.10, $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ is in the G_Δ-orbit of $\mathfrak{l}_{S,T,d,V}$.

Q.E.D.

3.6. **Examples of smooth G_Δ-orbit closures in $\mathcal{L}$.** The closure of a G_Δ-orbit in $\mathcal{L}$ is not necessarily smooth. We now look at two cases for which such a closure is smooth.

Proposition 3.19. *If $\mathfrak{l} \in \mathcal{L}$ is such that $\mathfrak{g}_\Delta \cap \mathfrak{l} = 0$, then $\overline{G_\Delta \cdot \mathfrak{l}} = \overline{(G \times G) \cdot \mathfrak{l}}$ is smooth.*

Proof. We only need to show that $\dim(G_\Delta \cdot \mathfrak{l}) = \dim((G \times G) \cdot \mathfrak{l})$. By Corollary 3.18, we may assume that $\mathfrak{l} = \mathfrak{l}_{S,T,d,V}$, where (S, T, d, V) is a Belavin-Drinfeld system, and so

$$\mathfrak{g}_\Delta \cap \mathfrak{r}_{S,T,d} = \mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S)),$$

where $V_S = \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}$. For a subspace A of $\mathfrak{h} \oplus \mathfrak{h}$, let

$$A^\perp = \{(x, y) \in \mathfrak{h} \oplus \mathfrak{h} : \langle (x, y), (x_1, y_1) \rangle = 0 \, \forall (x_1, y_1) \in A\}.$$

Then $(\mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S)))^\perp = \mathfrak{h}_\Delta + V_S$. Since $\mathfrak{h}_\Delta \cap V_S = 0$, we have

$$\dim(\mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S))) = 2 \dim \mathfrak{h} - \dim \mathfrak{h} - \dim \mathfrak{h}_S = \dim \mathfrak{z}_S.$$

Thus $\dim(G_\Delta \cdot \mathfrak{l}) = \dim \mathfrak{g} - \dim \mathfrak{z}_S = \dim((G \times G) \cdot \mathfrak{l})$ by Proposition 2.17.

Q.E.D.

We now show that the De Concini-Procesi compactification of a complex symmetric space of G can be embedded into $\mathcal{L}$ as the closure of a G_Δ-orbit in $\mathcal{L}$.

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an involution with lift σ to G, and let $\mathfrak{g}^\sigma$ and G^σ be the fixed subalgebra and subgroup of σ. Let again $\mathfrak{l}_\sigma \in \mathcal{L}$ be the graph of σ. The orbit $G_\Delta \cdot \mathfrak{l}_\sigma$ may be identified with the complex symmetric space G/G^σ. We will show that the closure $\overline{G_\Delta \cdot \mathfrak{l}_\sigma}$ is isomorphic to the De Concini-Procesi compactification of G/G^σ, which as defined as follows. Let $\dim(\mathfrak{g}^\sigma) = m$, so $\mathfrak{g}^\sigma \in \mathrm{Gr}(m, \mathfrak{g})$. Then $G \cdot \mathfrak{g}^\sigma \cong G/G^\sigma$, and $X_\sigma := \overline{G \cdot \mathfrak{g}^\sigma}$, the closure of $G \cdot \mathfrak{g}^\sigma$ in $\mathrm{Gr}(m, \mathfrak{g})$, is the De Concini–Procesi compactification. It is smooth with finitely many G-orbits [D-P].

We recall some basic results about involutions. Choose a σ-stable maximal split Cartan subalgebra $\mathfrak{h}_s$ of $\mathfrak{g}$, i.e., a σ-stable Cartan subalgebra $\mathfrak{h}_s$ such that $\mathfrak{h}_s^{-\sigma}$ has maximal dimension. There is an induced action of σ on the roots of $\mathfrak{h}_s$ in $\mathfrak{g}$, and there is a positive root system $\Sigma^+(\mathfrak{h}_s)$ for $\mathfrak{h}_s$ with the property that if $\alpha \in \Sigma^+(\mathfrak{h}_s)$, then either $\sigma(\alpha) = \alpha$ and $\sigma|_{\mathfrak{g}_\alpha} = \mathrm{id}$, or $\sigma(\alpha) \notin \Sigma^+(\mathfrak{h}_s)$. A weight $\lambda \in \mathfrak{h}_s^*$ is called a regular special dominant weight if λ is nonnegative on roots in $\Sigma^+(\mathfrak{h}_s)$, $\sigma(\lambda) = -\lambda$, and $\lambda(H_\alpha) = 0$ for α simple implies that $\sigma(\alpha) = \alpha$. If λ and μ are weights, we say $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\alpha \in \Sigma^+(\mathfrak{h}_s), n_\alpha \geq 0} n_\alpha \alpha$. For a weight μ, let $\overline{\mu} = \frac{1}{2}(\mu - \sigma(\mu))$.

Lemma 3.20. *[De Concini-Procesi,* [D-P]*, Lemmas 4.1 and 6.1] Let V be a representation of G, and suppose there exists a vector $v \in V$ such that G^σ is the stabilizer of the line through v. Suppose that when we decompose v into a sum of weight vectors for $\mathfrak{h}_s$, $v = v_\lambda + \sum v_i$ where v_λ has regular special dominant weight λ and each v_i has weight μ_i where $\lambda \geq \overline{\mu_i}$. Let $[v]$ be class of v in* Proj(V) *and let X' be the closure of $G \cdot [v]$ in* Proj(V)*. Then $X' \cong X_\sigma$.*

Proposition 3.21. *There is a G-equivariant isomorphism $\overline{G_\Delta \cdot \mathfrak{l}_\sigma} \cong X_\sigma$.*

Proof. To apply Lemma 3.20, let $n = \dim(\mathfrak{g})$ and consider the diagonal action of G on $V = \wedge^n(\mathfrak{g} \oplus \mathfrak{g})$ and the vector $v_\sigma = \wedge^n(\mathfrak{l}_\sigma)$. In order to represent v_σ as a sum of weight vectors in $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$, we choose a basis. Let $U_1, \dots, U_l$ be a basis of $\mathfrak{h}_s$. Let $\beta_1, \dots, \beta_s$ be the roots of $\Sigma^+(\mathfrak{h}_s)$ such that $\sigma(\beta_i) = \beta_i$, and let $\alpha_1, \dots, \alpha_t$ be the other roots in $\Sigma^+(\mathfrak{h}_s)$. For each root α, choose a root vector X_α. Then

$$\{(U_i, \sigma(U_i)) | i = 1, \dots, l\} \cup \{(X_{\pm\beta_i}, X_{\pm\beta_i}) | i = 1, \cdots, s\} \cup \{(X_{\pm\alpha_j}, \sigma(X_{\pm\alpha_j})) | i = 1, \cdots t\}$$

is clearly a basis of $\mathfrak{l}_\sigma$. Now v_σ is the wedge of the vectors $(Y_i, \sigma(Y_i))$ as Y_i runs through the above basis, and v_σ contains the summand

$$u : \bigwedge_{i=1,\dots,l} (U_i, \sigma(U_i)) \bigwedge_{i=1,\dots,s} (X_{\beta_i}, 0) \wedge (X_{-\beta_i}, 0) \bigwedge_{j=1,\dots,t} (X_{\alpha_i}, 0) \wedge (0, \sigma(X_{-\alpha_i})).$$

It is easy to see that u is a weight vector for the diagonal Cartan subalgebra with weight $\nu := \sum_{i=1,\dots,t} \alpha_i - \sigma(\alpha_i)$, and $\nu = 2 \sum_{i=1,\dots,t} \alpha_i$ on the subspace $\mathfrak{h}_s^{-\sigma}$. Thus, ν is a regular special dominant weight by Lemma 6.1 in [D-P]. Moreover, the other weight vectors appearing in v_σ have weights ψ such that $\overline{\psi}$ is of the form $\nu - \sum_{n_\alpha \geq 0, \alpha \in \Sigma^+(\mathfrak{h}_s)} n_\alpha \alpha$. Thus, by Lemma 3.20, $\overline{G \cdot v_\sigma} \cong X_\sigma$.

Note that using the Plucker embedding of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \hookrightarrow \mathrm{Proj(V)}$, we can identify $\overline{G \cdot v_\sigma}$ with $\overline{G_\Delta \cdot \mathfrak{l}_\sigma}$. Thus, $\overline{G_\Delta \cdot \mathfrak{l}_\sigma} \cong X_\sigma$.

Q.E.D.

Remark 3.22. Let d be the automorphism of the Dynkin diagram of $\mathfrak{g}$ such that $\sigma = \gamma_d \mathrm{Ad}_{g_0}$ for some g_0. Consider the embedding

$$G/G^\sigma \longrightarrow G: \ gG^\sigma \longmapsto \gamma_d^{-1}(g) g_0 g^{-1},$$

which in turn gives an embedding of G/G^σ into the De Concini-Procesi compactification Z_d of G. Proposition 3.21 then says that the closure of G/G^σ in Z_d is isomorphic to the De Concini-Procesi compactification of G/G^σ.

4. The Poisson structure Π_0 on $\mathcal{L}$

4.1. **Lagrangian splittings of $\mathfrak{g}\oplus\mathfrak{g}$.** By a *Lagrangian splitting* of $\mathfrak{g}\oplus\mathfrak{g}$ we mean a decomposition $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Lagrangian subalgebras of $\mathfrak{g}\oplus\mathfrak{g}$. By [E-L2], every Lagrangian splitting $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ gives rise to a Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}$ as follows: let $\{x_j\}$ be a basis for $\mathfrak{l}_1$ and $\{\xi_j\}$ the basis for $\mathfrak{l}_2$ such that $\langle x_j, \xi_k\rangle = \delta_{jk}$ for $1 \leq j,k \leq n = \dim \mathfrak{g}$. Set

$$R = \frac{1}{2}\sum_{j=1}^{n}(\xi_j \wedge x_j) \in \wedge^2(\mathfrak{g}\oplus\mathfrak{g}). \tag{4.1}$$

The action of $G\times G$ on $\mathcal{L}$ defines a Lie algebra anti-homomorphism κ from $\mathfrak{g}\oplus\mathfrak{g}$ to the space of vector fields on $\mathcal{L}$. Set

$$\Pi_{\mathfrak{l}_1,\mathfrak{l}_2} = (\kappa\wedge\kappa)(R) = \frac{1}{2}\sum_{j=1}^{n}(\kappa(\xi_j)\wedge\kappa(x_j)).$$

Proposition 4.1. [E-L2] *For any Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$, the bi-vector field $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}$ is Poisson with the property that all L_1 and L_2-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$, where, for $i = 1,2$, L_i is the connected subgroup of $G\times G$ with Lie algebra $\mathfrak{l}_i$.*

The rank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ can be computed as in the following Lemma 4.2. A version of Lemma 4.2 first appeared in [E-L2], and a generalization of Lemma 4.2 can be found in [Lu-Y2].

Lemma 4.2. *For $\mathfrak{l} \in \mathcal{L}$, let $\mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l})$ be the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{g}\oplus\mathfrak{g}$, and let $(\mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l}))^\perp = \{x \in \mathfrak{g}\oplus\mathfrak{g} : \langle x, y\rangle = 0 \ \forall y \in \mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l})\}$. Set*

$$\mathcal{T}(\mathfrak{l}) = \mathfrak{l}_1 \cap \mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l}) + (\mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l}))^\perp \subset \mathfrak{g}\oplus\mathfrak{g}.$$

Then $\mathcal{T}(\mathfrak{l}) \in \mathcal{L}$, and the rank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at $\mathfrak{l}$ is equal to $\dim(L_1\cdot\mathfrak{l}) - \dim(\mathfrak{l}_2 \cap \mathcal{T}(\mathfrak{l}))$, where $L_1\cdot\mathfrak{l}$ is the orbit in $\mathcal{L}$ of L_1 through $\mathfrak{l}$.

Example 4.3. It is also clear from the definition of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ that $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ is tangent to every $(G\times G)$-orbit in $\mathcal{L}$. Thus every $(G\times G)$-orbit in $\mathcal{L}$ is a Poisson submanifold of $(\mathcal{L}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$, and its closure is a Poisson subvariety. For example, let d be a diagram automorphism and consider the embedding of G into $\mathcal{L}$ as the $(G\times G)$-orbit through $\mathfrak{l}_{\gamma_d} = \{(x, \gamma_d(x)) : x \in \mathfrak{g}\}$:

$$G \hookrightarrow \mathcal{L}: \quad g \longmapsto \{(x, \gamma_d \mathrm{Ad}_g(x)) : x \in \mathfrak{g}\}. \tag{4.2}$$

By Proposition 4.1, every Lagrangian splitting of $\mathfrak{g}\oplus\mathfrak{g}$ gives rise to a Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on G which extends to the closure $Z_d(G)$ of G in $\mathcal{L}$.

Example 4.4. By a *Belavin-Drinfeld splitting* we mean a Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ in which $\mathfrak{l}_1 = \mathfrak{g}_\Delta$. By Corollary 3.18, $\mathfrak{l}_2$ is conjugate by an element in G_Δ to an $\mathfrak{l}_{S,T,d,V}$, where (S,T,d,V) is a Belavin-Drinfeld system (Definition 3.17). We will also denote a Belavin-Drinfeld splitting by $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{BD}$, and denote by Π_{BD} the corresponding Poisson structure on $\mathcal{L}$. By Proposition 4.1, all the G_Δ-orbits in $\mathcal{L}$ as well as their closures are Poisson submanifolds with respect to any Π_{BD}. For example, for a diagram automorphism d, equip G and $Z_d(G)$ with the Poisson structure Π_{BD} via the embedding (4.2). Then every d-twisted conjugacy class in G, as well as its closure in $Z_d(G)$, is a Poisson subvariety with respect to every Π_{BD}. As a special case, every complex symmetric space G/G^σ, as well as its De Concini-Procesi compactification, inherits the Poisson structure Π_{BD} this way. See Remark 3.22.

Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$ up to conjugation by elements in $G \times G$ have been classified by P. Delorme [De]. A study of the Poisson structures $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ defined by arbitrary Lagrangian splittings $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ will be carried out in [Lu-Y2].

For the rest of this section, we will only be concerned with the *standard Lagrangian splitting* of $\mathfrak{g} \oplus \mathfrak{g}$, namely, the splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}^*_{\mathrm{st}}$, where

$$\mathfrak{g}^*_{\mathrm{st}} = \mathfrak{h}_{-\Delta} + (\mathfrak{n} \oplus \mathfrak{n}^-).$$

We will denote by Π_0 the Poisson structure on $\mathcal{L}$ determined by the standard Lagrangian splitting. We will compute the rank of Π_0 everywhere on $\mathcal{L}$. As a consequence, we will see that every non-empty intersection $\mathcal{O} \cap \mathcal{O}'$ of a G_Δ-orbit $\mathcal{O}$ and a $(B \times B^-)$-orbit $\mathcal{O}'$ is a regular Poisson subvariety of Π_0, and that the subgroup $H_\Delta = \{(h,h) : h \in H\}$ of G_Δ acts transitively on the set of all symplectic leaves in $\mathcal{O} \cap \mathcal{O}'$, where $H = B \cap B^-$.

4.2. **The rank of the Poisson structure Π_0.** Let $\mathcal{O}$ be a G_Δ-orbit in $\mathcal{L}$ and $\mathcal{O}'$ a $(B \times B^-)$-orbit in $\mathcal{L}$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. Since $(\mathfrak{b} \oplus \mathfrak{b}^-) + \mathfrak{g}_\Delta = \mathfrak{g} \oplus \mathfrak{g}$, $\mathcal{O}$ and $\mathcal{O}'$ intersect transversally in their $(G \times G)$-orbit. Since both $\mathcal{O}$ and $\mathcal{O}'$ are Poisson submanifolds for Π_0, the intersection $\mathcal{O} \cap \mathcal{O}'$ is a Poisson submanifold of $(\mathcal{L}, \Pi_0)$. Thus, it is enough to compute the rank of Π_0 as a Poisson structure in $\mathcal{O} \cap \mathcal{O}'$. By Theorem 2.16, there exists a generalized Belavin-Drinfeld triple (S,T,d) and $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$ such that $\mathcal{O}, \mathcal{O}' \subset (G \times G) \cdot \mathfrak{l}_{S,T,d,V}$ with $\mathfrak{l}_{S,T,d,V}$ given in (2.4). By Corollaries 2.21 and 3.10, there exist $w \in W$, $v, v_1 \in W^T$, and $m \in M_{S(v,d)}$ such that

$$\mathcal{O} = G_\Delta \cdot \mathrm{Ad}_{(m,\dot{v})} \mathfrak{l}_{S,T,d,V}, \qquad \mathcal{O}' = (B \times B^-) \cdot \mathrm{Ad}_{(\dot{w},\dot{v}_1)} \mathfrak{l}_{S,T,d,V} \tag{4.3}$$

where $\dot{w}, \dot{v}$ and $\dot{v}_1$ are representatives of w, v, and v_1 in G respectively. Set

$$X_{S,T,d,v} = \{(z, v^{-1}z) : z \in \mathfrak{z}_{S(v,d)}, \gamma_d(\chi_S(z)) = \chi_T(v^{-1}z)\} + V_S \subset \mathfrak{h} \oplus \mathfrak{h} \tag{4.4}$$

with $V_S = \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}$. One shows directly that $X_{S,T,d,v}$ is a Lagrangian subspace of $\mathfrak{h} \oplus \mathfrak{h}$.

Theorem 4.5. *Let $\mathcal{O}$ and $\mathcal{O}'$ be as in (4.3), and suppose that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. The rank of Π_0 at every $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$ is equal to*

$$\dim(\mathcal{O} \cap \mathcal{O}') - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1) X_{S,T,d,v}),$$

where $X_{S,T,d,v}$ is given in (4.4). In particular, $\mathcal{O} \cap \mathcal{O}'$ is a regular Poisson submanifold of Π_0.

Proof. Let $\mathfrak{l} = \mathrm{Ad}_{(g,g)}\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V} \in \mathcal{O}$, where $g \in G$, and let $\mathcal{T}(\mathfrak{l})$ be as in Lemma 4.2. Then the rank of Π_0 at $\mathfrak{l}$ is equal to $\dim(G_\Delta \cdot \mathfrak{l}) - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathcal{T}(\mathfrak{l}))$. It follows from the definition of $\mathcal{T}(\mathfrak{l})$ that $\mathcal{T}(\mathfrak{l}) = \mathrm{Ad}_{(g,g)}\mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V})$. Let $\mathfrak{r}'_{S,T,d} = (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\}$, and let

$$\mathfrak{l}_{S,T,d,v} = X_{S,T,d,v} + \mathfrak{r}'_{S,T,d}. \tag{4.5}$$

By Theorem 3.13,

$$\begin{aligned} \mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}) &= \mathfrak{g}_\Delta \cap \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}_{S,T,d} + \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}'_{S,T,d} \\ &= (\mathfrak{z}'_{S(v,d)})_\Delta + \left(\mathfrak{g}^\phi_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta + \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}'_{S,T,d}. \end{aligned}$$

Since $\mathrm{Ad}^{-1}_{(m,\dot{v})}\left(\mathfrak{g}^\phi_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta \subset \mathfrak{r}'_{S,T,d}$, we have

$$\mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}) = \mathrm{Ad}_{(m,\dot{v})}\left(\mathrm{Ad}^{-1}_{(m,\dot{v})}(\mathfrak{z}'_{S(v,d)})_\Delta + \mathfrak{r}'_{S,T,d}\right) = \mathrm{Ad}_{(m,\dot{v})}(\mathfrak{l}_{S,T,d,v}).$$

Thus the rank of Π_0 at $\mathfrak{l}$ is equal to $\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v})$. Let

$$\delta = \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d}) - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}).$$

Then $\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} + \delta - \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d})$. Since

$$\dim \mathcal{O}' = \dim(\mathfrak{b} \oplus \mathfrak{b}^-) - \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d}),$$

we have

$$\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} + \dim \mathcal{O}' + \delta - \dim(\mathfrak{b} \oplus \mathfrak{b}^-) = \dim \mathcal{O} + \dim \mathcal{O}' + \delta - 2\dim \mathfrak{b}.$$

Since $\mathcal{O}$ and $\mathcal{O}'$ intersect transversally at $\mathfrak{l}$ inside the $(G \times G)$-orbit through $\mathfrak{l}$, and since $\dim(G \times G) \cdot \mathfrak{l} = \dim \mathfrak{g} - \dim \mathfrak{z}_S$ by Proposition 2.17, we have

$$\begin{aligned} \mathrm{Rank}_{\Pi_0}(\mathfrak{l}) &= \dim(\mathcal{O} \cap \mathcal{O}') + \dim((G \times G) \cdot \mathfrak{l}) + \delta - 2\dim \mathfrak{b} \\ &= \dim(\mathcal{O} \cap \mathcal{O}') - (\dim \mathfrak{z}_S + \dim \mathfrak{h}) + \delta. \end{aligned}$$

It remains to compute δ. Since $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$, there exist $r \in R_{S,T,d}$ and $(b, b^-) \in B \times B^-$ such that $(gm, g\dot{v}) = (b, b^-)(\dot{w}, \dot{v}_1)r$. Using $\mathrm{Ad}_{(b,b^-)}(\mathfrak{b} \oplus \mathfrak{b}^-) = \mathfrak{b} \oplus \mathfrak{b}^-$ and $\mathrm{Ad}_{(b,b^-)}\mathfrak{g}_{\mathrm{st}^*} = \mathfrak{g}_{\mathrm{st}^*}$, we have

$$\delta = \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{r}_{S,T,d}) - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}).$$

Set $Y = (\mathfrak{n} \oplus \mathfrak{n}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\left((\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \mathrm{span}_{\mathbb{C}}\{(E_\alpha, \gamma_d(E_\alpha)) : \alpha \in [S]\}\right)$. Then

$$(\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{r}_{S,T,d} = (w, v_1)(\mathfrak{z}_S \oplus \mathfrak{z}_T + V_S) + Y.$$

Since $Y \subset \mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}$, we have $\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v} = Y + \mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}$. Thus

$$\begin{aligned} \delta &= \dim(\mathfrak{z}_S \oplus \mathfrak{z}_T + V_S) - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}) \\ &= \dim \mathfrak{z}_S + \dim \mathfrak{h} - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}). \end{aligned}$$

Thus the rank of Π_0 at $\mathfrak{l}$ is equal to $\dim(\mathcal{O} \cap \mathcal{O}') - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v})$.

Q.E.D.

Remark 4.6. Our conclusion that $\mathcal{O} \cap \mathcal{O}'$ is a regular Poisson manifold for Π_0 follows immediately from our computation of the rank of Π_0. It will be shown in [Lu-Y2] that a similar result holds for the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}$ defined by any Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$.

Corollary 4.7. *Equip G with the Poisson structure Π_0 via the embedding of G into $\mathcal{L}$ in (4.2) for $d = 1$. Let C be a conjugacy class in G and let $w \in W$ be such that $C \cap (B^- wB) \neq \emptyset$. Then the rank of Π_0 at every point in $C \cap (B^- wB)$ is*

$$\dim C - l(w) - \dim(\mathfrak{h}^{-w}),$$

where $l(w)$ is the length of w, and $\mathfrak{h}^{-w} = \{x \in \mathfrak{h} : w(x) = -x\}$. In particular, $C \cap B^- B$ is an open dense leaf for C, and Π_0 is degenerate on the complement of $B^- B \cap C$ in C.

Proof. By Proposition 4.1, C is a Poisson submanifold of (G, Π_0). By the Bruhat decomposition, $C = \cup_{w \in W}(C \cap (B^- wB))$. Since $B^- B$ is open in G and $C \cap B \neq \emptyset$ (Theorem 1 on P. 69 of [St]), $C \cap B^- B$ is open and dense in C. The rank formula follows from Theorem 4.5, and it follows easily that $C \cap B^- B$ is a symplectic leaf and Π_0 is degenerate on $C \cap (B^- wB)$ if $w \neq 1$.

Q.E.D.

Remark 4.8. By Corollary 4.7, any unipotent conjugacy class (and its closure in $Z_1(G)$) has an induced Poisson structure Π_0 with an open symplectic leaf, although the structure is not symplectic unless the orbit is a single point. Since the unipotent variety is isomorphic to the nilpotent cone in $\mathfrak{g}^*$, it follows that every nilpotent orbit in $\mathfrak{g}^*$ has an induced Poisson structure with the same properties. It would be quite interesting to compare this structure with the Kirillov-Kostant symplectic structure.

Example 4.9. Consider the closed $(G \times G)$-orbit through a Lagrangian subalgebra of the form $V + (\mathfrak{n} \oplus \mathfrak{n}^-)$, where V is any Lagrangian subspace of $\mathfrak{h} \oplus \mathfrak{h}$. Such an orbit can be identified with $G/B \times G/B^-$, so we can regard Π_0 as a Poisson structure on $G/B \times G/B^-$. Let $\mathcal{O}$ be a G_Δ-orbit and let $\mathcal{O}'$ be a $(B \times B^-)$-orbit in $G/B \times G/B^-$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. By the Bruhat decomposition of G, there are elements $w, u, v \in W$ such that

$$\mathcal{O} = G_\Delta \cdot (B, wB^-), \qquad \mathcal{O}' = (B \times B^-) \cdot (uB, vB^-).$$

The stabilizer subgroup of $G_\Delta \cong G$ at the point $(B, \dot{w}B^-) \in G/B \times G/B^-$ is $B \cap w(B^-)$. Identify $\mathcal{O} \cong G/(B \cap w(B^-))$, and let $p : G \to \mathcal{O} \cong G/(B \cap w(B^-))$ be the projection. It is then easy to see that $\mathcal{O} \cap \mathcal{O}' = p(G_w^{u,v}) \subset \mathcal{O}$, where

$$G_w^{u,v} = (BuB) \cap (B^- vB^- w^{-1}).$$

We will refer to $G_w^{u,v}$ as the *shifted double Bruhat cell* in G determined by u, v and w. Note that $B \cap w(B^-)$ acts freely on $G_w^{u,v}$ by right multiplications, so

$$\mathcal{O} \cap \mathcal{O}' \ \cong\ G_w^{u,v}/(B \cap w(B^-)).$$

Since $\dim \mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{h} - l(w)$ and $\dim \mathcal{O}' = l(u) + l(v)$, we have

$$\dim(\mathcal{O} \cap \mathcal{O}') = \dim \mathcal{O} + \dim \mathcal{O}' - \dim(G/B \times G/B^-) = l(u) + l(v) - l(w),$$

and $\dim G_w^{u,v} = l(u) + l(v) + \dim \mathfrak{h}$. By Theorem 4.5, the rank of Π_0 at every point of $\mathcal{O} \cap \mathcal{O}'$ is

$$l(u) + l(v) - l(w) - \dim \mathfrak{h}^{-u^{-1}vw^{-1}},$$

where $\mathfrak{h}^{-u^{-1}vw^{-1}} = \{x \in \mathfrak{h} : u^{-1}vw^{-1}x = -x\}$. When $w = 1$, we have $\mathcal{O} \cong G/H$, and $\mathcal{O} \cap \mathcal{O}' \cong G^{u,v}/H$, where $G^{u,v} = G_1^{u,v}$ is the double Bruhat cell in G determined by u and v. The set $G^{u,v}/H$ is called a *reduced double Bruhat cell* in [Z]. In [K-Z], Kogan and Zelevinsky constructed toric charts on symplectic leaves of Π_0 in $\mathcal{O} \cap \mathcal{O}'$ (for the case when $w = 1$) by using the so-called *twisted minors* that are developed in [Fm-Z], and they also constructed integrable systems on the symplectic leaves. It would very interesting to generalize the Kogan-Zelevinsky construction to all symplectic leaves of Π_0 in $G/B \times G/B^-$.

4.3. **The action of H_Δ on the set of symplectic leaves of the Poisson structure Π_0.**

Proposition 4.10. *Let D be a connected complex algebraic group with connected algebraic subgroups A and C. Suppose there exists a connected algebraic subgroup $C_1 \subset C$ such that the multiplication morphism $A \times C_1 \to D$ is an isomorphism to a connected open set U of D. Let X be a homogeneous space for D such that the stabilizer in D of a point in X is connected. Then any nonempty intersection of an A-orbit in X with a C-orbit in X is smooth and connected.*

Proof. Let $A \cdot x \cap C \cdot x$ be a nonempty intersection of orbits in X, and note that this intersection is smooth since the hypotheses imply that the orbits intersect transversely. We show there is a fiber bundle $\pi : V \to U$, with fiber $\pi^{-1}(e) \cong A \cdot x \cap C \cdot x$ over the identity and V connected, that is trivial in the Zariski topology. This implies the connectedness of the intersection, and hence the proposition. The proof is inspired by the proof of Kleiman's transversality theorem.

Let $Y = C \cdot x$ and $Z = A \cdot x$. Let $h : D \times Y \to X$ be the action map and let $i : Z \to X$ be the obvious embedding. Let $W = (D \times Y) \times_X Z$ be the fiber product. Then h is a smooth fiber bundle (see the proof of 10.8 in [Hart]) and the fibers $h^{-1}(x)$ are connected. For the second claim, note that $h^{-1}(x) = \{(d, c \cdot x) : dc \cdot x = x\}$ and $\psi : h^{-1}(x) \to D_x \cdot C$ given by $\psi(d, c \cdot x) = d$ is an isomorphism. Since D_x and C are connected, the claim follows. Thus, the induced morphism from $W \to Z$ also has connected fibers. Since Z is connected, it follows that W is connected. Moreover, W is smooth (again by the proof of 10.8 in [Hart]), so W is irreducible.

Let $\pi : W \to D \times Y \to D$ be the composition of the induced fiber product map with projection to the first factor. Since $\pi^{-1}(U)$ is open in W, it is smooth and irreducible, and thus connected. Note also that $\pi^{-1}(e) \cong Y \cap Z$. It remains to show that $\pi : \pi^{-1}(U) \to U$ is a trivial fiber bundle. We define a free left A action and a free right C_1 action on W by the formulas

$$a \cdot (d, y, z) = (ad, y, a \cdot z)$$
$$c \cdot (d, y, z) = (dc, c^{-1} \cdot y, z)$$
$$a \in A, c \in C, d \in D, y \in Y, z \in Z$$

A and C_1 have the obvious free left and right multiplication actions on U, and $\pi : \pi^{-1}(U) \to U$ is equivariant for these actions. It follows that the morphism

$$\phi : A \times C_1 \times (A \cdot x \cap C \cdot x) \to \pi^{-1}(U), \ (a, c, v) \mapsto (ac, c^{-1} \cdot v, a \cdot v), \ a \in A, c \in C_1, v \in Y \cap Z$$

is a bijection, and hence is an isomorphism since $\pi^{-1}(U)$ is smooth. Thus the fiber bundle is trivial.

Q.E.D.

Remark 4.11. We thank Michel Brion for suggesting this approach. Note that the Proposition 4.10 is false as stated if we only assume that $A \cdot C$ is open in D. For example, let $A = G_\Delta$, and let $C = \{(nh, h^{-1}n^-) : n \in N, h \in H, n^- \in N^-\}$ be the connected subgroup of $D = G \times G$ corresponding to $\mathfrak{g}_{\rm st}^*$. Let $X = D$ and let D act on X by left translation. Then the intersection of the A-orbit and the C-orbit through the identity element of D is $A \cap C$ which is disconnected.

Proposition 4.12. *The intersection of any G_Δ-orbit and any $(B \times B^-)$-orbit in $\mathcal{L}$ is either empty or a smooth connected subvariety of $\mathcal{L}$.*

Proof. This is a consequence of Proposition 4.10. Indeed, we take $A = G_\Delta$, $C = B \times B^-$, $D = G \times G$, and $C_1 = B \times N^-$. The fact that the stabilizer of a point in $\mathcal{L}$ is connected follows from Lemma 2.19.

Q.E.D.

Let $H = B \cap B^-$ and let $H_\Delta = \{(h,h) : h \in H\}$. For every G_Δ-orbit $\mathcal{O}$ and every $(B\times B^-)$-orbit $\mathcal{O}'$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, H_Δ clearly leaves $\mathcal{O}\cap\mathcal{O}'$ invariant. It is easy to show that the element $R \in \wedge^2(\mathfrak{g}\oplus\mathfrak{g})$ given in (4.1) is invariant under $\mathrm{Ad}_{(h,h)}$ for every $h \in H$. Thus the Poisson structure Π_0 on $\mathcal{L}$ is H_Δ-invariant. In particular, for every $h \in H$, $\mathrm{Ad}_{(h,h)}\mathcal{E}$ is a symplectic leaf of Π_0 in $\mathcal{O}\cap\mathcal{O}'$ if $\mathcal{E}$ is.

Lemma 4.13. *Let $\mathcal{O}$ be a G_Δ-orbit and $\mathcal{O}'$ a $(B\times B^-)$-orbit in $\mathcal{L}$ such that $\mathcal{O}\cap\mathcal{O}'\neq\emptyset$. Let $\mathcal{E}$ be any symplectic leaf of Π_0 in $\mathcal{O}\cap\mathcal{O}'$. Then the map*

$$\sigma:\ H\times\mathcal{E} \longrightarrow \mathcal{O}\cap\mathcal{O}':\ (h,\mathfrak{l}) \longmapsto \mathrm{Ad}_{(h,h)}\mathfrak{l}$$

is a submersion.

Proof. Let e be the identity element of H and let $\mathfrak{l}\in\mathcal{E}$. It is enough to show that

$$\dim\ker\sigma_*(e,\mathfrak{l}) = \dim\mathfrak{h} + \dim\mathcal{E}_\mathfrak{l} - \dim\mathcal{O}\cap\mathcal{O}',$$

where $\sigma_*(e,\mathfrak{l}) : \mathfrak{h}\times T_\mathfrak{l}\mathcal{E} \to T_\mathfrak{l}(\mathcal{O}\cap\mathcal{O}')$ is the differential of σ at $(e,\mathfrak{l})$.

We may assume that $\mathcal{O}$ and $\mathcal{O}'$ are given in (4.3), and that

$$\mathfrak{l} = \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,V} = \mathrm{Ad}_{(b\dot{w},b^-\dot{v}_1)}\mathfrak{l}_{S,T,d,V}$$

for some $g\in G$ and $(b,b^-)\in B\times B^-$. By Theorem 4.5, it is enough to show that

$$\dim(\ker\sigma_*(e,\mathfrak{l})) = \dim\mathfrak{h} - \dim(\mathfrak{h}_{-\Delta}\cap(w,v_1)X_{S,T,d,v}),$$

where $X_{S,T,d,v}$ is given in (4.4). Identify the tangent space of $\mathcal{O}$ at $\mathfrak{l}$ as

$$T_\mathfrak{l}\mathcal{O} \cong \mathfrak{g}_\Delta/(\mathfrak{g}_\Delta\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d})$$

and let $q:\mathfrak{g}_\Delta\to\mathfrak{g}_\Delta/(\mathfrak{g}_\Delta\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d})$ be the projection. Let $p:\mathfrak{g}\oplus\mathfrak{g}\to\mathfrak{g}_\Delta$ be the projection with respect to the decomposition $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}^*_{\mathrm{st}}$. By the computation of $\mathcal{T}(\mathfrak{l})$ in the proof of Theorem 4.5, the tangent space of $\mathcal{E}$ at $\mathfrak{l}$ is given by

$$T_\mathfrak{l}\mathcal{E} = (q\circ p)\left(\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}\right),$$

where $\mathfrak{l}_{S,T,d,v}$ is given in (4.5). For $x\in\mathfrak{h}$, let κ_x be the vector field on $\mathcal{O}\cap\mathcal{O}'$ that generates the action of $\mathrm{Ad}_{(\exp tx,\exp tx)}$. Then $\ker\sigma_*(e,\mathfrak{l}) \cong \{x\in\mathfrak{h} : \kappa_x(\mathfrak{l})\in T_\mathfrak{l}\mathcal{E}\}$. Let $x\in\mathfrak{h}$. If $\kappa_x(\mathfrak{l})\in T_\mathfrak{l}\mathcal{E}$, then there exists $y\in\mathfrak{g}$ and $(y_1,y_2)\in\mathfrak{g}^*_{\mathrm{st}}$ with $(y+y_1,y+y_2)\in\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}$ such that

$$(x-y,x-y)\in\mathfrak{g}_\Delta\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d} = \mathfrak{g}_\Delta\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}.$$

It follows that $(x+y_1,x+y_2)\in(\mathfrak{b}\oplus\mathfrak{b}^-)\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}$. Let $r\in R_{S,T,d}$ be such that $(gm,g\dot{v}) = (b\dot{w},b^-\dot{v}_1)r$. Then

$$(\mathfrak{b}\oplus\mathfrak{b}^-)\cap\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v} = \mathrm{Ad}_{(b,b^-)}\left((\mathfrak{b}\oplus\mathfrak{b}^-)\cap\mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}\right).$$

Thus there exists $(y'_1,y'_2)\in\mathfrak{g}^*_{\mathrm{st}}$ such that

$$(x+y'_1,x+y'_2)\in(\mathfrak{b}\oplus\mathfrak{b}^-)\cap\mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}.$$

If $(y', -y')$ is the $\mathfrak{h}_{-\Delta}$-component of $(y_1', y_2') \in \mathfrak{g}_{\mathrm{st}}^*$, then $(x+y', x-y') \in (w, v_1)X_{S,T,d,v}$. Thus $(x,x) \in p((w,v_1)X_{S,T,d,v})$, where p denotes the projection $\mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h}_\Delta$ along $\mathfrak{h}_{-\Delta}$. Conversely, if $x \in \mathfrak{h}$ is such that $(x,x) \in p((w,v_1)X_{S,T,d,v})$, then there exists $y' \in \mathfrak{h}$ such that

$$(x+y', x-y') \in (w,v_1)X_{S,T,d,v} \subset \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v},$$

and thus $\mathrm{Ad}_{(b,b^-)}(x+y', x-y') \in \mathrm{Ad}_{(gm, g\dot{v})}\mathfrak{l}_{S,T,d,v}$. Since

$$\mathrm{Ad}_{(b,b^-)}(x+y', x-y') = (x+y', x-y') \mathrm{mod}(\mathfrak{n} \oplus \mathfrak{n}^-),$$

we see that $p(\mathrm{Ad}_{(b,b^-)}(x+y', x-y')) = (x,x)$, so $\kappa_x(\mathfrak{l}) \in T_{\mathfrak{l}}\mathcal{E}$. Thus we have shown that

$$\ker \sigma_*(e, \mathfrak{l}) \cong \{x \in \mathfrak{h} : (x,x) \in p((w,v_1)X_{S,T,d,v})\}.$$

It follows that $\dim(\ker \sigma_*(e,\mathfrak{l})) = \dim \mathfrak{h} - \dim(\mathfrak{h}_{-\Delta} \cap (w,v_1)X_{S,T,d,v})$. The lemma now follows from Theorem 4.5.

Q.E.D.

Theorem 4.14. *For every G_Δ-orbit $\mathcal{O}$ and every $(B \times B^-)$-orbit $\mathcal{O}'$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, H_Δ acts transitively on the set of symplectic leaves of Π_0 in $\mathcal{O} \cap \mathcal{O}'$.*

Proof. For $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$, let $\mathcal{E}_{\mathfrak{l}}$ be the symplectic leaf of Π_0 through $\mathfrak{l}$, and let

$$\mathcal{F}_{\mathfrak{l}} = \bigcup_{h \in H} \mathrm{Ad}_{(h,h)}\mathcal{E}_{\mathfrak{l}} \subset \mathcal{O} \cap \mathcal{O}'.$$

Then it is easy to see that either $\mathcal{F}_{\mathfrak{l}} \cap \mathcal{F}_{\mathfrak{l}'} = \emptyset$ or $\mathcal{F}_{\mathfrak{l}} = \mathcal{F}_{\mathfrak{l}'}$ for any $\mathfrak{l}, \mathfrak{l}' \in \mathcal{O} \cap \mathcal{O}'$. It follows from Lemma 4.13 that $\mathcal{F}_{\mathfrak{l}}$ is open in $\mathcal{O} \cap \mathcal{O}'$ for every $\mathfrak{l}$. Since $\mathcal{O} \cap \mathcal{O}'$ is connected by Proposition 4.12, $\mathcal{O} \cap \mathcal{O}' = \mathcal{F}_{\mathfrak{l}}$ for every $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$.

Q.E.D.

5. Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{h}$

Let again $\mathfrak{g}$ be a complex semi-simple Lie algebra with Killing form $\ll\, , \,\gg$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. In this section, we will consider the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$, together with the symmetric, non-degenerate, and ad-invariant bilinear form

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \ll x_1, x_2 \gg - \ll y_1, y_2 \gg, \qquad x_1, x_2 \in \mathfrak{g}, y_1, y_2 \in \mathfrak{h}. \tag{5.1}$$

We wish to describe the variety $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{h}$ with respect to $\langle\, , \,\rangle$. We can describe all such Lagrangian subalgebras by using a theorem of Delorme [De].

Definition 5.1. [De] Let $\mathfrak{m}$ be a complex reductive Lie algebra with simple factors $\mathfrak{m}_i, i \in I$. A complex linear involution σ of $\mathfrak{m}$ is called an f-involution if σ does not preserve any $\mathfrak{m}_i$.

Theorem 5.2. [De] *Let $\mathfrak{u}$ be a complex reductive Lie algebra with a symmetric, non-degenerate, and ad-invariant bilinear form β.*

1). Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{u}$ with Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$, and decompose $\mathfrak{m}$ into $\mathfrak{m} = \overline{\mathfrak{m}} + \mathfrak{z}$, where $\overline{\mathfrak{m}}$ is its semisimple part and $\mathfrak{z}$ its center. Let σ be an f-involution of $\overline{\mathfrak{m}}$ such that $\overline{\mathfrak{m}}^\sigma$ is a Lagrangian subalgebra of $\overline{\mathfrak{m}}$ with respect to the restriction of β, and let V be a Lagrangian subspace of $\mathfrak{z}$ with respect to the restriction of β. Then $\mathfrak{l}(\mathfrak{p}, \sigma, V) := \overline{\mathfrak{m}}^\sigma \oplus V \oplus \mathfrak{n}$ is a Lagrangian subalgebra of $\mathfrak{u}$ with respect to β.

2). Every Lagrangian subalgebra of $\mathfrak{u}$ is $\mathfrak{l}(\mathfrak{p}, \sigma, V)$ for some $\mathfrak{p}$, σ, and V as in 1).

Proposition 5.3. *Every Lagrangian subalgebra of* $\mathfrak{g} \oplus \mathfrak{h}$ *with respect to* $\langle \, , \, \rangle$ *given in (5.1) is of the form* $\mathfrak{n}+V$*, where* $\mathfrak{n}$ *is the nilradical of a Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$*,* V *is a Lagrangian subspace of* $\mathfrak{h} \oplus \mathfrak{h}$*, and* $\mathfrak{n} + V = \{(x + y_1, y_2) : x \in \mathfrak{n}, (y_1, y_2) \in V\}$.

Proof. Applying Delorme's theorem to our case of $\mathfrak{u} = \mathfrak{g} \oplus \mathfrak{h}$ and $\langle \, , \, \rangle$ as the bilinear form β, every Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ is of the form

$$\mathfrak{l} = \{(x + y_1, y_2) : \, x \in \overline{\mathfrak{m}}^\sigma + \mathfrak{n}, \, (y_1, y_2) \in V\}$$

for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} = \overline{\mathfrak{m}} + \mathfrak{z} + \mathfrak{n}$, an f-involution σ on $\overline{\mathfrak{m}}$, and a Lagrangian subspace V of $\mathfrak{z} \oplus \mathfrak{h}$. We will now show that if $\overline{\mathfrak{m}} \neq 0$ and if σ is an f-involution of $\overline{\mathfrak{m}}$, then $\overline{\mathfrak{m}}^\sigma$ is not an isotropic subspace of $\overline{\mathfrak{m}}$ for the restriction of the Killing form $\ll \, , \, \gg$ of $\mathfrak{g}$ to $\overline{\mathfrak{m}}$. It follows that $\mathfrak{p}$ must be Borel, which gives Proposition 5.3.

Assume that $\overline{\mathfrak{m}} \neq 0$. Let $\mathfrak{m}_i$ be a simple factor of $\overline{\mathfrak{m}}$. Then since $\mathfrak{m}_i$ is simple, it has a unique nondegenerate invariant form up to scalar multiplication. Hence the Killing form $\ll \, , \, \gg$ of $\mathfrak{g}$ restricts to a scalar multiple of the Killing form of $\mathfrak{m}_i$. Recall that the Killing form on a maximal compact subalgebra of a semisimple Lie algebra is negative definite. It follows that the Killing form of $\mathfrak{g}$ restricts to a nonzero positive scalar multiple of the Killing form on $\mathfrak{m}_i$. Suppose that σ is an involution of $\overline{\mathfrak{m}}$ mapping $\mathfrak{m}_i$ to $\mathfrak{m}_j$ with $i \neq j$. Then σ is an isometry with respect to the Killing form of $\mathfrak{m}_i$ and the Killing form of $\mathfrak{m}_j$. Thus, there exists a nonzero positive scalar μ such that $\ll \sigma(x), \sigma(y) \gg = \mu \ll x, y \gg, \forall x, y \in \mathfrak{m}_i$. The fixed point set $\overline{\mathfrak{m}}^\sigma$ contains the subspace $\{x + \sigma(x) : \, x \in \mathfrak{m}_i\}$. Let x be a nonzero element of a maximal compact subalgebra of $\mathfrak{m}_i$. Then $\ll x + \sigma(x), x + \sigma(x) \gg = (1 + \mu) \ll x, x, \gg \neq 0$. Thus $\overline{\mathfrak{m}}^\sigma$ cannot be isotropic with respect to $\ll \, , \, \gg$.

Q.E.D.

Let G be the adjoint group of $\mathfrak{g}$, and let B be the Borel subgroup of G corresponding to a Borel subalgebra $\mathfrak{b}$.

Theorem 5.4. *The variety* $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ *is isomorphic to the trivial fiber bundle over* G/B *with fibre* $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle \, , \, \rangle)$*. In particular,* $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ *is smooth with two disjoint irreducible components, corresponding to the two connected components of* $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle \, , \, \rangle)$.

Proof. Identify G/B with the variety of all Borel subalgebras of $\mathfrak{g}$. We map $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ to G/B by mapping a Lagrangian algebra $\mathfrak{l} = \mathfrak{n} + V$ to the unique Borel subalgebra with nilradical $\mathfrak{n}$. The fiber over $\mathfrak{n}$ may be identified with $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle \, , \, \rangle)$. The claim about connected components follows from the fact the bundle is trivial.

Q.E.D.

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