Poisson harmonic forms, Kostant harmonic forms, and the S^1-equivariant cohomology of K/T

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Dedicated to Professor Bertram Kostant for his 70th birthday

Abstract

We characterize the harmonic forms on a flag manifold K/T defined by Kostant in 1963 in terms of a Poisson structure. Namely, they are "Poisson harmonic" with respect to the so-called Bruhat Poisson structure on K/T. This enables us to give Poisson geometrical proofs of many of the special properties of these harmonic forms. In particular, we construct explicit representatives for the Schubert basis of the S^1-equivariant cohomology of K/T, where the S^1-action is defined by ρ. Using a simple argument in equivariant cohomology, we recover the connection between the Kostant harmonic forms and the Schubert calculus on K/T that was found by Kostant and Kumar in 1986. By using a family of symplectic structures on K/T, we also show that the Kostant harmonic forms are limits of the more familiar Hodge harmonic forms with respect to a family of Hermitian metrics on K/T.

Contents

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1 Introduction

Let K be a compact semi-simple Lie group and $T \subset K$ a maximal torus. In [Ko2], Kostant introduced a degree -1 operator ∂ with $\partial^2 = 0$ on the space of K-invariant complex valued differential forms on the flag manifold K/T. He then constructed certain forms s^w on K/T, for $w \in W$, the Weyl group of (K, T), that are (d, ∂)-harmonic, where d is the de Rham differential. In this work, we establish connections between these harmonic forms and the geometry of certain Poisson structures on K/T. A first set of such connections have been reported in [Lu2], where some explicit formulas for the form s^w are given in coordinates.

We think it is fair to say that, although introduced more than 30 years ago, the Kostant harmonic forms have remained mysterious to this day. What is especially mysterious is the fact that they possess many special properties (to be reviewed later in this introduction) that other kinds of harmonic forms do not. As was emphasized in [Ko2], this is all due to the fact that the forms s^w are ∂-closed. Thus a conceptual understanding of the operator ∂ is the key to unveil the mystery about the Kostant harmonic forms. We point out that ∂ is in general not the adjoint operator of d with respect to any Hermitian metric on K/T (see Example 3.20), so the harmonicity of the forms s^w is not the same as that in the sense of Hodge.

In this paper, we characterize the Kostant operator ∂, and thus the Kostant harmonic forms as well, using the so-called Bruhat Poisson structure on K/T [L-W]. We also show that ∂ is the limit of the adjoint operators of d with respect to a family of Hermitian metrics that come from a family of symplectic structures on K/T.

Given an orientable manifold P with a Poisson structure π and a volume form μ, we consider the degree -1 **Koszul-Brylinski operator** $\partial_{\pi,\mu}$ on the space of differential forms on P defined by

$$\partial_{\pi,\mu} = i_\pi d - d i_\pi + i_{\theta_\mu},$$

where θ_μ is the modular vector field of π with respect to μ, and i_π and i_{θ_μ} are the contraction operators defined by π and θ_μ respectively (see Section 2 for details). We say that a form ξ on P is "**Poisson harmonic with respect to** π **and** μ" if $d\xi = \partial_{\pi,\mu}\xi = 0$. This is a modified version of a notion introduced by Brylinski in [B].

The Bruhat Poisson structure on K/T, first introduced in [L-W] (see also [S]), has its origin from quantum groups. It is so named because its symplectic leaves are exactly all the Bruhat (or Schubert) cells in K/T. In this paper, the Bruhat Poisson structure is denoted by π_∞ (for reasons given below) and its Koszul-Brylinski operator (using a K-invariant volume form) by ∂_∞. Our Theorem 3.10 says that the Kostant operator ∂ is related to ∂_∞ by $\partial = J\partial_\infty J^{-1}$ on the space C of K-invariant complex valued differential forms on K/T, where J is a standard complex structure on K/T. Consequently (Corollary 3.15), the Kostant harmonic forms s^w, for $w \in W$, which have pure bi-degrees with respect to the bi-grading on C defined by J, are Poisson harmonic with respect to the Bruhat Poisson structure. In fact, Theorem 4.5 in [Ko2] can be reformulated as saying that every de Rham cohomology class of K/T has a unique K-invariant representative that is Poisson harmonic. Once this is proved, we show that the special properties of these Kostant harmonic forms follow from fairly general arguments in Poisson geometry. In particular, we construct from the forms $s^w, w \in W$, explicit representatives for the Schubert basis of the S^1-equivariant cohomology of K/T, where the S^1 action is defined by ρ, half of the sum of all positive roots (for a choice of such roots). Using a simple argument in equivariant cohomology, we then show geometrically how the Kostant harmonic forms can be used to describe the ring structure on the de Rham cohomology $H^\bullet(K/T, \mathbb{C})$ of K/T, a fact first proved by Kostant and Kumar in [K-K]. To make connections between the Kostant harmonic forms and Hodge harmonic forms, we employ a family of symplectic structures π_λ on K/T for $\lambda \in \mathfrak{t}^*$ regular, where $\mathfrak{t}$ is the Lie algebra of T. The symplectic structure π_λ degenerates to the Bruhat Poisson structure when $\lambda \to \infty$ within the positive Weyl chamber (and thus the notation π_∞ for the latter). Each π_λ gives rise to a Hermitian metric h_λ on K/T. Using the Koszul-Brylinski operators for the π_λ's, we show that the Kostant operator ∂ is the limit of the adjoint operators of d with respect to the Hermitian metrics h_λ as $\lambda \to \infty$. Correspondingly, the Kostant harmonic forms are shown to be limits of ordinary Hodge harmonic forms.

To further explain the content of this paper, we recall Kostant's construction of the harmonic forms in [Ko2]. In fact, we will recall the main results in [Ko2].

Since K is compact, the (complex-valued) de Rham cohomology $H^\bullet(K/T, \mathbb{C})$ of K/T can be calculated from the space $C = \Omega(K/T, \mathbb{C})^K$ of K-invariant complex valued differential forms on K/T. Denote by $\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of K and T respectively and by $\mathfrak{g} = \mathfrak{k}_\mathbb{C}$ the complexification of $\mathfrak{k}$. Then we can use the Killing form of $\mathfrak{g}$ to identify the complexified cotangent space of K/T at the base point e with the vector space $\mathfrak{n}_- \oplus \mathfrak{n}_+$, where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are the Lie subalgebras of $\mathfrak{g}$ spanned respectively by the positive and negative root vectors. Consequently, we have the identification (see (18))

$$I: \ (\wedge^\bullet(\mathfrak{n}_- \oplus \mathfrak{n}_+))^T \xrightarrow{\sim} C^\bullet \ = \ \Omega^\bullet(K/T)^K. \tag{1}$$

Now equip $\mathfrak{n}_- \oplus \mathfrak{n}_+$ with the direct sum Lie algebra structure, where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ have the Lie subalgebra structures of $\mathfrak{g}$. Let $b_{\mathfrak{n}_- \oplus \mathfrak{n}_+}$ be the Chevalley-Eilenberg boundary operator for this Lie algebra. Then Kostant introduced in [Ko2] the degree -1 operator ∂ on $C = \Omega(K/T, \mathbb{C})^K$ by

$$\partial := -Ib_{\mathfrak{n}_- \oplus \mathfrak{n}_+} I^{-1}: \ C^q \longrightarrow C^{q-1}.$$

The first main theorem in [Ko2] says that the two operators d and ∂, where d is the de Rham differential, are "disjoint" in the sense that $\mathrm{Im}(d) \cap \mathrm{Ker}(\partial) = \mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$. Set $S = d\partial + \partial d$ and call it the "Laplacian" of d and ∂. Then it follows immediately from the disjointness of d and ∂ that $\mathrm{Ker}(S) = \mathrm{Ker}(d) \cap \mathrm{Ker}(\partial)$ and that the natural maps

$$\begin{aligned} \psi_{d,S}: & \qquad \mathrm{Ker}(S) \longrightarrow H(C, d): \ \xi \longmapsto [\xi]_d \\ \psi_{\partial,S}: & \qquad \mathrm{Ker}(S) \longrightarrow H(C, \partial): \ \xi \longmapsto [\xi]_\partial \end{aligned}$$

are isomorphisms of graded vector spaces. Elements in $\mathrm{Ker}(S)$ are said to be (d, ∂)-harmonic or simply harmonic in [Ko2] [1]. Our Example 3.20 shows that the operator S is in general not semi-simple, so ∂ is in general not the adjoint of d with respect to any Hermitian metric on K/T.

What is done next in [Ko2] is the construction of a special basis of $\mathrm{Ker}(S)$ which we explain now. The Lie algebra homology $H_\bullet(\mathfrak{n}_+)$ of $\mathfrak{n}_+$ as a T-module had been determined earlier by Kostant in [Ko1] to have weights exactly of the form $\rho - w\rho$, for $w \in W$, and with multiplicity one for each weight. By viewing $\mathfrak{n}_-$ as the contragradient of $\mathfrak{n}_+$ as T-modules via the Killing form of $\mathfrak{g}$, we see by Schur's Lemma that there is a canonical basis for the Lie algebra homology $(H_\bullet(\mathfrak{n}_- \oplus \mathfrak{n}_+))^T \cong (H_\bullet(\mathfrak{n}_-) \otimes H_\bullet(\mathfrak{n}_+))^T$ and thus for $H(C, \partial)$, the homology of C with respect to the operator ∂. Denote this basis for $H(C, \partial)$ by $\mathbf{h}^w$, $w \in W$. Then its inverse image under the map $\psi_{\partial,S}$ is a basis for $\mathrm{Ker}(S)$. In other words, for each $w \in W$, set

$$s^w \ = \ \psi_{\partial,S}^{-1}(\mathbf{h}^w) \ \in \ \mathrm{Ker}(S).$$

We will refer to the forms s^w, for $w \in W$, as the **Kostant harmonic forms**. The form s^w is of pure degree $2l(w)$, where $l(w)$ is the length of w. In fact, it has bi-degree $(l(w), l(w))$

[1] We will say throughout this paper that such forms are "Kostant harmonic" or "harmonic in the sense of Kostant"

in the bi-grading of C defined by the complex structure J on $K/T \cong G/B_-$, where G is the complexification of K and B_- the Borel subgroup of G corresponding to the Lie algebra $\mathfrak{b}_- = \mathfrak{h} + \mathfrak{n}_-$.

The rest of [Ko2] is devoted to the proofs of several special properties of the forms s^w, $w \in W$. We list some of these properties here.

1. The basis $\{[s^w] : w \in W\}$ of $H^\bullet(K/T, \mathbb{C})$ is, up to scalar multiples, dual to the basis in the homology of K/T defined by the Schubert varieties. In other words, if we use Σ_w to denote the Schubert cell in K/T corresponding to w (defined as the orbits of the B_+-action on $K/T \cong G/B_+$, where B_+ is the Borel subgroup of G corresponding to the Lie algebra $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+$), then

$$\int_{\Sigma_{w_1}} s^w = \begin{cases} 0 & \text{if } w_1 \neq w \\ \lambda_w \neq 0 & \text{if } w_1 = w \end{cases}.$$

   The number λ_w was later calculated by Kostant and Kumar in [K-K], where they also treated the general Kac-Moody case.

2. For $w \in W$, denote by $i_w : \Sigma_w \hookrightarrow K/T$ the inclusion map. Then, in fact, when $l(w) = l(w_1)$ but $w \neq w_1$, the form $i_{w_1}^* s^w$ on Σ_{w_1} is identically equal to zero.

3. Kostant actually constructed harmonic forms on G/P, where $P \subset G$ is any parabolic subgroup containing B_+. Denote by $W_P \subset W$ the subgroup of W corresponding to P and by W^P the set of minimal length representatives of the right coset space $W_P \backslash W$. Then for each $w \in W^P$, Kostant constructed a K-invariant harmonic form s_P^w on $G/P \cong K/(K \cap P)$. Under the natural projection $\nu : G/B_+ \to G/P$, the harmonic form s_P^w on G/P goes to the harmonic form s^w on G/B_+ for each $w \in W^P$.

4. In [K-K], Kostant and Kumar described a connection between the harmonic forms $s^w, w \in W$, and the Schubert calculus on K/T. Namely, a certain D-matrix, which encodes the structure constants of the ring structure on $H^\bullet(K/T, \mathbb{C})$ in the Schubert basis, can be constructed using these harmonic forms. This is reviewed in Section 4.

As we mentioned earlier, what we achieve in this paper is a Poisson geometrical interpretation of Kostant's operator ∂ and the forms s^w for $w \in W$. This will allow us to give Poisson geometrical proofs of the above properties of these forms. We now explain our results and the organization of the paper in more detail.

The notion of Poisson harmonic forms is introduced in Section 2, where we also prove a few general facts about such forms that will be applied to the Kostant harmonic forms in later sections.

The definition of the Bruhat Poisson structure π_∞ on the flag manifold K/T is reviewed in Section 3.1. The Koszul-Brylinski operator $\partial_{\pi_\infty,\mu_0}$ defined by π_∞ and a K-invariant volume form μ_0 on K/T is denoted by ∂_∞:

$$\partial_\infty := i_{\pi_\infty} d - d i_{\pi_\infty} + i_{\theta_0}.$$

The modular vector field $\theta_0 := \theta_{\mu_0}$ in this case turns out to be the infinitesimal generator of the T-action on K/T (by left translations) in the direction of $iH_\rho \in \mathfrak{t}$ (see Section 3.1). The operator ∂_∞ leaves invariant the subspace $C = \Omega(K/T, \mathbb{C})^K$ of K-invariant differential forms on K/T, and on C, the two operators d and ∂_∞ anti-commute.

In Section 3.2, using an identification given in (19):

$$I_\infty : (\wedge^\bullet(\mathfrak{n}_- \oplus \mathfrak{n}_+))^T \xrightarrow{\sim} C^\bullet,$$

the operator ∂_∞ is identified with the Chevalley-Eilenberg boundary operator $b_{\mathfrak{n}_-\oplus\mathfrak{n}_+}$ for the Lie algebra $\mathfrak{n}_- \oplus \mathfrak{n}_+$:

- **Theorem 3.7** *As operators on $C = \Omega(K/T, \mathbb{C})^K$, we have*

$$\partial_\infty = I_\infty b_{\mathfrak{n}_-\oplus\mathfrak{n}_+} I_\infty^{-1} : C^q \longrightarrow C^{q-1}.$$

This fact is the key reason why the Bruhat Poisson structure should be related to Kostant's harmonic forms: Recall that the Kostant operator ∂ is defined as $\partial = -I b_{\mathfrak{n}_-\oplus\mathfrak{n}_+} I^{-1}$, where I is the identification in (1) between the same two spaces. The two identifications I and I_∞ are different, but their difference is expressed via the standard complex structure J on $K/T \cong G/B_-$, namely $I_\infty I^{-1} = J$ (Lemma 3.6). The immediate consequences of this important point are the following three of our main theorems, where $C = \oplus_{p,q} C^{p,q}$ is the bi-grading on C defined by J:

- **Theorem 3.10** *As operators on $C = \Omega(K/T, \mathbb{C})^K$, the Kostant operator ∂ and the Koszul-Brylinski operator ∂_∞ are related by*

$$\partial = J \partial_\infty J^{-1}$$

  where J is the standard complex structure on $K/T \cong G/B_-$.

- **Theorem 3.14** *A form $\xi \in C^{p,q}$ is ∂-closed if and only if it is ∂_∞-closed.*

- **Corollary 3.15** *A form $\xi \in C^{p,q}$ is harmonic in the sense of Kostant if and only if it is Poisson harmonic with respect to the Bruhat Poisson structure π_∞ (and any K-invariant volume form). In particular, the Kostant harmonic forms s^w, for $w \in W$, are all Poisson harmonic.*

Using this Poisson characterization of the forms s^w for $w \in W$, we immediately derive their first three properties listed earlier. This is done in Section 3.4.

In Section 4, we consider the S^1-equivariant cohomology $H_{S^1}(K/T)$ of K/T, where the S^1-action on K/T is defined by the element $iH_\rho \in \mathfrak{t}$, or, in other words, by the modular vector field θ_0. More precisely, for each $w \in W$, we define the $\Omega(K/T, \mathbb{C})$-valued function on $\mathbb{R}$ by

$$s^w(u) = i_{\exp_\wedge(-u\pi_\infty)} s^w = s^w - u i_{\pi_\infty} s^w + \frac{u^2}{2!} i_{\pi_\infty \wedge \pi_\infty} s^w + \cdots .$$

Following from an easy fact about Poisson harmonic forms (Theorem 2.10 in Section 2), we have

- **Theorem 4.5** *Each $s^w(u)$ is S^1-equivariantly closed, and, up to scalar multiples, the set $\{[s^w(u)] : w \in W\}$ is the $\mathbb{C}[u]$-Schubert basis for the S^1-equivariant cohomology of K/T.*

This theorem is then used to give another proof of the connection between the Kostant harmonic forms and the Schubert calculus on K/T that was found by Kostant and Kumar in [K-K] (the fourth property of these forms listed earlier): By evaluating the forms $s^w(u)$ at the T-fixed points on K/T, we get a matrix D that encodes the structure constants of the $\mathbb{C}[u]$-algebra structure on $H_{S^1}(K/T, \mathbb{C})$ in the Schubert basis. Since $H_{S^1}(K/T, \mathbb{C})$ specialized at $u = 0$ is the ordinary de Rham cohomology $H^\bullet(K/T, \mathbb{C})$, we get an explicit description of the ring structure on $H^\bullet(K/T, \mathbb{C})$ in terms of the Kostant harmonic forms. We remark that a closely related algorithm for computing the ring structure was given in the seminal paper [BGG], and that the argument in [BGG] is based on consideration of certain $\mathbb{C}P^1$ bundles over G/B_+. We are not using $\mathbb{C}P^1$ bundles in any explicit way to obtain our formulas, although we believe that recovering formulas based on $\mathbb{C}P^1$ bundles from our arguments should not be very difficult. We plan to pursue this topic in a future paper.

Section 5 is devoted to a family of Poisson (in fact symplectic) structures on K/T denoted by π_λ, where the parameter λ runs over all regular elements in $\mathfrak{t}^*$, and the symplectic structure π_λ comes from the dressing orbit of K in its dual group through the point $e^{-\lambda}$. Very importantly, the Poisson structures π_λ tend to the Bruhat Poisson structure as $\lambda \to \infty$. We study Poisson harmonic forms for π_λ. It turns out that the Koszul-Brylinski operator $\partial_\lambda := \partial_{\pi_\lambda, \mu_0}$ for π_λ and the adjoint $d_{*,\lambda}$ of d with respect to a certain Hermitian metric h_λ are related by $d_{*,\lambda} = J\partial_\lambda J^{-1}$ (compare with $\partial = J\partial_\infty J^{-1}$). Consequently, we have (Theorem 5.16)

$$\partial = \lim_{t \to +\infty} d_{*,\lambda + tH_\rho} \qquad \text{and} \qquad S = \lim_{t \to +\infty} S_{\lambda + tH_\rho},$$

where $S_\lambda = dd_{*,\lambda} + d_{*,\lambda}d$ is the usual (Hodge) Laplacian for d with respect to the Hermitian metric h_λ. Thus, although ∂ is in general not the adjoint of d with respect to any Hermitian metric, it is the limit of a family of such operators.

The construction of the forms s^w for $w \in W$ in [Ko2] relies on the fact that the two operators d and ∂ are disjoint in the sense that $\mathrm{Im}(d) \cap \mathrm{Ker}(\partial) = \mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$. Notice that d is always disjoint from to its adjoint operator with respect to any Hermitian metric. Using Theorem 5.16 and some simple linear algebra arguments, we give another proof of the disjointness of d and ∂. This is done in Section 5.4.

Finally, for each π_λ, we construct Poisson harmonic forms $s_\lambda^w, w \in W$. We show that $s^w_{\lambda + tH_\rho} \to s^w$ as $t \to +\infty$. Thus the Kostant harmonic forms are limits of usual Hodge harmonic forms.

We conclude this introduction by mentioning a few other works that are related to this paper.

The work in [Lu2] is our first attempt to relate Kostant's harmonic forms to the geometry of the Bruhat Poisson structure. In [Lu2], for each $w \in W$, we explicitly express the Kostant harmonic form s^w as a Duistermaat-Heckman type volume form on the cell Σ_w (as a symplectic leaf of π_∞). In fact, the form s^w, as well as some other related quantities such as the moment map for the T action on Σ_w, are written down in [Lu2] by explicit formulas in certain Bott-Samelson type coordinates on the cell. The calculation there for the integral $\lambda_w = \int_{\Sigma_w} s^w$ takes only a couple of lines by using the explicit formula in coordinates. Some of the results in [Lu2], for example, Theorem 4.3 there, follow from the global Poisson properties of the forms s^w we find in this paper. This is explained in Section 3.4.

The notion of the modular vector field θ_μ of a Poisson structure π associated to a volume form μ, which plays a central role in our work (see Sections 2 and 4), has been actively investigated only recently: although it appeared in some earlier work of Koszul, it was rediscovered by Weinstein [Ws] and independently by Brylinski and Zuckerman [B-Z] in 1995. See also the paper [P] by Polishchuk. A related paper is [E-L-W], in which we study modular vector fields for arbitrary Lie algebroids. Results in [E-L-W] show that there is a notion of Poisson harmonic forms on a Poisson manifold P for each rank 1 representation of the cotangent bundle Lie algebroid T^*P. The notion of Poisson harmonic forms we use in this paper corresponds to the trivial rank 1 representation of T^*P, while that in [B] corresponds to the representation of T^*P on the canonical line bundle of P. See [E-L-W] [P] and [X] for more details.

The Bruhat Poisson structure π_∞ on K/T is an example of a (K, π)-homogeneous Poisson structure, where π is the Poisson structure on K given in (8) which makes (K, π) into a Poisson Lie group. Poisson homogeneous spaces for Poisson Lie groups were first studied by Drinfeld [D], where he shows that a certain Lie algebra determines the Poisson structure on the underlying homogeneous space. For the Bruhat Poisson structure, this Lie algebra is $\mathfrak{t} + \mathfrak{n}_+$ as a real Lie subalgebra of $\mathfrak{g}$. In [Lu1], we extend the work of Drinfeld and show that the (real) invariant Poisson cohomology of the Poisson homogeneous space is isomorphic to certain relative Lie algebra cohomology of the Lie algebra defined by Drinfeld–it is $H^\bullet(\mathfrak{t} + \mathfrak{n}_+; \mathfrak{t}) \cong (H^\bullet(\mathfrak{n}_+))^T$ in the case of π_∞ (note that $\mathfrak{n}_+$ is considered as a real Lie algebra here). It is clear at this point that Kostant's results in [Ko2] are statements about the Bruhat Poisson structure and its (complex) K-invariant Poisson cohomology. This is the starting point of our project to give Poisson interpretations of Kostant's work.

Ideas in Section 5 come from [Lu3], where a family of Poisson structures on K/T, of which the π_λ's considered in Section 5 constitute only a subfamily, is studied. The Poisson structures in this family all come from the solutions to the Classical Dynamical Yang Baxter Equation for the pair $(\mathfrak{g}, \mathfrak{h})$ that have recently been classified by Etingof and Varchenko [E-V]. The fact that the Bruhat Poisson structure is the limit of this family is first proved in [Lu3]. Many Poisson geometrical properties of members of this family, such as their symplectic leaves, their K-invariant Poisson cohomology, and the Drinfeld Lie algebras, are studied in [Lu3].

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2 Poisson harmonic forms

Recall that a Poisson structure on a manifold P is a bi-vector field π on P such that $[\pi, \pi] = 0$, where $[\pi, \pi]$ is the Schouten bracket (see the Appendix) of π with itself. Denote by $\chi^q(P)$ the space of q-vector fields on P, i.e., smooth sections of the vector bundle $\wedge^q TP$, and by $\Omega^q(P)$ the space of q-differential forms on P. Define

$$\begin{aligned} \delta_\pi &= [\pi, \bullet] : \ \chi^q(P) \longrightarrow \chi^{q+1}(P), \\ \partial_\pi &= i_\pi d - d i_\pi : \ \Omega^q(P) \longrightarrow \Omega^{q-1}(P). \end{aligned}$$

It follows from the graded Jacobi identity for the Schouten bracket that $\delta_\pi^2 = 0$, and it follows from (49) in the Appendix that $\partial_\pi^2 = 0$. The **Poisson cohomology** [Li] of (P, π) is defined to be the cohomology of the cochain complex $(\chi^\bullet(P), \delta_\pi)$ and is denoted by $H^\bullet(P, \pi)$. The **Poisson homology** [B] of (P, π) is defined to be the homology of the chain complex $(\Omega^\bullet(P), \partial_\pi)$. See [V] for more details on Poisson (co)homology.

Assume now that (P, π) is an orientable Poisson manifold of dimension n and that μ is a volume form on P. The map

$$\lrcorner\mu : \chi^q(P) \longrightarrow \Omega^{n-q}(P) : \; X \longmapsto X \lrcorner \mu \, = \, i_X\mu$$

is an isomorphism of vector spaces, where $X \lrcorner \mu = i_X\mu$ is the contraction of X with μ. [2]

Lemma 2.1 *For any q-vector field X on P,*

$$(\delta_\pi X) \lrcorner \mu \, = \, (-1)^{q-1}(\partial_\pi + i_{\theta\mu})(X \lrcorner \mu),$$

where θ_μ is the unique vector field on P such that $\theta_\mu \lrcorner \mu = d(\pi \lrcorner \mu) = -\partial_\pi\mu$.

Proof. It follows from (48) in the Appendix that

$$\begin{aligned}
(\delta_\pi X) \lrcorner \mu &= i_{[\pi,\, X]}\mu \\
&= (-1)^{q-1}(\partial_\pi i_X\mu \, - \, (-1)^q i_X \partial_\pi\mu) \\
&= (-1)^{q-1}(\partial_\pi i_X\mu \, + \, (-1)^q i_X i_{\theta_\mu}\mu) \\
&= (-1)^{q-1}(\partial_\pi \, + \, i_{\theta_\mu})(X \lrcorner \mu).
\end{aligned}$$

Q.E.D.

Definition 2.2 *[Ws]* The vector field θ_μ is called the **modular vector field** of π associated to the volume form μ.

The modular vector field is always a Poisson vector field, i.e., $[\theta_\mu, \, \pi] = 0$. The class it defines in $H^1(P, \pi)$ is independent of the choice of the volume form μ, and this class is called the modular class of π ([Ws] [E-L-W] [X]).

Notation 2.3 We set

$$\partial_{\pi,\mu} \, = \, \partial_\pi \, + \, i_{\theta_\mu} \, = \, i_\pi d \, - \, di_\pi \, + \, i_{\theta_\mu} : \; \Omega^q(P) \longrightarrow \Omega^{q-1}(P).$$

Definition 2.4 We say that a differential form ξ on P is **Poisson harmonic** with respect to the Poisson structure π and the volume form μ if it satisfies

$$d\xi \, = \, 0 \qquad \text{and} \qquad \partial_{\pi,\mu}\xi \, = \, 0.$$

We will also say that such forms are $(d, \partial_{\pi,\mu})$-harmonic.

[2] Throughout this paper, we will use both $X \lrcorner \xi$ and $i_X\xi$ to denote the contraction of a multi-vector field X with a differential form ξ.

Example 2.5 The volume form μ is always $(d, \partial_{\pi,\mu})$-harmonic, and a general top degree form $f\mu$ is $(d, \partial_{\pi,\mu})$-harmonic if and only if f is a Casimir function on P, i.e., $df \lrcorner \pi = 0$. This is easily seen from Lemma 2.1.

Example 2.6 Let (P, ω) be a symplectic manifold. Then there is a unique non-degenerate Poisson bi-vector field π on P characterized by

$$\omega(\tilde{\pi}(\alpha),\, \tilde{\pi}(\beta)) \,=\, \pi(\alpha,\, \beta) \tag{2}$$

for any 1-forms α and β, where $\tilde{\pi}(\alpha) = \alpha \lrcorner \pi$. Let $\mu_\omega = \frac{\omega^m}{m!}$, where $2m = \dim P$, be the Liouville volume form on P defined by ω. Then the modular vector field of π associated to μ_ω is zero. In general, let $\mu = f\mu_\omega$ be an arbitrary volume form on P, where f is a nowhere vanishing function on P. Then the modular vector field of π associated to μ is $\theta_\mu = -d(\log|f|) \lrcorner \pi$, or, in other words, the Hamiltonian vector field of the function $\log|f|$. As a particular case of Example 2.5, when P is connected, a top degree form s on P is $(d, \partial_{\pi,\mu})$-harmonic if and only if $s = c\mu$ for a constant function c. This simple fact will be used in Section 3.4.

Remark 2.7 Note that the volume form μ enters into our definition of Poisson harmonic forms. The operator $\partial_\pi = i_\pi d - d i_\pi$ was first studied by Koszul [Kz] and Brylinski [B] and is generally called the Koszul-Brylinski operator associated to π. It is thus appropriate to call $\partial_{\pi,\mu} = \partial_\pi + i_{\theta_\mu}$ the **Koszul-Brylinski operator associated to π and μ**. Of course, it also satisfies $\partial_{\pi,\mu}^2 = 0$, but its homology is isomorphic to the Poisson cohomology of π. Poisson harmonic forms are defined in [B] to be (d, ∂_π)-harmonic. One can show by using our results in Section 3.4 that for the Bruhat Poisson structure on the flag manifold K/T, a non-zero de Rham cohomology class has a (d, ∂_π)-harmonic representative only when it is in degree 0 while this is not the case for $(d, \partial_{\pi,\mu})$-harmonic forms. The following question is a modification of a question asked by Brylinski in [B].

Question 2.8 *Given an orientable manifold P with a Poisson structure π and a volume form μ, does every de Rham cohomology class of P have a representative that is Poisson harmonic with respect to π and μ?*

When π is non-degenerate and when μ is the Liouville volume form corresponding to ω given by (2), the modular vector field θ_μ is zero, and thus $\partial_{\pi,\mu} = i_\pi d - d i_\pi$ is the same as ∂_π. A result of O. Mathieu [M] says that the answer to Question 2.8 is "yes" if and only if for any $k \leq m = \frac{1}{2}\dim P$, the cup product by $[\omega]^k : H^{m-k}(P) \to H^{m+k}(P)$ is an isomorphism. A simpler proof of this fact is given in [Y].

Our results in Section 3.4 will show that for the Bruhat Poisson structure on the flag manifold K/T and a K-invariant volume form μ_0 on K/T, the answer to Question 2.8 is "yes". Other examples for which the answer to Question 2.8 is "yes" are given in Section 5.2. See Remark 5.14.

We now look at the connection between Poisson harmonic forms and S^1-equivariant cohomology.

Let again (P, π) be a Poisson manifold. Introduce the elements

$$\exp_\wedge \pi \;=\; 1 + \pi + \frac{1}{2!}\pi \wedge \pi + \frac{1}{3!}\pi \wedge \pi \wedge \pi + \cdots \in \chi(P) \tag{3}$$

$$\exp_\wedge(-\pi) \;=\; 1 - \pi + \frac{1}{2!}\pi \wedge \pi - \frac{1}{3!}\pi \wedge \pi \wedge \pi + \cdots \in \chi(P). \tag{4}$$

They are inverse to each other with respect to the wedge product on $\chi(P)$. Correspondingly, the operator

$$i_{\exp_\wedge \pi} : \; \Omega^k(P) \longrightarrow \Omega^k(P) \;\oplus\; \Omega^{k-2}(P) \;\oplus\; \Omega^{k-4}(P) \;\oplus \cdots$$

given by

$$i_{\exp_\wedge \pi}(\alpha) \;=\; \alpha \;+\; i_\pi \alpha \;+\; \frac{1}{2!} i_{\pi\wedge\pi}\alpha \;+\; \frac{1}{3!} i_{\pi\wedge\pi\wedge\pi}\alpha \;+\; \cdots \tag{5}$$

has the operator $i_{\exp_\wedge(-\pi)}$ as its inverse. Since $i_{\pi\wedge\pi} = i_\pi^2$, we have, $i_{\exp_\wedge \pi} = \mathrm{Exp}(i_\pi)$, where $\mathrm{Exp}(i_\pi)$ is the exponential of the operator i_π. It is the sum of only finitely many terms. The following is also proved by Polishchuk [P].

Proposition 2.9 *On $\Omega(P)$, we have*

$$d \;+\; \partial_\pi \;=\; (i_{\exp_\wedge \pi}) \; d \; (i_{\exp_\wedge \pi})^{-1}. \tag{6}$$

Proof. Denote by [ ] the commutator bracket between operators on $\Omega(P)$. We know from (48) in the Appendix that $[i_\pi, \partial_\pi] = [i_\pi, [i_\pi, d]] = 0$. Thus

$$\begin{aligned} i_{\exp_\wedge \pi} \; d \; i_{\exp_\wedge(-\pi)} \;&=\; \mathrm{Exp}(i_\pi)\, d\, \mathrm{Exp}(-i_\pi) \\ &=\; d \;+\; [i_\pi,\; d] \;+\; \frac{1}{2!}[i_\pi,\; [i_\pi,\; d]] \;+\; \frac{1}{3!}[i_\pi,\; [i_\pi,\; [i_\pi,\; d]]] \;+\; \cdots \\ &=\; d \;+\; \partial_\pi. \end{aligned}$$

Q.E.D.

Assume now that P is orientable. Let μ be a volume form, and let θ_μ be the modular vector field of π associated to μ. Recall that a θ_μ-equivariantly closed differential form on P is an $\Omega(P)^{\theta_\mu}$-valued polynomial function $\xi(u)$ on $\mathbb{R}$ such that

$$d\xi(u) \;+\; i_{u\theta_\mu}\xi(u) \;=\; 0, \qquad\qquad \forall u \in \mathbb{R},$$

where $\Omega(P)^{\theta_\mu}$ is the space of forms on P whose Lie derivatives in the direction of θ_μ are 0. When the vector field θ_μ integrates to an S^1-action on P, such forms are said to be S^1-equivariantly closed. The following observation is the key to our construction of S^1-equivariantly closed forms on the flag manifolds in Section 4.

Theorem 2.10 *For any differential form ξ on P of pure degree, define*

$$\xi(u) \;=\; i_{\exp_\wedge(-u\pi)}\xi,$$

for $u \in \mathbb{R}$. Then ξ is $(d, \partial_{\pi,\mu})$-harmonic if and only if $\xi(u)$ is θ_μ-equivariantly closed.

Proof. By Proposition 2.9,

$$\begin{aligned} d\xi(u) &= di_{\exp_\wedge(-u\pi)}\xi \\ &= i_{\exp_\wedge(-u\pi)}(d + \partial_{u\pi})(\xi) \\ &= i_{\exp_\wedge(-u\pi)}(d + u\partial_{\pi,\mu} - ui_{\theta_\mu})(\xi). \end{aligned}$$

Thus

$$d\xi(u) + i_{u\theta_\mu}\xi(u) \;=\; i_{\exp_\wedge(-u\pi)}(d + u\partial_{\pi,\mu})(\xi).$$

Since ξ is of pure degree, we see that $d\xi(u) + i_{u\theta_\mu}\xi(u) = 0$ for all $u \in \mathbb{R}$ if and only if $d\xi = 0$ and $\partial_{\pi,\mu}\xi = 0$, i.e., if and only if ξ is $(d, \partial_{\pi,\mu})$-harmonic.

Q.E.D.

Example 2.11 Let π be a non-degenerate Poisson structure on a manifold P of dimension $2m$. Assume that θ is a Hamiltonian vector field on P with a Hamiltonian function ϕ, i.e., $\theta = -d\phi \,\lrcorner\, \pi$. Consider the volume form $\mu = e^\phi \frac{\omega^m}{m!}$, where ω is the symplectic form on P defined by π as given in (2). Then the modular vector field of π associated to μ is θ, and thus by Example 2.5 and Theorem 2.10, we have the θ-equivariantly closed form $\mu(u)$ on P given by

$$\begin{aligned} \mu(u) &= i_{\exp_\wedge(-u\pi)}\mu \\ &= e^\phi\left(\frac{\omega^m}{m!} - u\frac{\omega^{m-1}}{(m-1)!} + u^2\frac{\omega^{m-2}}{(m-2)!} + \cdots + (-1)^{m-1}u^{m-1}\omega + (-1)^m u^m\right) \\ &= (-u)^m \exp(\phi - \frac{\omega}{u}). \end{aligned}$$

This is the well-known θ-equivariant form used by Atiyah and Bott in [A-B] to prove the Duistermaat-Heckman integral formula from the localization theorem for equivariant cohomology.

3 The Bruhat Poisson structure and the Kostant harmonic forms

3.1 The Bruhat Poisson structure π_∞ and the Koszul-Brylinski operator ∂_∞

Let K be a compact semi-simple Lie group and $T \subset K$ a maximal torus. Denote by $\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of K and T respectively. Let $\mathfrak{g}$ be the complexification of $\mathfrak{k}$. Then the complexification $\mathfrak{h}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by Φ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and by Φ^+ a choice of positive roots. We will also write $\alpha > 0$ for $\alpha \in \Phi^+$.

Let $\ll\ \gg$ be the Killing form of $\mathfrak{g}$. For each positive root α, denote by H_α the image of α under the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ via $\ll\ ,\ \gg$, i.e., for any $H \in \mathfrak{h}$,

$$\ll H_\alpha,\ H \gg\ =\ \alpha(H).$$

Choose $X_\alpha, Y_\alpha \in \mathfrak{k}$ such that $\ll E_\alpha,\ E_{-\alpha} \gg\ =\ 1$, where

$$E_\alpha\ =\ \frac{1}{2}(X_\alpha\ -\ iY_\alpha), \qquad \text{and} \qquad E_{-\alpha}\ =\ -\frac{1}{2}(X_\alpha\ +\ iY_\alpha)$$

are root vectors for α and $-\alpha$ respectively. It follows that $[E_\alpha, E_{-\alpha}] = H_\alpha$. Set

$$r\ =\ \frac{1}{4}\sum_{\alpha>0} X_\alpha \wedge Y_\alpha\ \in\ \mathfrak{k} \wedge \mathfrak{k}. \tag{7}$$

This is the well-known r-matrix for the Lie algebra $\mathfrak{k}$. Let r^R and r^L be respectively the right and left invariant bi-vector fields on K with values r at the identity element e. Define a bi-vector field π on K by

$$\pi\ =\ r^R\ -\ r^L. \tag{8}$$

Then π is a Poisson structure on K, and (K, π) becomes a Poisson Lie group [L-W].

Since the Poisson structure π on K is invariant by the right translations by elements in T, there is a unique Poisson structure on K/T, which will be denoted by π_∞ (the reason for this notation will become clear in Section 5), such that the natural projection $(K, \pi) \to (K/T, \pi_\infty)$ is a Poisson map. This Poisson structure π_∞ is called the Bruhat Poisson structure on K/T. It is so named because its symplectic leaves are exactly all the Bruhat cells in K/T [L-W]. In [Lu2], explicit formulas for π_∞ on each Bruhat cell are written down in coordinates.

Note that although the Bruhat Poisson structure is invariant under the left translations by elements in T, it is not invariant for elements in K. In fact, the action map

$$K\ \times K/T \longrightarrow K/T:\ (k_1,\ kT) \longmapsto k_1 kT \tag{9}$$

is a Poisson map, where $K \times K/T$ is equipped with the direct product Poisson structure $\pi \oplus \pi_\infty$. This is an example of a (K,π)-homogeneous Poisson space [D]. The classification of all (K,π)-homogeneous Poisson spaces is given by Karolinsky [Ka]. Some geometric properties of all (K,π)-homogeneous Poisson structures on K/T are studied in [Lu3]. A family of (K,π)-homogeneous Poisson structures π_λ on K/T will be discussed in Section 5.

Let μ_0 be a K-invariant volume form on K/T. The modular vector field θ_{μ_0} of π_∞ associated to μ_0 is shown in [E-L-W] to be the infinitesimal generator of the T action on K/T in the direction of the element $iH_\rho \in \mathfrak{t}$, where $2\rho = \sum_{\alpha>0} \alpha \in i\mathfrak{t}$.

Notation 3.1 Throughout this paper, the modular vector field θ_{μ_0} of π_∞ associated to μ_0 will be denoted by θ_0. Thus

$$\theta_0(kT) \;=\; \frac{d}{dt}|_{t=0} \exp(tiH_\rho)kT. \tag{10}$$

Notice that θ_0 is independent of the choice of the K-invariant volume form.

Consider now the Koszul-Brylinski operator defined by π_∞ and μ_0

$$\partial_{\pi_\infty,\mu_0} \;=\; i_{\pi_\infty} d \;-\; d i_{\pi_\infty} \;+\; i_{\theta_0} :\; \Omega^q(K/T) \longrightarrow \Omega^{q-1}(K/T).$$

Notation 3.2 For notational simplicity, we set

$$\partial_\infty \;=\; \partial_{\pi_\infty,\mu_0} :\; \Omega^q(K/T) \longrightarrow \Omega^{q-1}(K/T).$$

By Lemma 2.1, the operator ∂_∞ is related to the operator $\delta_{\pi_\infty} = [\pi_\infty, \bullet]$ on $\chi(K/T)$ via

$$(\delta_{\pi_\infty} X) \,\lrcorner\, \mu_0 \;=\; (-1)^{q-1} \partial_\infty (X \,\lrcorner\, \mu_0) \tag{11}$$

for $X \in \chi^q(K/T)$. Clearly, $d\partial_\infty + \partial_\infty d = L_{\theta_0}$, where L_{θ_0} is the Lie derivative operator by the vector field θ_0. The homology of the chain complex $(\Omega^\bullet(K/T),\, \partial_\infty)$, which is isomorphic to the Poisson cohomology of π_∞ by (11), can be shown to be isomorphic to the Lie algebra $\mathfrak{n}$-cohomology with coefficients in a certain infinite dimensional module. The calculation of this cohomology will be carried out in a separate paper. For now, we restrict ourselves to the space of K-invariant differential forms on K/T.

3.2 The mixed complex $(\Omega(K/T)^K,\, d,\, \partial_\infty)$

Denote by $\chi(K/T)^K$ the space of K invariant (real) multi-vector fields on K/T. A general fact about Poisson actions (see, for example, §7 in [Lu1]) is that the subspace $\chi(K/T)^K \subset \chi(K/T)$ is invariant under the operator $\delta_{\pi_\infty} = [\pi_\infty, \bullet]$. The cohomology of the cochain complex $(\chi(K/T)^K,\, \delta_{\pi_\infty})$ is called the K**-invariant Poisson cohomology** of $(K/T,\, \pi_\infty)$ (see [Lu1]).

Denote by $\Omega(K/T)^K$ the space of (real) K-invariant differential forms on K/T. Then since μ_0 is K-invariant, the map

$$\lrcorner\mu_0 :\ \chi^q(K/T)^K \longrightarrow \Omega^{n-q}(K/T)^K :\ X \longmapsto X \lrcorner\, \mu_0,$$

where $n = \dim(K/T)$, is an isomorphism of vector spaces. We know from (11) that the operator ∂_∞ on $\Omega(K/T)$ leaves the subspace $\Omega(K/T)^K$ invariant. It follows that $d\partial_\infty + \partial_\infty d = L_{\theta_0} = 0$ as operators on $\Omega(K/T)^K$. Thus on the graded vector space $\Omega(K/T)^K = \oplus_q \Omega^q(K/T)^K$, we have two anti-commuting operators d and ∂_∞, of degrees 1 and -1 respectively, such that $d^2 = 0$ and $\partial_\infty^2 = 0$. In other words, we have a mixed complex $(\Omega(K/T)^K,\, d,\, \partial_\infty)$ [Lo].

In Section 3.3, we will compare the operator ∂_∞ with the operator ∂ introduced by Kostant in [Ko2]. Since Kostant's operator ∂ is the Chevalley-Eilenberg boundary operator for a certain Lie algebra, we will need to first identify ∂_∞ as a such. Since Kostant considered complex valued differential forms on K/T, we will also need to complexify the mixed complex $(\Omega(K/T)^K,\, d,\, \partial_\infty)$. These two tasks occupy this section.

We first recall the Manin triple corresponding to the Poisson Lie group (K, π). Consider the real Lie subalgebra $\mathfrak{a}+\mathfrak{n}$ of $\mathfrak{g}$, where $\mathfrak{a} = i\mathfrak{t}$ and $\mathfrak{n} = \mathfrak{n}_+$ is the subalgebra of $\mathfrak{g}$ spanned by all the positive root vectors (but considered here as a real Lie algebra). Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$ (as a real semi-simple Lie algebra). The triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}+\mathfrak{n})$, together with twice the imaginary part of the Killing form $\langle\, ,\, \rangle := 2\mathrm{Im} \ll ,\, \gg$, is the Manin triple for the Poisson Lie group (K, π) [L-W].

Set $e = eT \in K/T$, where e is the identity element in K. Identify $T_e(K/T) \cong \mathfrak{k}/\mathfrak{t}$. The pairing $2\mathrm{Im} \ll ,\, \gg$ between $\mathfrak{k}$ and $\mathfrak{a}+\mathfrak{n}$ then induces a non-degenerate pairing between $\mathfrak{k}/\mathfrak{t}$ and $\mathfrak{n}$, and we get an identification (of real vector spaces)

$$I_\infty :\ \mathfrak{n} \stackrel{\sim}{\longrightarrow} T_e^*(K/T) :\ (I_\infty(x),\, y) \;=\; 2\mathrm{Im} \ll x,\, y \gg \tag{12}$$

for $x \in \mathfrak{n}$ and $y \in \mathfrak{k}/\mathfrak{t} \cong T_e(K/T)$. Note that I_∞ is T-equivariant, where T acts on $\mathfrak{n}$ by the Adjoint action and on $T_e^*(K/T)$ by the linearization at e of the T action on K/T. Since

$$\Omega(K/T)^K \;\cong\; \wedge(T_e^*(K/T))^T,$$

we get an identification, still denoted by I_∞:

$$I_\infty :\ (\wedge^q \mathfrak{n})^T \stackrel{\sim}{\longrightarrow} \Omega^q(K/T)^K$$

for each $0 \le q \le n = \dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}}(K/T)$. Let $I_\infty^* : T_e(K/T) \to \mathfrak{n}^*$ be the dual of I_∞. It gives rise to

$$I_\infty^* : \ \chi^q(K/T)^K \cong (\wedge^q T_e(K/T))^T \stackrel{\sim}{\longrightarrow} (\wedge^q \mathfrak{n}^*)^T.$$

Let

$$d_{\mathfrak{n}} : \ \wedge^q \mathfrak{n}^* \longrightarrow \wedge^{q+1} \mathfrak{n}^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra $\mathfrak{n}$ (considered as a real Lie algebra). Its restriction to the subspace $(\wedge \mathfrak{n}^*)^T \subset \wedge \mathfrak{n}^*$ is also denoted by $d_{\mathfrak{n}}$. Let

$$b_{\mathfrak{n}} : \ \wedge^q \mathfrak{n} \longrightarrow \wedge^{q-1} \mathfrak{n} : \ b_{\mathfrak{n}}(x_1 \wedge x_2 \wedge \cdots \wedge x_q) \ = \ \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q$$

be the Chevalley-Eilenberg boundary operator for $\mathfrak{n}$. Its restriction to $(\wedge \mathfrak{n})^T \subset \wedge \mathfrak{n}$ will also be denoted by $b_{\mathfrak{n}}$. The two operators $d_{\mathfrak{n}}$ and $b_{\mathfrak{n}}$ are dual to each other. We remark that our definition of the Chevalley-Eilenberg boundary operator for a Lie algebra differs from that used by Kostant in [Ko2] by a minus sign.

Theorem 3.3 *We have*

$$(I_\infty^*)^{-1} d_{\mathfrak{n}} I_\infty^* \ = \ \delta_{\pi_\infty} \ = \ [\pi_\infty, \bullet] \tag{13}$$

as degree 1 *operators on* $\chi(K/T)^K$*, and*

$$I_\infty b_{\mathfrak{n}} I_\infty^{-1} \ = \ \partial_\infty \tag{14}$$

as degree -1 *operators on* $\Omega(K/T)^K$*.*

A proof of a general fact about (K, π)-homogeneous Poisson structures on K/T, of which Theorem 3.3 is a special case, is given in [Lu3]. Here, we give a direct proof for the sake of completeness. We first prove a lemma about the Poisson Lie group (K, π). Denote by $\mathcal{I} : \mathfrak{k} \stackrel{\sim}{\longrightarrow} (\mathfrak{a} + \mathfrak{n})^*$ the identification via the pairing $2\mathrm{Im} \ll , \gg$, and use the same letter to denote the induced map

$$\mathcal{I} : \ \chi^q(K)^K \ \cong \ \wedge^q \mathfrak{k} \stackrel{\sim}{\longrightarrow} \wedge^q (\mathfrak{a} + \mathfrak{n})^*,$$

where $\chi^q(K)^K \cong \wedge^q \mathfrak{k}$ is the space of left invariant q-vector fields on K. Let

$$d_{\mathfrak{a}+\mathfrak{n}} : \ \wedge^q (\mathfrak{a} + \mathfrak{n})^* \longrightarrow \wedge^{q+1} (\mathfrak{a} + \mathfrak{n})^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra $\mathfrak{a} + \mathfrak{n}$. The following lemma says that the (left) K-invariant Poisson cohomology of (K, π) is isomorphic to the Lie algebra cohomology of the Lie algebra $\mathfrak{a} + \mathfrak{n}$.

Lemma 3.4 *$\delta_\pi = \mathcal{I}^{-1} d_{\mathfrak{a}+\mathfrak{n}} \mathcal{I}$ as degree 1 operators on $\chi(K)^K \cong \wedge \mathfrak{k}$.*

Proof. This is again a general fact about Poisson Lie groups specialized to the Poisson Lie group (K, π). To give a direct proof, we first notice that both operators are derivations of degree 1 on $\wedge\mathfrak{k}$. Thus it is enough to show that they are the same on $\mathfrak{k}$. This can be checked directly.

Q.E.D.

Proof of Theorem 3.3. Denote by $\chi(K)^{K;T}$ the space of multi-vector fields on K that are both left K-invariant and right T-invariant. It is invariant under the operator δ_π. By Lemma 3.4, we know that

$$\mathcal{I}: \ (\chi(K)^{K;T}, \, \delta_\pi) \longrightarrow \big((\wedge(\mathfrak{a}+\mathfrak{n})^*)^T, \, d_{\mathfrak{a}+\mathfrak{n}}\big) \tag{15}$$

is an isomorphism of cochain complexes. Now the projection from K to K/T induces a surjective cochain complex morphism

$$p_1: \ (\chi(K)^{K;T}, \, \delta_\pi) \longrightarrow (\chi(K/T)^K, \, \delta_{\pi_\infty}).$$

Similarly, the map

$$p_2: \ \big((\wedge(\mathfrak{a}+\mathfrak{n})^*)^T, \, d_{\mathfrak{a}+\mathfrak{n}}\big) \longrightarrow \big((\wedge\mathfrak{n}^*)^T, \, d_{\mathfrak{n}}\big)$$

induced by the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{a}+\mathfrak{n}$ is a surjective cochain complex morphism. Since $I_\infty^* p_1 = p_2 \mathcal{I}$, it follows from (15) that

$$I_\infty^*: \ (\chi(K/T)^K, \, \delta_{\pi_\infty}) \longrightarrow \big((\wedge\mathfrak{n}^*)^T, \, d_{\mathfrak{n}}\big)$$

is a cochain complex isomorphism. This proves (13).

We now show that (14) follows from (13). Let $\mu_0' := I_\infty^{-1}(\mu_0) \in \wedge^n\mathfrak{n}$, where $n = \dim_{\mathbb{R}} \mathfrak{n}$. Since $\mathfrak{n}$ is unimodular, a general fact about Lie algebra cohomology says that

$$(d_{\mathfrak{n}} X) \lrcorner \, \mu_0' \ = \ (-1)^{q-1} b_{\mathfrak{n}}(X \lrcorner \, \mu_0') \tag{16}$$

for any $X \in \wedge^q\mathfrak{n}^*$. Since μ_0' is T-invariant, the map $\lrcorner\mu_0'$ from $\wedge^q\mathfrak{n}$ to $\wedge^{n-q}\mathfrak{n}^*$ is T-equivariant. Thus it induces an isomorphism from $(\wedge^q\mathfrak{n}^*)^T$ to $(\wedge^{n-q}\mathfrak{n})^T$. Now (14) follows from (13) by comparing (11) and (16).

Q.E.D.

Remark 3.5 There is a simpler expression for ∂_∞ as an operator on the space $\Omega(K/T)^K$ (as opposed to on all of $\Omega(K/T)$). Recall the definition of ∂_∞ in (11) and the definition of the r-matrix r in (7). Use the same letter r to denote the K-invariant bi-vector field on K/T whose value at $e = eT \in K/T$ is r considered as in $\wedge^2(\mathfrak{k}/\mathfrak{t})$. Then, if $X \in \chi^q(K/T)$ is K-invariant, we have $[\pi_\infty, X] = -[r, X]$. A proof similar to that of Lemma 2.1 shows that $[r, X] \lrcorner \mu_0 = (-1)^{q-1}(i_r d - d i_r)(X \lrcorner \mu_0)$. Thus we have

$$\partial_\infty \;=\; -i_r d \,+\, d i_r$$

on $\Omega(K/T)^K$.

We now look at the complexification of the mixed complex $(\Omega(K/T)^K,\, d,\, \partial_\infty)$. We first need the complexification of the Manin triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}+\mathfrak{n}, 2\mathrm{Im} \ll\,,\,\gg)$. For this purpose, we use J_0 to denote the complex structure on $\mathfrak{g}$ as the complexification of $\mathfrak{k}$, and let $c_0 : \mathfrak{g} \to \mathfrak{g}$ be the complex conjugation on $\mathfrak{g}$ defined by $\mathfrak{k}$. Then the map

$$\Psi :\, (\mathfrak{g}_{\mathbb{C}},\, i) \longrightarrow (\mathfrak{g} \oplus \mathfrak{g},\, J_0 \oplus J_0) :\, x_1 + i x_2 \longmapsto (c_0(x_1 - J_0 x_2),\; x_1 + J_0 x_2)$$

is an isomorphism of complex Lie algebras. Accordingly, the complexifications of the various (real) Lie subalgebras of $\mathfrak{g}$ can be identified via Ψ with (complex) Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$. For example, denoting by

$$\mathfrak{g}_1^{\Delta} \;:=\; \{\, (x,\, x) :\; x \in \mathfrak{g}_1 \} \;\subset \mathfrak{g} \oplus \mathfrak{g}$$

for any complex subspace $\mathfrak{g}_1$ of $\mathfrak{g}$, we have the isomorphism

$$\Psi :\, \mathfrak{k}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g}^{\Delta}.$$

Similarly, for $H_1, H_2 \in \mathfrak{a}$,

$$\Psi(H_1 + iH_2) \;=\; (-H_1 - J_0 H_2,\; H_1 + J_0 H_2),$$

and thus we have the isomorphism

$$\Psi :\, \mathfrak{a}_{\mathbb{C}} \xrightarrow{\sim} \{(-H,\, H) :\, H \in \mathfrak{h}\}.$$

For $\mathfrak{n}$ as a real Lie algebra, we have the isomorphism

$$\Psi :\, \mathfrak{n}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{n}_- \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_+ = \mathfrak{n}$ and $\mathfrak{n}_- = c_0(\mathfrak{n})$ (both regarded as complex Lie subalgebras of $\mathfrak{g}$), with

$$\Psi(-\frac{1}{2}(E_\alpha + iJ_0 E_\alpha)) \;=\; (E_{-\alpha},\, 0) \qquad \Psi(\frac{1}{2}(E_\alpha - iJ_0 E_\alpha)) \;=\; (0,\, E_\alpha).$$

Set

$$\mathfrak{b}_- \oplus_{\mathfrak{h}} \mathfrak{b}_+ \;=\; \{(x_- - H,\; H + x_+) :\, x_\pm \in \mathfrak{n}_\pm, H \in \mathfrak{h}\}.$$

Then we have

$$\Psi:\ (\mathfrak{a}+\mathfrak{n})_{\mathbb{C}} \stackrel{\sim}{\longrightarrow} \mathfrak{b}_- \oplus_{\mathfrak{h}} \mathfrak{b}_+.$$

Denote by $\langle\,,\rangle_{\mathbb{C}}$ the complex-linear extension of the bilinear form $\langle\,,\,\rangle = 2\mathrm{Im} \ll , \gg$ from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$. Then under the identification of $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}\oplus\mathfrak{g}$ by Ψ, it becomes

$$\langle (x_1,x_2),\,(y_1,y_2)\rangle_{\mathbb{C}} \;=\; i(\ll x_1,y_1 \gg \;-\; \ll x_2,y_2 \gg).$$

The two Lie subalgebras $\mathfrak{g}^{\Delta}$ and $\mathfrak{b}_-\oplus_{\mathfrak{h}}\mathfrak{b}_+$ are both isotropic with respect to $\langle\,,\rangle_{\mathbb{C}}$. The triple $(\mathfrak{g}\oplus\mathfrak{g},\ \mathfrak{g}^{\Delta},\ \mathfrak{b}_-\oplus_{\mathfrak{h}}\mathfrak{b}_+)$, together with $\langle\,,\rangle_{\mathbb{C}}$, is a Manin triple of complex Lie algebras. It is isomorphic to the complexification $(\mathfrak{g}_{\mathbb{C}},\ \mathfrak{k}_{\mathbb{C}},\ (\mathfrak{a}+\mathfrak{n})_{\mathbb{C}},\ \langle\,,\,\rangle_{\mathbb{C}})$ of the Manin triple $(\mathfrak{g},\ \mathfrak{k},\ \mathfrak{a}+\mathfrak{n},\ \langle\,,\,\rangle)$ via the map Ψ.

Using the above complexification, we have the natural identification

$$(T_e(K/T))_{\mathbb{C}} \;\cong\; (\mathfrak{k}/\mathfrak{t})_{\mathbb{C}} \;\cong\; \mathfrak{g}/\mathfrak{h}.$$

Using $\mathfrak{n}_{\mathbb{C}}\cong\mathfrak{n}_-\oplus\mathfrak{n}_+$ given by Ψ, the complexification of the map I_∞ in (12), which we will still denote by I_∞, becomes

$$\begin{aligned} I_\infty:\quad & \mathfrak{n}_-\oplus\mathfrak{n}_+ \longrightarrow (T_e^*(K/T))_{\mathbb{C}}: \\ & (I_\infty(x_-,\,x_+),\ x) \;=\; i\ll x,\ x_- \;-\; x_+ \gg = \ll x,\ ix_- \;-\; ix_+ \gg \end{aligned} \tag{17}$$

for $x\in\mathfrak{g}/\mathfrak{h}\cong(T_e(K/T))_{\mathbb{C}}$. Notice the minus sign in front of ix_+. Notice also that there is another more standard identification

$$I:\ \mathfrak{n}_-\oplus\mathfrak{n}_+ \longrightarrow (T_e^*(K/T))_{\mathbb{C}}:\ (I(x_-,\,x_+),\ x) \;= \ll x,\ x_- \,+\, x_+ \gg \tag{18}$$

for $x\in\mathfrak{g}/\mathfrak{h}\cong(T_e(K/T))_{\mathbb{C}}$. As we will see, it is very important that our identification I_∞ is different from the standard one I.

At this point, we introduce a complex structure J on K/T: at $e\in K/T$, it is the complex structure on $T_e(K/T)\cong\mathfrak{k}/\mathfrak{t}$ coming from the identification $\mathfrak{k}/\mathfrak{t}\cong\mathfrak{g}/\mathfrak{b}_-$, where $\mathfrak{b}_-=\mathfrak{n}_-+\mathfrak{h}$. We extend J to all of K/T by left translations by elements in K. We will use the same letter J to denote its dual map on $T^*(K/T)$ as well as its complex linear extension to the complexified cotangent bundle $T^*_{\mathbb{C}}(K/T)$ and its multi-linear extension to $\wedge T^*_{\mathbb{C}}(K/T)$. If we identify $\mathfrak{k}/\mathfrak{t}\cong\mathrm{span}_{\mathbb{R}}\{X_\alpha,Y_\alpha:\alpha>0\}$, then

$$JX_\alpha \;=\; Y_\alpha, \qquad JY_\alpha \;=\; -X_\alpha.$$

Thus, if we use $\{X^\alpha,Y^\alpha:\ \alpha>0\}$ to denote the dual basis for $T_e^*(K/T)\cong(\mathfrak{k}/\mathfrak{t})^*$, then

$$JX^\alpha \;=\; -Y^\alpha, \qquad JY^\alpha \;=\; X^\alpha.$$

Lemma 3.6 *1) Under the identifications I_∞ in (17) or I in (18), the operator J on $(T_e^*(K/T))_{\mathbb{C}}$ becomes the following one on $\mathfrak{n}_- \oplus \mathfrak{n}_+$:*

$$I^{-1}JI = I_\infty^{-1}JI_\infty : \mathfrak{n}_- \oplus \mathfrak{n}_+ \longrightarrow \mathfrak{n}_- \oplus \mathfrak{n}_+ : (x_-, x_+) \longmapsto (ix_-, -ix_+).$$

2) The identifications I_∞ and I are related by

$$I_\infty I^{-1} = J : (T_e^*(K/T))_{\mathbb{C}} \longrightarrow (T_e^*(K/T))_{\mathbb{C}}.$$

Proof. 1) is straightforward, and 2) follows from the definitions of I_∞ and I and from 1).

Q.E.D.

Denote by $C = \Omega(K/T, \mathbb{C})^K$ the space of all K-invariant complex valued differential forms on K/T, and by $C^q = \Omega^q(K/T, \mathbb{C})^K$ the space of those of degree q. Denote the multilinear extension of I_∞ to $(\wedge^\bullet(\mathfrak{n}_- \oplus \mathfrak{n}_+))^T$ by the same letter:

$$I_\infty : (\wedge^q(\mathfrak{n}_- \oplus \mathfrak{n}_+))^T \xrightarrow{\sim} C^q, \tag{19}$$

and denote also by ∂_∞ the complex linear extension of ∂_∞ to C. By Theorem 3.3, we have

Theorem 3.7 *As operators on C,*

$$\partial_\infty = I_\infty b_{\mathfrak{n}_- \oplus \mathfrak{n}_+} I_\infty^{-1} : C^q \longrightarrow C^{q-1}, \tag{20}$$

where $b_{\mathfrak{n}_- \oplus \mathfrak{n}_+}$ is the Chevalley-Eilenberg boundary operator for the Lie algebra $\mathfrak{n}_- \oplus \mathfrak{n}_+$ as well as it restriction to $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$.

The complex structure J on K/T defines a bi-grading on $C : C = \oplus_{p,q} C^{p,q}$, where $C^{p,q}$ is the space of K-invariant differential forms on K/T of holomorphic degree p and anti-holomorphic degree q. By Lemma 3.6,

$$I_\infty \text{ and } I : (\wedge^p \mathfrak{n}_- \otimes \wedge^q \mathfrak{n}_+)^T \longrightarrow C^{p,q}$$

are both isomorphisms.

We now look at how the operator ∂_∞ respects the bi-grading of C. Denote by $\Pi_{p,q}$ the projection from $(\wedge \mathfrak{n}_- \otimes \wedge \mathfrak{n}_+)^T$ to $(\wedge^p \mathfrak{n}_- \otimes \wedge^q \mathfrak{n}_+)^T$, and set $\tau = \sum_{p,q} (-1)^p \Pi_{p,q}$. Since $\mathfrak{n}_- \oplus \mathfrak{n}_+$ has the direct sum Lie algebra structure, we have

$$b_{\mathfrak{n}_- \oplus \mathfrak{n}_+} = b_{\mathfrak{n}_-} \otimes 1 + \tau(1 \otimes b_{\mathfrak{n}_+}),$$

where $b_{\mathfrak{n}_-}$ and $b_{\mathfrak{n}_+}$ are the Chevalley-Eilenberg boundary operators for the Lie algebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ respectively. Set

$$\begin{aligned}
b^{'} &= b_{\mathfrak{n}_-} \otimes 1 : \ (\wedge^p \mathfrak{n}_- \otimes \wedge^q \mathfrak{n}_+)^T \longrightarrow (\wedge^{p-1} \mathfrak{n}_- \otimes \wedge^q \mathfrak{n}_+)^T \\
b^{''} &= \tau(1 \otimes b_{\mathfrak{n}_+}) : \ (\wedge^p \mathfrak{n}_- \otimes \wedge^q \mathfrak{n}_+)^T \longrightarrow (\wedge^p \mathfrak{n}_- \otimes \wedge^{q-1} \mathfrak{n}_+)^T \\
\partial^{'}_{\infty} &= I_\infty b^{'} I_\infty^{-1} : \ C^{p,q} \longrightarrow C^{p-1,q} \\
\partial^{''}_{\infty} &= I_\infty b^{''} I_\infty^{-1} : \ C^{p,q} \longrightarrow C^{p,q-1}.
\end{aligned}$$

Then, by Theorem 3.7, we have

$$\partial_\infty = \partial^{'}_\infty + \partial^{''}_\infty.$$

On the other hand, since J is integrable, we can write d as $d = d' + d^{''}$, where $d^{'} : C^{p,q} \to C^{p+1,q}$ and $d^{''} : C^{p,q} \to C^{p,q+1}$. Recall that the two operators d and ∂_∞ anti-commute on C, i.e., $d\partial_\infty + \partial_\infty d = 0$. Thus by reasons of degree, we have

Proposition 3.8 *The following holds on $C = \Omega(K/T, \mathbb{C})^K$:*

$$\begin{aligned}
(d^{'} \partial^{'}_\infty + \partial^{'}_\infty d^{'}) + (d^{''} \partial^{''}_\infty + \partial^{''}_\infty d^{''}) &= 0; \\
d^{'} \partial^{''}_\infty + \partial^{''}_\infty d^{'} = 0 &= d^{''} \partial^{'}_\infty + \partial^{'}_\infty d^{''}.
\end{aligned}$$

Remark 3.9 By Remark 3.5, we also have

$$\partial^{'}_\infty = -i_r d^{''} + d^{''} i_r, \quad \partial^{''}_\infty = -i_r d^{'} + d^{'} i_r.$$

The statements in Proposition 3.8 also follow from these two identities.

3.3 Kostant's operator ∂ versus the Koszul-Brylinski operator ∂_∞

We now recall the operator ∂ on $C = \Omega(K/T, \mathbb{C})^K$ introduced by Kostant in [Ko2]. In [Ko2], the map I given in (18) is used to identify $\mathfrak{n}_- \oplus \mathfrak{n}_+$ with $(T_e^*(K/T))_{\mathbb{C}}$. Set

$$\partial = -I b_{\mathfrak{n}_- \oplus \mathfrak{n}_+} I^{-1} : \ C^q \longrightarrow C^{q-1}. \tag{21}$$

This is the operator ∂ in [Ko2]. We will call ∂ the **Kostant operator**. The relation between the Kostant operator ∂ and the Koszul-Brylinski operator ∂_∞ defined by the Bruhat Poisson structure π_∞ is now clear from (20) and Lemma 3.6:

Theorem 3.10 *The Kostant operator ∂ and the Koszul-Brylinski operator ∂_∞ on $C = \Omega(K/T, \mathbb{C})^K$ are related by*

$$\partial = -J^{-1} \partial_\infty J = J \partial_\infty J^{-1},$$

where J is the complex structure on K/T (defined before Lemma 3.6).

Corollary 3.11 *On $C = \Omega(K/T, \mathbb{C})^K = \oplus_{p,q} C^{p,q}$, we have*

$$\partial \,=\, -i\partial_\infty' \,+\, i\partial_\infty''.$$

Proof. This follows immediately from Theorem 3.10 and the fact that $J|_{C^{p,q}} = i^{p-q}\mathrm{id}$.

Q.E.D.

Set

$$\partial' \,=\, -\,i\partial_\infty', \qquad \partial'' \,=\, i\partial_\infty'',$$

so that $\partial = \partial' + \partial''$. By Proposition 3.8 and Corollary 3.11, we have

Corollary 3.12 *The following hold on $C = \Omega(K/T, \mathbb{C})^K = \oplus_{p,q} C^{p,q}$:*

$$\begin{aligned} d'\partial' \,+\, \partial' d' \,&=\, d''\partial'' \,+\, \partial'' d''; \\ d'\partial'' \,+\, \partial'' d' \,&=\, 0 \,=\, d''\partial' \,+\, \partial' d''. \end{aligned}$$

Remark 3.13 The above facts are in Proposition 4.2 of [Ko2].

Theorem 3.14 *A form $\xi \in C^{p,q}$ is ∂-closed if and only if it is ∂_∞-closed.*

Proof. Let $\xi \in C^{p,q}$. Then

$$\partial\xi \,=\, -\,i\partial_\infty'\xi \,+\, i\partial_\infty''\xi \qquad \text{and} \qquad \partial_\infty\xi \,=\, \partial_\infty'\xi \,+\, \partial_\infty''\xi.$$

Thus $\partial\xi = 0$ if and only if $\partial_\infty'\xi = 0$ and $\partial_\infty''\xi = 0$, or, if and only if $\partial_\infty\xi = 0$.

Q.E.D.

In [Ko2], a K-invariant form ξ on K/T is said to be harmonic if it satisfies $d\xi = \partial\xi = 0$. We will also say that such forms are (d, ∂)-harmonic or say that they are "harmonic in the sense of Kostant".

Corollary 3.15 *A K-invariant form ξ on K/T of pure bi-degree is harmonic in the sense of Kostant if and only if it is harmonic with respect to the Bruhat Poisson structure π_∞ (and a K-invariant volume form μ_0) on K/T.*

3.4 Poisson geometrical proofs of Kostant's theorems

In this section, we will recall the main theorems in [Ko2]. We will give new proofs of some of them using the Bruhat Poisson structure.

Kostant's Theorem 1 (Theorem 4.5 in [Ko2]) *The operators d and ∂ on $C = \Omega(K/T, \mathbb{C})^K$ are disjoint, i.e.,*

$$\mathrm{Im}(d) \cap \mathrm{Ker}(\partial) \;=\; \mathrm{Im}(\partial) \cap \mathrm{Ker}(d) \;=\; 0.$$

In Section 5.4, we will give a proof of this theorem by using the Koszul-Brylinski operators for a family of symplectic structures on K/T. In this section, we assume this theorem and proceed to prove the other main results in [Ko2] using our Poisson interpretation of the operator ∂. Set

$$S \;=\; d\partial \,+\, \partial d \,\in\, \mathrm{End}(C).$$

It follows immediately from the disjointness of d and ∂ that $\xi \in \mathrm{Ker}(S)$ if and only if $d\xi = \partial\xi = 0$, i.e., ξ is (d, ∂)-harmonic, and that the maps

$$\psi_{d,S} :\; \mathrm{Ker}(S) \longrightarrow H(C, d) :\; \xi \longmapsto [\xi]_d, \tag{22}$$

$$\psi_{\partial,S} :\; \mathrm{Ker}(S) \longrightarrow H(C, \partial) :\; \xi \longmapsto [\xi]_\partial \tag{23}$$

are isomorphisms, where $H(C, d)$ and $H(C, \partial)$ are the cohomology groups of C defined by d and ∂, and $[\xi]_d$ and $[\xi]_\partial$ are the cohomology classes defined by ξ in $H(C, d)$ and $H(C, \partial)$ respectively. It follows from Corollary 3.12 that

$$S \;=\; 2(d^{'}\partial^{'} \,+\, \partial^{'}d^{'}) \;=\; 2(d^{''}\partial^{''} \,+\, \partial^{''}d^{''}) :\; C^{p,q} \longrightarrow C^{p,q}$$

(this is Proposition 4.2 in [Ko2]). Consequently, all the three spaces $\mathrm{Ker}(S)$, $H(C, d)$ and $H(C, \partial)$ are bi-graded, and the maps $\psi_{d,S}$ and $\psi_{\partial,S}$ are isomorphisms of bi-degree $(0, 0)$. The space $H(C, d)$ is of course the de Rham cohomology of K/T (with complex coefficients), and it is clear from the definition of ∂ that

$$I :\; (H_\bullet(\mathfrak{n}_-) \otimes H_\bullet(\mathfrak{n}_+))^T \longrightarrow H(C, \partial)$$

is a bi-degree $(0, 0)$ isomorphism, where $H_\bullet(\mathfrak{n}_-)$ and $H_\bullet(\mathfrak{n}_+)$ are respectively the Lie algebra homology groups of the Lie algebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$.

The space $(H_\bullet(\mathfrak{n}_-) \otimes H_\bullet(\mathfrak{n}_+))^T$ has a distinguished basis. Indeed, the space $H_\bullet(\mathfrak{n}_+)$ as a T-module is shown in [Ko1] to have weights exactly of the form $\rho - w\rho$, where w is in the Weyl group W, and each such weight has multiplicity 1. By pairing $\mathfrak{n}_-$ and $\mathfrak{n}_+$ by the Killing form $\ll\, , \gg$, we can regard $H_\bullet(\mathfrak{n}_-)$ as the contragradient of $H_\bullet(\mathfrak{n}_+)$ as T-modules.

The weight decomposition of $H_\bullet(\mathfrak{n}_+)$ then gives a basis for $(H_\bullet(\mathfrak{n}_-)\otimes H_\bullet(\mathfrak{n}_+))^T$ by Schur's Lemma. Explicitly, for $w \in W$, let

$$\Phi_w = \{\alpha_1, \alpha_2, ..., \alpha_{l(w)}\} = \{\alpha > 0 : w^{-1}\alpha < 0\},$$

where $l(w)$ is the length of w. Set $\beta_j = -w^{-1}\alpha_j$ for $j = 1, .., l(w)$ and

$$h^{w^{-1}} = i^{l(w)^2} E_{-\beta_1} \wedge E_{-\beta_2} \wedge \cdots \wedge E_{-\beta_{l(w)}} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_{l(w)}}. \tag{24}$$

Then $b_{\mathfrak{n}_-\oplus\mathfrak{n}_+}(h^{w^{-1}}) = 0$, and, up to scalar multiples, the set $\{[h^{w^{-1}}] : w \in W\}$ is the basis of $(H_\bullet(\mathfrak{n}_-)\otimes H_\bullet(\mathfrak{n}_+))^T$ determined by the weight decomposition of $H_\bullet(\mathfrak{n}_+)$.

Consider now the K-invariant forms $I(h^{w^{-1}}) \in C^{l(w),l(w)}$. They are ∂-closed and thus also ∂_∞-closed by Theorem 3.14. The following lemma will be used later.

Lemma 3.16 *Let $w, w_1 \in W$ be such that $l(w) = l(w_1)$. Regard w and w_1 as points in K/T. Then*

$$\left(I(h^{w_1^{-1}}),\ \frac{\pi_\infty^{l(w_1)}}{l(w_1)!} \right)(w) = \delta_{w,w_1}, \tag{25}$$

where $\pi_\infty^{l(w_1)}$ is the wedge product of π_∞ with itself $l(w_1)$-times.

Proof. Since $I(h^{w_1^{-1}})$ is K-invariant, we need to find $L_{\dot{w}^{-1}}\pi_\infty^{l(w)}(w) \in \wedge^{2l(w)}T_e(K/T)$, where $\dot{w} \in K$ is any representative of w in K, and $L_{\dot{w}^{-1}}$ is the map $K/T \to K/T : kT \mapsto \dot{w}^{-1}kT$ as well as its differential. By the definition of the Poisson structure π on K, we easily see that

$$L_{\dot{w}^{-1}}\pi(\dot{w}) = -i(E_{\beta_1} \wedge E_{-\beta_1} + \cdots + E_{\beta_{l(w)}} \wedge E_{-\beta_{l(w)}}) \in \mathfrak{g} \wedge \mathfrak{g}.$$

Here $L_{\dot{w}^{-1}}$ also denotes the left translation on K by $\dot{w}^{-1}$. Hence

$$\frac{L_{\dot{w}^{-1}}\pi_\infty^{l(w)}(w)}{l(w)!} = (-i)^{l(w)}(-1)^{\frac{l(w)(l(w)-1)}{2}} E_{\beta_1}\wedge E_{\beta_2}\wedge\cdots\wedge E_{\beta_{l(w)}} \otimes E_{-\beta_1}\wedge E_{-\beta_2}\wedge\cdots\wedge E_{-\beta_{l(w)}}.$$

Now (25) follows from the definitions of I and of $h^{w_1^{-1}}$.

Q.E.D.

The differential forms $I(h^{w^{-1}})$, for $w \in W$, although ∂_∞-closed, are in general not d-closed, and thus not harmonic (in the sense of Kostant or with respect to π_∞). But recall that the map $\psi_{\partial,S} : \mathrm{Ker}(S) \to H(C, \partial)$ in (23) is a bi-degree $(0, 0)$ isomorphism. It is used in [Ko2] to define harmonic forms:

Definition 3.17 For $w \in W$, set

$$s^w = \psi_{\partial,S}^{-1}\left([I(h^{w^{-1}})]_\partial\right) \in \mathrm{Ker}(S).$$

These are called the **Kostant harmonic forms**.

Thus, by definition, the form s^w has bi-degree $(l(w), l(w))$, and it is (d, ∂) as well as (d, ∂_∞)-harmonic. In other words, the Kostant harmonic forms are Poisson harmonic. We now review the explicit formulas for $s^w, w \in W$, as given in [Ko2].

Introduce the operators E_I and L_0 on $(\wedge \mathfrak{n}_- \otimes \wedge \mathfrak{n}_+)^T$ by:

$$E_I = 2 \sum_{\alpha>0} ad_{E_{-\alpha}} \otimes ad_{E_\alpha} \tag{26}$$

and

$$L_0|_{\mathbb{C}E_{-\alpha_1} \wedge E_{-\alpha_2} \wedge \cdots \wedge E_{-\alpha_p} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_q}}$$
$$= \begin{cases} 0 & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 = 0 \\ \frac{1}{\|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2} \mathrm{id} & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 \neq 0 \end{cases},$$

where $\|\lambda\|^2 = \ll \lambda, \lambda \gg$ for $\lambda \in \mathfrak{h}^*$. Set $R = -L_0 E_I$.

Kostant's Theorem 2 (Theorem 5.6 in [Ko2]) *The harmonic forms s^w, for $w \in W$, are explicitly given by*

$$s^w = I\left((1 - R)^{-1} h^{w^{-1}}\right) = I(h^{w^{-1}} + R h^{w^{-1}} + R^2 h^{w^{-1}} + \cdots). \tag{27}$$

The proof of this theorem is relatively easy, and it uses a fact that we will need later, so we review the main points in the proof now. We first introduce the Hermitian inner product h on $C = \Omega(K/T, \mathbb{C})^K$ as follows: the negative of the Killing form of $\mathfrak{k}$ gives rise to a symmetric positive definite inner product g^{-1} on $T_e(K/T) \cong \mathfrak{k}/\mathfrak{t}$. Correspondingly, we get a symmetric positive definite inner product g on $T_e^*(K/T)$. Now define h to be the Hermitian extension of g from $T_e^*(K/T)$ to $(T_e^*(K/T))_\mathbb{C}$, as well as its multi-linear extension to $C = \Omega(K/T, \mathbb{C})^K$. We will also regard h as a K-invariant Hermitian metric on K/T. In the basis $\{X_\alpha, Y_\alpha : \alpha > 0\}$ of $T_e(K/T) \cong \mathfrak{k}/\mathfrak{t}$ and its dual basis $\{X^\alpha, Y^\alpha : \alpha > 0\}$ of $T_e^*(K/T)$, we have

$$g^{-1}(X_\alpha, X_\alpha) = g^{-1}(Y_\alpha, Y_\alpha) = 2, \qquad g(X^\alpha, X^\alpha) = g(Y^\alpha, Y^\alpha) = \frac{1}{2}, \qquad \forall \alpha > 0,$$

and 0 otherwise. Under the identification I given in (18), the induced Hermitian inner product on $\mathfrak{n}_- \oplus \mathfrak{n}_+$ is given by

$$(h \circ I)(E_\alpha,\, E_\alpha) \;=\; (h \circ I)(E_{-\alpha},\, E_{-\alpha}) \;=\; 1 \qquad \forall \alpha > 0, \tag{28}$$

and 0 otherwise. Notice that, since the complex structure J is an isometry for h, we have $h \circ I = h \circ I_\infty$ by Lemma 3.6.

Let δ be the adjoint operator of ∂ with respect to the Hermitian inner product h. Set

$$L \;=\; \partial\delta \,+\, \delta\partial :\; C^q \longrightarrow C^q.$$

The eigenvalues and eigenvectors of this Laplacian L were determined by Kostant in [Ko1]. They are easier to describe if we identify C with $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$ by I: For notational simplicity, set $L_I = I^{-1}LI$, so it is an operator on $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$. Kostant showed in [Ko1] that with respect to the basis of $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$ given by

$$\{E_{-(\alpha)} \otimes E_{(\beta)} :\; (\alpha) = \{\alpha_1, ..., \alpha_p\},\, (\beta) = \{\beta_1, ..., \beta_q\} :\, \alpha_1 + \cdots + \alpha_p = \beta_1 + \cdots + \beta_q\},$$

where

$$E_{-(\alpha)} \;=\; E_{-\alpha_1} \wedge \cdots \wedge E_{-\alpha_p} \qquad \text{and} \qquad E_{(\beta)} \;=\; E_{\beta_1} \wedge \cdots \wedge E_{\beta_q},$$

the Laplacian L_I is diagonal, and that $L_I(E_{-(\alpha)} \otimes E_{(\beta)}) = 0$ if and only if $(\alpha) = \Phi_w = \{\alpha > 0 : w^{-1}\alpha < 0\}$ for some $w \in W$ and otherwise

$$L_I(E_{-(\alpha)} \otimes E_{(\beta)}) \;=\; (\|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2) E_{-(\alpha)} \otimes E_{(\beta)}.$$

Thus the operator L_0 is nothing but Green's operator for L_I. Define $E \in \mathrm{End}(C)$ by

$$S \;=\; L \,+\, E. \tag{29}$$

Let $E_I = I^{-1}EI$ be the corresponding operator on $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$.

Lemma 3.18 *The operator E_I is explicitly given by (26).*

This is Proposition 2.8 in [Ko2] (applied to the case when $\mathfrak{r} = \mathfrak{n}_- + \mathfrak{n}_+$ in the notation of [Ko2]). The proof as given in [Ko2] is simple, and we omit it here.

The explicit formula for the form s^w given in (27) is now easy to derive: By definition, the element $h^{w^{-1}}$ is the projection of $I^{-1}(s^w)$ to $\mathrm{Ker}(L_I)$ with respect to the decomposition $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T = \mathrm{Im}(L_I) + \mathrm{Ker}(L_I)$. Thus

$$\begin{aligned} I^{-1}(s^w) \;&=\; h^{w^{-1}} \,+\, L_0 L_I(I^{-1}(s^w)) \\ &=\; h^{w^{-1}} \,-\, L_0 E_I(I^{-1}(s^w)). \end{aligned}$$

Hence $s^w = I(1 + L_0 E_I)^{-1}(h^{w^{-1}})$.

Remark 3.19 It is clear that ∂ commutes with L and thus also with Green's operator of L on C. Since ∂ commutes with S, it also commutes with $E = S - L$. This fact will be used in Section 5.4.

Example 3.20 The operator S on C is in general not semi-simple, as is shown in the following example. Consider the case when $\mathfrak{g} = sl(4, \mathbb{C})$. Consider the space spanned by the three vectors

$$x = E_{32} \wedge E_{41} \otimes E_{23} \wedge E_{14} \quad y = E_{31} \wedge E_{41} \otimes E_{13} \wedge E_{14} \quad z = E_{42} \wedge E_{41} \otimes E_{24} \wedge E_{14}$$

in $(\wedge^2 \mathfrak{n}_- \otimes \wedge^2 \mathfrak{n}_+)^T$. Denote by S_I the operator $I^{-1}SI$ corresponding to S. It is easy to check that the Laplacian L_I acts by the same scalar on x, y, and z while E_I is strictly lower triangular with respect to the ordered basis (x, y, z). It follows that $S_I = L_I + E_I$ is not semisimple on the space spanned by these three vectors, and hence that S is not semisimple on C. This example shows that the Kostant operator ∂ is in general not adjoint to the de Rham operator d with respect to any Hermitian metric on K/T, for the operator S would be semi-simple if this were the case.

We now state the second main result in [Ko2], which is the first two properties of the forms $s^w, w \in W$ listed in the Introduction.

Kostant's Theorem 3 (Theorem 6.15 and Corollary 6.15 in [Ko2]) *The de Rham cohomology classes $[s^w]_d$, for $w \in W$, form a basis of $H(K/T, \mathbb{C})$ that is, up to scalar multiples λ_w, dual to the basis of the homology of K/T formed by the Schubert varieties. In fact, when $l(w) = l(w_1)$ but $w \neq w_1$, the form $i^*_{w_1} s^w$ on the Schubert cell Σ_{w_1} is identically zero, where $i_{w_1} : \Sigma_{w_1} \hookrightarrow K/T$ is the inclusion map.*

The numbers λ_w such that $\{[s^w]_d / \lambda_w : w \in W\}$ is the Schubert basis for $H(K/T, \mathbb{C})$ were later calculated by Kostant and Kumar in [K-K] to be

$$\lambda_w = \prod_{j=1}^{l(w)} \frac{2\pi}{\ll \rho,\, \alpha_j \gg}. \tag{30}$$

(Here, unlike elsewhere in the paper, π denotes the irrational number 3.14159....)

We now give a proof of the above properties using the Poisson interpretation of the harmonic forms s^w.

A Poisson Geometrical Proof of Kostant's Theorem 3. Since the maps $\psi_{d,S}$ and $\psi_{\partial,S}$ in (22) and (23) are both isomorphisms, we know that $\{[s^w]_d : w \in W\}$ is a basis for the de Rham cohomology space $H(K/T, \mathbb{C})$. It remains to show that

$$\int_{\Sigma_{w_1}} s^w = \begin{cases} 0 & \text{if } w_1 \neq w \\ \lambda_w & \text{if } w_1 = w \end{cases}$$

where λ_w is given in (30). This is clearly true if $l(w) \neq l(w_1)$. Assume now that $l(w) = l(w_1)$. For notational simplicity, we set

$$s_{w_1}^w = i_{w_1}^* s^w \in \Omega^{2l(w_1)}(\Sigma_{w_1}).$$

Denote by Ω_{w_1} the symplectic 2-form on Σ_{w_1} induced from π_∞ and by $\mu_{w_1} = \Omega_{w_1}^{l(w_1)}/l(w_1)!$ the Liouville volume form on Σ_{w_1} defined by Ω_{w_1}. We now relate $s_{w_1}^w$ and μ_{w_1}.

Recall that the Bruhat Poisson structure π_∞ is invariant under the T-action on K/T by left translations. All the Schubert cells are invariant under this T action. Thus, since Σ_{w_1} is simply-connected, there is a unique moment map, denoted by $\phi_{w_1} : \Sigma_{w_1} \to \mathfrak{t}^*$, for the T-action on Σ_{w_1} such that $\phi_{w_1}(w_1) = 0$. The function $\langle \phi_{w_1}, iH_\rho \rangle$ is then a Hamiltonian function for the vector field θ_0 on the symplectic manifold $(\Sigma_{w_1}, \Omega_{w_1})$. Consider the volume form $\mu = e^{\langle \phi_{w_1}, iH_\rho \rangle} \mu_{w_1}$ on Σ_{w_1}. Then by Example 2.6 in Section 2, the modular vector field with respect to the volume form μ and the Poisson structure π_∞ on Σ_{w_1} is iH_ρ. It follows that $\partial_1 i_{w_1}^* = i_{w_1}^* \partial_\infty$, where ∂_1 is the Koszul-Brylinski operator on Σ_{w_1} associated to μ. Hence, $i_{w_1}^*(s^w)$ is Poisson harmonic. But again by Example 2.6 in Section 2, a top degree form s on Σ_{w_1} is harmonic with respect to $\pi|_{\Sigma_{w_1}}$ and μ if and only if $s = c\mu$ for a constant function c on Σ_{w_1}. Hence $s_{w_1}^w$ is such a form on Σ_{w_1}. Thus there exists a constant $C_{w_1}^w$ such that

$$s_{w_1}^w = C_{w_1}^w e^{\langle \phi_{w_1}, iH_\rho \rangle} \mu_{w_1}.$$

By Lemma 3.16 and by the explicit formula for s^w, we can show easily that

$$\left(s^w, \ \frac{\pi_\infty^{l(w)}}{l(w)!} \right)(w_1) = \delta_{w, w_1}.$$

Thus $C_{w_1}^w = \delta_{w, w_1}$ and

$$i_{w_1}^* s^w = \begin{cases} 0 & \text{if } l(w) = l(w_1), w \neq w_1 \\ e^{\langle \phi_{w_1}, iH_\rho \rangle} \mu_{w_1} & \text{if } w = w_1. \end{cases}$$

The fact that $\int_{\Sigma_w} s^w = \lambda_w$ as given in (30) now follows from the Duistermaat-Heckman formula. See Remark 3.21.

Q.E.D.

Remark 3.21 Explicit formulas for the quantities s^w, μ_w and $\langle \phi_{w_1}, iH_\rho \rangle$ are written down in [Lu2] in certain coordinates on the Schubert cell Σ_w. In particular, the function $\langle \phi_{w_1}, iH_\rho \rangle$ goes to $-\infty$ towards the boundary of the closure of Σ_w in K/T. The integral $\int_{\Sigma_w} s^w$ can also be calculated directly using the formula for s^w in these coordinates. See [Lu2].

In [Ko2], Kostant actually constructed harmonic forms on G/P for any parabolic subgroup P of G that contains B_+. He showed that with respect to the projection $\nu : G/B_+ \to G/P$ induced by the inclusion $B_+ \to P$, the harmonic forms on G/P go into those on G/B_+. More precisely, let $W_P \subset W$ be the subgroup of W corresponding to P and let W^P be the set of minimal length representatives of the right coset space $W^P \backslash W$. For each $w \in W^P$, Kostant constructed a harmonic form s_P^w, and they have the properties stated in Kostant's Theorem 3 when K/T is replaced by $K/K \cap P$ and W by W^P. Proposition 6.10 of [Ko2] says that $\nu^*(s_P^w) = s^w$ for $w \in W^P$. This is the third property of the Kostant harmonic forms listed in the Introduction. We now give a Poisson geometrical proof of this fact.

Let $K_P = K \cap P$. Then there is a natural identification between K/K_P and G/P. It is shown in [L-W] that $K_P \subset K$ is a Poisson Lie subgroup, and thus there is an induced Poisson structure on K/K_P such that the natural projection $\nu : K/T \to K/K_P$ is a Poisson map. This induced Poisson structure on K/K_P is also called the Bruhat Poisson structure, because its symplectic leaves are also exactly the Bruhat (or Schubert) cells, i.e., the B_+-orbits, in $K/K_P \cong G/P$, which are now indexed by elements in W^P. We denote this Bruhat Poisson structure on K/K_P by $\pi_{P,\infty}$.

With respect to any K-invariant volume form on K/K_P, the modular vector field of $\pi_{P,\infty}$ is again the infinitesimal generator of the T-action on K/K_P in the direction of iH_ρ [E-L-W]. We will again use θ_0 to denote this vector field. Then the same arguments we have been using so far show that the forms s_P^w, for $w \in W^P$, are Poisson harmonic with respect to the Bruhat Poisson structure $\pi_{P,\infty}$ on K/K_P, i.e.,

$$(i_{\pi_{P,\infty}} d - d i_{\pi_{P,\infty}} + i_{\theta_0}) s_P^w = 0.$$

Because $\nu_* \pi_\infty = \pi_{P,\infty}$ and $\nu_* \theta_0 = \theta_0$, it follows immediately that $\nu^* s_P^w$ is also (d, ∂_∞)-harmonic. But since $\nu^* s_P^w$ and s^w have the same integrals on the Schubert cells in K/T (Lemma 6.6 in [Ko2]), they define the same de Rham cohomology classes in $H^\bullet(K/T, \mathbb{C})$. Thus they must be the same.

4 The S^1-equivariant cohomology of K/T

In this section, we give a Poisson geometrical proof of the fourth property of the Kostant harmonic forms listed in the Introduction. Namely, we show how the ring structure on the de Rham cohomology of K/T can be described using the Kostant harmonic forms. We do this by using equivariant cohomology.

Consider the S^1-action on K/T defined by the modular vector field θ_0. We will use $H_{S^1}(K/T, \mathbb{C})$ to denote the corresponding S^1-equivariant cohomology of K/T. The S^1-equivariant cohomology of a one point space is the polynomial ring $\mathbb{C}[u]$, and $H_{S^1}(K/T, \mathbb{C})$ is a $\mathbb{C}[u]$-algebra. Since the CW-complex structure on K/T defined by the Bruhat Decomposition is S^1-equivariant, we know [Ar] that $H_{S^1}(K/T, \mathbb{C})$ is in fact a free $\mathbb{C}[u]$-module with rank equal to the dimension of the ordinary de Rham cohomology of K/T, and correspondingly, there is a distinguished $\mathbb{C}[u]$-basis $\{\sigma^{(w)} : w \in W\}$, called the **Schubert basis**, of $H_{S^1}(K/T, \mathbb{C})$ that is characterized by the following properties:

1. $\deg \sigma^{(w)} = 2l(w)$;

2. $\int_{X_w} \sigma^{(w_1)} = \delta_{w,w_1}$ for $w, w_1 \in W$. Here, X_w is the closure of Σ_w in K/T.

3. Under the evaluation at 0:

$$e_0 : \; H_{S^1}(K/T, \mathbb{C}) \longrightarrow \mathbb{C} \otimes_{\mathbb{C}[u]} H_{S^1}(K/T, \mathbb{C}) \cong H^\bullet(K/T, \mathbb{C}) \tag{31}$$

   we have $\sigma^{(w)} \mapsto \sigma^w$, where $\{\sigma^w : w \in W\}$ is the Schubert basis for $H^\bullet(K/T, \mathbb{C})$.

Remark 4.1 In fact, this basis has the further property that $i_w^* \sigma^{(w_1)} = 0 \in H_{S^1}(X_w, \mathbb{C})$ unless $w_1 \leq w$, where i_w is the inclusion $X_w \hookrightarrow K/T$, and $H_{S^1}(X_w, \mathbb{C})$ is the S^1-equivariant cohomology of X_w. The partial order $\leq$ is the Bruhat order on W, i.e., $w_1 \leq w$ if and only if $\Sigma_{w_1} \subset X_w$.

The $\mathbb{C}[u]$-algebra structure on $H_{S^1}(K/T, \mathbb{C})$ is determined by its structure constants in the Schubert basis: For $w_1, w_2 \in W$, write

$$\sigma^{(w_1)} \sigma^{(w_2)} = \sum_{w \in W} c_w^{w_1,w_2} \sigma^{(w)} \tag{32}$$

with $c_w^{w_1,w_2} \in \mathbb{C}[u]$. Then, since $\deg \sigma^{(w)} = 2l(w)$, we know that $c_w^{w_1,w_2}$ is homogeneous and has degree $l(w_1) + l(w_2) - l(w)$ in u. Moreover, since

$$c_w^{w_1,w_2} = \int_{X_w} i_w^* \sigma^{(w_1)} i_w^* \sigma^{(w_2)},$$

it follows from Remark 4.1 that $c_w^{w_1,w_2} = 0$ unless $w_1 \leq w$ and $w_2 \leq w$. Since the map e_0 in (31) is a surjective homomorphism between algebras over $\mathbb{C}$, the values of the $c_w^{w_1,w_2}$'s at $u = 0$ are the structure constants for the algebra structure on the ordinary de Rham cohomology $H^\bullet(K/T, \mathbb{C})$ in the Schubert basis $\{\sigma^w : w \in W\}$. Thus we need to determine the polynomials $c_w^{w_1,w_2}$.

Introduce the $\mathbb{C}[u]$-module $\mathrm{Hom}_{\mathbb{C}[u]}(H_{S^1}(K/T,\mathbb{C}),\, \mathbb{C}[u])$. It has the obvious $\mathbb{C}[u]$-basis $\{\sigma_{(w)} : w \in W\}$ that is dual to the Schubert basis $\{\sigma^{(w)} : w \in W\}$. On the other hand, for each $w \in W$, by regarding w as an element in K/T (via any representative of w in K), we have the S^1-equivariant map

$$\{\mathrm{pt}\} \longrightarrow K/T : \ \mathrm{pt} \longmapsto w$$

which induces a $\mathbb{C}[u]$-algebra homomorphism $\psi_w \in \mathrm{Hom}_{\mathbb{C}[u]}(H_{S^1}(K/T,\mathbb{C}),\, \mathbb{C}[u])$. Write

$$\psi_w \;=\; \sum_{w_1 \in W} d_{w_1,w} \sigma_{(w_1)}$$

with $d_{w_1,w} \in \mathbb{C}[u]$. Then, by definition,

$$d_{w_1,w} \;=\; \psi_w(\sigma^{(w_1)}), \tag{33}$$

so $d_{w_1,w}$ has degree $l(w_1)$ in u. It again follows from Remark 4.1 that $d_{w_1,w} = 0$ unless $w_1 \leq w$.

Example 4.2 If α is a simple root and if $r_\alpha \in W$ is the simple reflection defined by α, then [Ar]

$$\psi_{r_\alpha} \;=\; \sigma_{(1)} \,+\, u \ll \alpha, \rho \gg, \sigma_{(r_\alpha)} \tag{34}$$

and thus,

$$d_{1,r_\alpha} \;=\; 1, \qquad d_{r_\alpha,r_\alpha} \;=\; u \ll \alpha,\, \rho \gg .$$

Notice that $\{\psi_w : w \in W\}$ is not a $\mathbb{C}[u]$-basis for $\mathrm{Hom}_{\mathbb{C}[u]}(H_{S^1}(K/T,\mathbb{C}),\, \mathbb{C}[u])$, as is seen from (34). Nevertheless, introduce the matrix

$$D \;=\; (d_{w_1,w})_{w_1,w \in W}. \tag{35}$$

Then, the localization theorem of equivariant cohomology [A-B] implies that the matrix D is invertible and that the entries of the matrix D^{-1} are in $\mathbb{C}(u)$, the algebra of Laurent polynomials in u. We now show how the structure constants $c_w^{w_1,w_2}$ can be determined by the matrices D and D^{-1}.

Proposition 4.3 *For each $w_1 \in W$, let D_{w_1} be the diagonal matrix with $\{d_{w_1,w}\}_{w \in W}$ as the diagonal. Let C_{w_1} be the matrix whose (w_2, w)-entry is $c_w^{w_1,w_2}$. Then*

$$C_{w_1} \;=\; D \cdot D_{w_1} \cdot D^{-1} \tag{36}$$

for any $w_1 \in W$.

Proof. Apply ψ_w to both sides of (32). We get

$$\begin{aligned}\text{l.h.s.} &= \psi_w(\sigma^{(w_1)}\sigma^{(w_2)}) = \psi_w(\sigma^{(w_1)})\psi_w(\sigma^{(w_2)}) = d_{w_1,w}d_{w_2,w}\\ \text{r.h.s.} &= \sum_{v\in W} d_{v,w}c_v^{w_1,w_2}.\end{aligned}$$

Thus $d_{w_1,w}d_{w_2,w} = \sum_{v\in W} d_{v,w}c_v^{w_1,w_2}$ for all w, w_1 and $w_2 \in W$, which is expressed in a more compact way in (36).

Q.E.D.

Therefore, to determine the structure constants $\{c_w^{w_1,w_2}\}$, it is enough to determine the matrix D.

Remark 4.4 If we consider the T-equivariant cohomology $H_T(K/T, \mathbb{C})$, a D-matrix, whose entries are polynomials on the Cartan subalgebra $\mathfrak{h}$, can be similarly defined. This is the D-matrix introduced in Section 4.21 of [K-K] by Kostant and Kumar. We refer to it as the full D-matrix. The matrix D we have here is the restriction of the full D-matrix to $uH_\rho \in \mathfrak{h}$.

To see how the Kostant harmonic forms $s^w, w \in W$, can be used to calculate the matrix D, we recall that these forms are also Poisson harmonic (Corollary 3.15) with respect to the Bruhat Poisson structure π_∞. Thus, we can apply Theorem 2.10 to get

Theorem 4.5 *For $w \in W$, set*

$$s^w(u) = i_{\exp_\wedge(-u\pi_\infty)}s^w = s^w - ui_{\pi_\infty}s^w + \frac{u^2}{2!}i_{\pi_\infty\wedge\pi_\infty}s^w + \cdots . \tag{37}$$

Then

1) Each $s^w(u)$ is S^1-equivariantly closed;

2) The set

$$\{\frac{[s^w(u)]}{\lambda_w} : \ w \in W\}, \tag{38}$$

where λ_w is the number given in (30), is the Schubert basis for $H_{S^1}(K/T, \mathbb{C})$.

Proof. 1) follows from Theorem 2.10. Clearly, $\deg s^w(u) = 2l(w)$ because $\deg u = 2$. Since $s^w(0) = s^w$, it follows from Kostant's Theorem 3 in Section 3.4 that the set in (38) goes to the Schubert basis of $H^\bullet(K/T, \mathbb{C})$ under the evaluation map e_0. We now need to show that $\int_{X_w} s^{w_1}(u) = \delta_{w,w_1}$. By replacing X_w with a T-equivariant resolution of singularities $Z \to X_w$ which is an isomorphism over Σ_w, we can show $\int_{X_w} s^{w_1}(u) = \int_{\Sigma_w} s^{w_1}(u)$. Thus, we only need to show $\int_{\Sigma_w} s^{w_1}(u) = \delta_{w,w_1}$. This is clearly true when $l(w_1) < l(w)$. When $l(w_1) = l(w)$, it follows again from Kostant's Theorem 3 in Section 3.4. It remains to show that $\int_{\Sigma_w} s^{w_1}(u) = 0$ when $l(w_1) > l(w)$. In this case, the only term in $s^{w_1}(u)$ that could possibly contribute to the integral is the term containing $i_{\pi_\infty^{l(w_1)-l(w)}} s^{w_1}$. But the pull-back of this term to Σ_w, which we will denote by τ, is zero for the following reason: since τ is a top degree form, its value at any point in Σ_w is determined by $i_{\pi_\infty^{l(w)}}\tau$ since π_∞ corresponds to a symplectic form on Σ_w. This last expression is zero because $\pi_\infty^{l(w_1)} = 0$ on Σ_w. Thus $\int_{\Sigma_w} s^{w_1}(u) = 0$. This shows that the set in (38) is the Schubert basis for $H_{S^1}(K/T, \mathbb{C})$.

Q.E.D.

Corollary 4.6 *The matrix D in (35) can be calculated from the Kostant harmonic forms $s^w, w \in W$, by*

$$d_{w_1,w} = \frac{1}{\lambda_{w_1}} u^{l(w_1)} \left(s^{w_1},\ \frac{\pi_\infty^{l(w_1)}}{l(w_1)!} \right)(w) = \frac{1}{\lambda_{w_1}} u^{l(w_1)} (s^{w_1},\ \exp_\wedge \pi_\infty)(w), \qquad (39)$$

where λ_w is given by (30).

Proof. Since the S^1-equivariantly closed form $\frac{s^{w_1}(u)}{\lambda_{w_1}}$ is a representative of the cohomology class $\sigma^{(w_1)}$, the polynomial $d_{w_1,w}$ is, by (33), the last term in the expansion of $s^{w_1}(u)$ in (37) evaluated at the point $w \in K/T$. Thus we have (39).

Q.E.D.

A description of how the entries $d_{w_1,w}$ of the matrix D can be obtained from the Kostant harmonic forms is given in Chapter 5 (Corollary 5.6) of [K-K]. Using the formula for $\pi_\infty(w_1)$ as given in the proof of Lemma 3.16, it is easy to see that our formula (39) for $d_{w_1,w}$ is the same as the one given in [K-K]. We have thus given a Poisson theoretical proof of the fourth property of the Kostant harmonic forms listed in the Introduction.

Remark 4.7 S. Kumar [Ku] has shown that the full D-matrix can be used to determine the singular locus of a Schubert variety and to determine the T-character of the tangent cone. We plan to extend our work in this paper to give a Poisson construction for the full D-matrix itself. It would be interesting to find other algebraic varieties which arise as closures of symplectic leaves in manifolds with Poisson structures. We remark that S. Billey [Bi] has given an explicit formula for $d_{w_1,w}$ using a reduced decomposition for w as a sum over reduced decompositions of w_1 that occur as subwords of w. D. Peterson [P] has a much more detailed analysis of the structures on the T-equivariant cohomology $H_T(K/T)$.

For each w, consider the function F^w on K/T given by

$$F^w(kT) \;=\; \left(s^w,\ \frac{\pi_\infty^{l(w)}}{l(w)!}\right)(kT) \;=\; (s^w,\ \exp_\wedge \pi_\infty)(kT). \tag{40}$$

Then Corollary 4.6 says that

$$d_{w_1,w} \;=\; \frac{1}{\lambda_{w_1}} u^{l(w_1)} F^{w_1}(w).$$

Thus, it is desirable to study the functions $F^w, w \in W$. In the following, we show that they are matrix entries of a finite dimensional representation of K that is equivalent to the Adjoint representation of K on $\wedge\mathfrak{g}$.

For each $w \in W$, let $\tilde{s}^w$ be the left invariant $2l(w)$-form on K that is the pull-back of s^w by the projection $K \to K/T$. Then $\tilde{s}^w(e)$ is a degree $2l(w)$ homogeneous element in $\wedge\mathfrak{g}^*$. Consider the operator $\mathcal{E}$ on $\wedge\mathfrak{g}$ defined by

$$\mathcal{E}(X) \;=\; (\exp_\wedge r) \wedge X \;=\; X + r \wedge X + \frac{1}{2!} r \wedge r \wedge X + \cdots,$$

where $r \in \mathfrak{k} \wedge \mathfrak{k} \subset \mathfrak{g} \wedge \mathfrak{g}$ is the r-matrix given in (7):

$$r \;=\; \frac{1}{4} \sum_{\alpha>0} X_\alpha \wedge Y_\alpha \;=\; \frac{i}{2} \sum_{\alpha>0} E_\alpha \wedge E_{-\alpha}.$$

Then $\mathcal{E}$ is invertible with $\mathcal{E}^{-1}(X) = (\exp_\wedge(-r)) \wedge X$. Define a representation of K on $\wedge\mathfrak{g}$ by making $k \in K$ act by $\mathcal{E}^{-1} \circ \mathrm{Ad}_k \circ \mathcal{E}$:

$$k \triangleright X \;:=\; \exp_\wedge(-r) \wedge \mathrm{Ad}_k((\exp_\wedge r) \wedge X) \;=\; \exp_\wedge(-r + \mathrm{Ad}_k r) \wedge \mathrm{Ad}_k X.$$

Then we have

Proposition 4.8 *For any $w \in W$ and $k \in K$,*

$$F^w(kT) \;=\; (\tilde{s}^w(e), \quad k^{-1} \triangleright 1),$$

where $k^{-1} \triangleright 1 = \exp_\wedge(-r + \mathrm{Ad}_{k^{-1}} r)$.

Proof. Recall that $\pi = r^R - r^L$ is the Poisson structure on K (see (8)) that makes (K, π) into a Poisson Lie group, where r^R and r^L are respectively the right and left invariant bi-vector fields on K with values r at the identity element e. Since, by the definition of π_∞, the natural projection $(K, \pi) \to (K/T, \pi_\infty)$ is a Poisson map, we have

$$\begin{aligned}
F^w(kT) &= (s^w,\ \exp_\wedge \pi_\infty)(kT) \\
&= (\tilde{s}^w,\ \exp_\wedge \pi)(k) \\
&= (\tilde{s}^w,\ (\exp_\wedge r)^R (\exp_\wedge(-r))^L) \\
&= (\tilde{s}^w(e),\ (\exp_\wedge(-r)) \wedge \mathrm{Ad}_{k^{-1}}(\exp_\wedge r)) \\
&= (\tilde{s}^w(e),\ k^{-1} \triangleright 1).
\end{aligned}$$

Q.E.D.

In a separate paper, we plan to study further properties of these functions F^w for $w \in W$, as well as similar functions for the full D-matrix.

5 Kostant's harmonic forms as limits of Hodge harmonic forms

In this section, we first introduce a family of Poisson (in fact symplectic) structures π_λ on K/T, where $\lambda \in \mathfrak{a}$ is regular. We study Poisson harmonic forms for each π_λ and show that, when $\lambda \in \mathfrak{h}^*$ is dominant, they are the same as Hodge harmonic forms for certain Hermitian metrics h_λ on K/T. We show that the Kostant operator ∂ is the limit as $\lambda \to \infty$ of the adjoint operators $d_{*,\lambda}$ of d with respect to the Hermitian metrics h_λ. As a consequence, the Laplacian $S = d\partial + \partial d$ defined by Kostant is the limit of the Hodge Laplacians of d with respect to the Hermitian metrics h_λ. This will enable us to give a new proof of Kostant's theorem that d and ∂ are disjoint. Finally, we define Hodge harmonic forms s_λ^w for $w \in W$ and show that they tend to the Kostant harmonic forms s^w as $\lambda \to \infty$.

5.1 The family of symplectic structures π_λ

We first recall some facts about the Poisson Lie group (K, π). Details can be found in [L-W].

Let $G = K_{\mathbb{C}}$ be the complexification of K, but here regarded as a real Lie group. Let A and N be the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then $G = KAN$ is an Iwasawa decomposition of G as a real semi-simple Lie group. The group AN has a unique Poisson structure π_{AN} making (AN, π_{AN}) into the dual Poisson Lie group of (K, π). The group G is then the double group for (K, π). The decomposition $G = KAN$ gives rise to the left action of K on AN:

$$K \times AN \longrightarrow AN: \ (k, b) \longmapsto k \cdot b := b_1, \qquad \text{if } bk^{-1} = k_1 b_1 \text{ for } k_1 \in K \text{ and } b_1 \in AN.$$

It is called the (left) dressing action of K on AN, and its orbits are exactly the symplectic leaves of π_{AN} in AN. Thus each dressing orbit inherits a symplectic, and thus Poisson, structure as a symplectic leaf. Since the dressing action is Poisson [STS] [L-W], these dressing orbits are examples of (K, π)-homogeneous Poisson spaces. Let $\lambda \in \mathfrak{a}$ be regular and consider the element $e^{-\lambda} \in A$. The stabilizer subgroup of K in AN at $e^{-\lambda}$ is T. Thus, by identifying K/T with the dressing orbit through $e^{-\lambda}$, we get a Poisson structure on K/T which is in fact symplectic.

Notation 5.1 For $\lambda \in \mathfrak{a}$ regular, we will use π_λ to denote the Poisson structure on K/T obtained by identifying K/T with the symplectic leaf in AN through the point $e^{-\lambda}$. We call it the *dressing orbit Poisson (or symplectic) structure corresponding to* $e^{-\lambda}$.

The following proposition is proved in [Lu3].

Proposition 5.2 *The Poisson structure π_λ on K/T is explicitly given by*

$$\pi_\lambda = \left(\sum_{\alpha>0} \frac{1}{1-e^{2\alpha(\lambda)}} \frac{X_\alpha \wedge Y_\alpha}{2} \right)^0 + \pi_\infty,$$

where the first term on the right hand side is the K-invariant bi-vector field on K/T whose value at $e = eT$ is the expression given in the parentheses. The modular vector field of π_λ with respect to a K-invariant volume form on K/T is again the vector field θ_0 given in (10).

Notation 5.3 We use ∂_λ to denote the Koszul-Brylinski operator defined by the Poisson structure π_λ and μ_0:

$$\partial_\lambda = \partial_{\pi_\lambda, \mu_0} = i_{\pi_\lambda} d - d i_{\pi_\lambda} + i_{\theta_0} : \Omega^q(K/T) \longrightarrow \Omega^{q-1}(K/T).$$

The following proposition explains the notation π_∞ and ∂_∞ we have given to the Bruhat Poisson structure and its associated Koszul-Brylinski operator:

Proposition 5.4 *For any $\lambda \in \mathfrak{a}$,*

$$\lim_{t\to+\infty} \pi_{\lambda+tH_\rho} = \pi_\infty \qquad \text{and} \qquad \lim_{t\to+\infty} \partial_{\lambda+tH_\rho} = \partial_\infty$$

in the C^∞-topology on the space of tensors on K/T.

Proof. This is because $\rho(\alpha) > 0$ for each $\alpha > 0$, and thus $\lim_{t\to+\infty} \frac{1}{1-e^{2\alpha(\lambda+tH_\rho)}} = 0$.

Q.E.D.

As in the case of the Bruhat Poisson structure π_∞, since π_λ is (K, π)-homogeneous and since the volume form μ_0 is K-invariant, the operator ∂_λ leaves invariant the space $\Omega(K/T)^K$ of K-invariant (real) differential forms on K/T, and we have the mixed complex $(\Omega(K/T)^K, d, \partial_\lambda)$ for each regular element $\lambda \in \mathfrak{a}$.

Remark 5.5 The restriction of ∂_λ to the space $\Omega(K/T)^K$ of K-invariant forms on K/T has a simpler expression, as for ∂_∞ (see Remark 3.5). Namely, consider the element

$$r_\lambda = \frac{1}{4} \sum_{\alpha>0} \frac{e^{\alpha(\lambda)} + e^{-\alpha(\lambda)}}{e^{\alpha(\lambda)} - e^{-\alpha(\lambda)}} X_\alpha \wedge Y_\alpha$$

as an element in $\wedge^2(\mathfrak{k}/\mathfrak{t})$ and extend it K-invariantly to a bi-vector field on K/T, which will be denoted by the same letter. Then

$$\partial_\lambda = -i_{r_\lambda} d + d i_{r_\lambda}$$

on $\Omega(K/T)^K$. Notice that this is not true on the space $\Omega(K/T)$ of all differential forms on K/T. The element r_λ is the skew-symmetric part of a *classical dynamical r-matrix* studied by Etingof and Varchenko in [E-V]. Connections between such r-matrices and homogeneous Poisson structures are given in [Lu3].

As in the case for the Bruhat Poisson structure π_∞, we will first identify the operator ∂_λ on $\Omega(K/T)^K$ with the Chevalley-Eilenberg boundary operator for some Lie algebra. For this purpose, we introduce the following real Lie subalgebra of $\mathfrak{g}$:

$$\mathfrak{l}_\lambda = \mathrm{Ad}_{e^\lambda}\mathfrak{k} \in \mathfrak{g}.$$

It is isotropic with respect to the bi-linear form $2\mathrm{Im} \ll , \gg$ and is maximal with this property since it has half of the real dimension of $\mathfrak{g}$. Moreover, $\mathfrak{l}_\lambda \cap \mathfrak{k} = \mathfrak{t}$. Under the correspondence established by Drinfeld [D] between maximal isotropic Lie algebras $\mathfrak{l}$ of $\mathfrak{g}$ satisfying $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{t}$ and (K, π)-homogeneous Poisson structures on K/T, the Lie algebra $\mathfrak{l}_\lambda$ corresponds to the Poisson structure π_λ. This fact is proved in [Lu1] and [Lu3].

Remark 5.6 In the case of π_∞, the corresponding maximal isotropic Lie subalgebra of $\mathfrak{g}$ is $\mathfrak{t} + \mathfrak{n}$. Since

$$\mathrm{Ad}_{e^\lambda} X_\alpha = e^{\alpha(\lambda)}(E_\alpha - e^{-2\alpha(\lambda)} E_{-\alpha}), \quad \mathrm{Ad}_{e^\lambda} Y_\alpha = e^{\alpha(\lambda)}(iE_\alpha + e^{-2\alpha(\lambda)} iE_{-\alpha}),$$

we have $\lim_{t\to+\infty} \mathfrak{l}_{\lambda+tH_\rho} = \mathfrak{t} + \mathfrak{n}$ in the Grassmannian of subspaces of $\mathfrak{g}$ of half of its (real) dimension. This fact corresponds to the statement in Proposition 5.4.

As always, we identify

$$T_e(K/T) \cong \mathfrak{k}/\mathfrak{t} \qquad \text{and} \qquad T_e^*(K/T) \cong (\mathfrak{k}/\mathfrak{t})^*.$$

The bi-linear form $2\mathrm{Im} \ll , \gg$ now gives an identification

$$I_\lambda : \ \mathfrak{l}_\lambda/\mathfrak{t} \stackrel{\sim}{\longrightarrow} T_e^*(K/T) : \ (I_\lambda(x+\mathfrak{t}), y+\mathfrak{t}) = 2\mathrm{Im} \ll x, y \gg, \tag{41}$$

where $x + \mathfrak{t} \in \mathfrak{l}_\lambda/\mathfrak{t}$ and $y + \mathfrak{t} \in \mathfrak{k}/\mathfrak{t}$. Its dual map

$$I_\lambda^* : \ T_e(K/T) \stackrel{\sim}{\longrightarrow} (\mathfrak{l}_\lambda/\mathfrak{t})^* \tag{42}$$

is defined by the same bi-linear form. They give rise to identifications, still denoted by the same letters,

$$I_\lambda : \ (\wedge^q(\mathfrak{l}_\lambda/\mathfrak{t}))^T \xrightarrow{\sim} \Omega^q(K/T)^K, \qquad I_\lambda^* : \ \chi^q(K/T)^K \xrightarrow{\sim} (\wedge^q(\mathfrak{k}/\mathfrak{t})^*)^T.$$

Denote by

$$d_{\mathfrak{l}_\lambda} : \ (\wedge^q(\mathfrak{l}_\lambda/\mathfrak{t})^*)^T \longrightarrow (\wedge^{q+1}(\mathfrak{l}_\lambda/\mathfrak{t})^*)^T$$

the Chevalley-Eilenberg coboundary operator that calculates the relative Lie algebra cohomology of $\mathfrak{l}_\lambda$ relative to $\mathfrak{t}$, and by

$$b_{\mathfrak{l}_\lambda} : \ (\wedge^q(\mathfrak{l}_\lambda/\mathfrak{t}))^T \longrightarrow (\wedge^{q-1}(\mathfrak{l}_\lambda/\mathfrak{t}))^T$$

the Chevalley-Eilenberg boundary operator that calculates the relative Lie algebra homology of $\mathfrak{l}_\lambda$ relative to T [Kn]. The following statement is a special case of a general fact about Poisson homogeneous spaces [Lu1] [Lu3].

Proposition 5.7 *We have*

$$(I_\lambda^*)^{-1} d_{\mathfrak{l}_\lambda} I_\lambda^* \ = \ \delta_{\pi_\lambda} \ = \ [\pi_\lambda, \bullet] \tag{43}$$

as degree 1 *operators on* $\chi(K/T)^K$, *and*

$$I_\lambda b_{\mathfrak{l}_\lambda} I_\lambda^{-1} \ = \ \partial_\lambda \tag{44}$$

as degree -1 *operators on* $\Omega(K/T)^K$.

Unlike $\mathfrak{t} + \mathfrak{n}$, the Lie algebra $\mathfrak{l}_\lambda$ is isomorphic to the Lie algebra $\mathfrak{k}$. As a result, the operator ∂_λ, when $\lambda \in \mathfrak{h}^*$ is dominant, is closely related to the adjoint operator of d with respect to a certain Riemannian metric on K/T. To show this, assume that $\lambda \in \mathfrak{h}^*$ is dominant. Introduce the following symmetric and positive definite inner product g_λ on $T_e^*(K/T) \cong (\mathfrak{k}/\mathfrak{t})^*$:

$$g_\lambda(X^\alpha, \, X^\alpha) \ = \ g_\lambda(Y^\alpha, \, Y^\alpha) \ = \ \frac{1}{2(e^{\alpha(\lambda)} - e^{-\alpha(\lambda)})}$$

and 0 otherwise, where $\{X^\alpha, Y^\alpha : \alpha > 0\}$ is the basis of $T_e^*(K/T)$ that is dual to the basis $\{X_\alpha, Y_\alpha : \alpha > 0\}$ of $T_e(K/T)$. Extend g_λ to all of K in a K-invariant manner, and denote by g_λ again the induced (symmetric and positive definite) inner product on $\Omega(K/T)^K$. Let J be the K-invariant complex structure on K/T defined in Section 3.2. Let $d_{*,\lambda}$ be the adjoint operator of d with respect to g_λ.

Theorem 5.8 *Assume that $\lambda \in \mathfrak{h}^*$ is dominant. Then, as operators on $\Omega(K/T)^K$, we have*

$$d_{*,\lambda} = J\partial_\lambda J^{-1}.$$

Proof. Denote by

$$b_{\mathfrak{k}}: \wedge^q(\mathfrak{k}/\mathfrak{t})^T \longrightarrow \wedge^{q-1}(\mathfrak{k}/\mathfrak{t})^T$$

the Chevalley-Eilenberg boundary operator that calculates the relative Lie algebra homology of $\mathfrak{k}$ relative to $\mathfrak{t}$ (see [Kn]). Then, since $\mathrm{Ad}_{e^\lambda}: \mathfrak{k} \to \mathfrak{l}_\lambda$ is a Lie algebra isomorphism, we know from Proposition 5.7 that

$$\begin{aligned} \partial_\lambda &= (I_\lambda \circ \mathrm{Ad}_{e^\lambda}) \circ b_{\mathfrak{k}} \circ (I_\lambda \circ \mathrm{Ad}_{e^\lambda})^{-1} \\ J\partial_\lambda J^{-1} &= (J \circ I_\lambda \circ \mathrm{Ad}_{e^\lambda}) \circ b_{\mathfrak{k}} \circ (J \circ I_\lambda \circ \mathrm{Ad}_{e^\lambda})^{-1}. \end{aligned}$$

It is easy to check that the map $(J \circ I_\lambda \circ \mathrm{Ad}_{e^\lambda})^{-1}: (\mathfrak{k}/\mathfrak{t})^* \to \mathfrak{k}/\mathfrak{t}$ is explicitly given by

$$X^\alpha \longrightarrow \frac{1}{2(e^{\alpha(\lambda)} - e^{-\alpha(\lambda)})} X_\alpha, \qquad Y^\alpha \longrightarrow -\frac{1}{2(e^{\alpha(\lambda)} - e^{-\alpha(\lambda)})} Y_\alpha, \qquad \alpha > 0.$$

Thus $(J \circ I_\lambda \circ \mathrm{Ad}_{e^\lambda})^{-1} = \tilde{g}_\lambda$, where $\tilde{g}_\lambda : (\mathfrak{k}/\mathfrak{t})^* \to \mathfrak{k}/\mathfrak{t}$ is given by $(\tilde{g}_\lambda(\xi), \eta) = g_\lambda(\xi, \eta)$ for $\xi, \eta \in (\mathfrak{k}/\mathfrak{t})^*$. Since the operators $b_{\mathfrak{k}}$ and d are dual to each other, Theorem 5.8 follows by letting $\sigma = J \circ I_\lambda \circ \mathrm{Ad}_{e^\lambda}$ in the following lemma.

Q.E.D.

Lemma 5.9 *Suppose that V is a finite dimensional real vector space and that $\sigma: V \to V^*$ is a linear isomorphism that is also symmetric (i.e., $\sigma = \sigma^*$). Define the symmetric bilinear form $\langle\, , \rangle_\sigma$ on V^* by $\langle \xi, \eta \rangle_\sigma = (\xi, \sigma^{-1}(\eta))$ for $\xi, \eta \in V^*$. Then for any $T \in \mathrm{End}(V)$ with $T^* \in \mathrm{End}(V^*)$ being its dual map, the linear map $\sigma T \sigma^{-1}$ on V^* is equal to the transpose of T^* with respect to $\langle\, , \rangle_\sigma$.*

Proof. For any $\xi, \eta \in V^*$, we have

$$\langle \sigma T \sigma^{-1}(\xi), \eta \rangle_\sigma = (T\sigma^{-1}(\xi), \eta) = (\sigma^{-1}(\xi), T^*\eta) = \langle \xi, T^*\eta \rangle_\sigma.$$

Q.E.D.

Consider now the complex picture. We complex linearly extend g_λ to a symmetric bilinear form on the complexified cotangent bundle $T^*_{\mathbb{C}}(K/T)$. All the operators: J, ∂_λ, d and $d_{*,\lambda}$ are also complex linearly extended to operators on $\Omega(K/T, \mathbb{C})$ but still denoted by the same letters. Let h_λ be the Hermitian extension of g_λ to $T^*_{\mathbb{C}}(K/T)$, i.e.,

$$h_\lambda(\xi_1, \xi_2) = g_\lambda(\xi_1, \bar{\xi}_2),$$

where $-$ is the complex conjugation on $T^*_{\mathbb{C}}(K/T)$. Then the operator $d_{*,\lambda}$ is also adjoint to d with respect to the Hermitian inner product on $C = \Omega(K/T, \mathbb{C})^K$ induced by h_λ, which we will also denote by h_λ. It is easy to see that the Hermitian metric h_λ on $(T^*_e(K/T))_{\mathbb{C}}$ becomes the following one on $\mathfrak{n}_- \oplus \mathfrak{n}_+$ under the identification I:

$$(h_\lambda \circ I)(E_\alpha, E_\alpha) = (h_\lambda \circ I)(E_{-\alpha}, E_{-\alpha}) = \frac{1}{e^{\alpha(\lambda)} - e^{\alpha(\lambda)}}$$

and 0 otherwise. Compare h_λ with the Hermitian metric h on $(T^*_e(K/T))_{\mathbb{C}}$ defined in Section 3.4 in (28). Note that $\lim_{t\to+\infty} h_{\lambda+tH_\rho} = 0$.

5.2 Poisson harmonic forms for π_λ

We continue to assume that $\lambda \in \mathfrak{h}^*$ is dominant. The decomposition of $C = \Omega(K/T, \mathbb{C})^K$ as $C = \oplus_{p,q} C^{p,q}$ is orthogonal with respect to the Hermitian inner product h_λ. Thus

$$d_{*,\lambda} = d^{'}_{*,\lambda} + d^{''}_{*,\lambda},$$

where $d^{'}_{*,\lambda} : C^{p,q} \longrightarrow C^{p-1,q}$ and $d^{''}_{*,\lambda} : C^{p,q} \longrightarrow C^{p,q-1}$ are respectively the adjoint operators of $d^{'}$ and $d^{''}$ with respect to h_λ. Since $\partial_\lambda = J^{-1} d_{*,\lambda} J$, we know that $\partial_\lambda(C^{p,q}) \subset C^{p-1,q} \oplus C^{p,q-1}$. (This fact also follows from Remark 5.5.) Thus, we can write

$$\partial_\lambda = \partial^{'}_\lambda + \partial^{''}_\lambda,$$

where $\partial^{'}_\lambda : C^{p,q} \to C^{p-1,q}$ and $\partial^{''}_\lambda : C^{p,q} \to C^{p,q-1}$ The following proposition now follows immediately from the fact that $J|_{C^{p,q}} = i^{p-q}$ id.

Proposition 5.10 *The two decompositions $d_{*,\lambda} = d^{'}_{*,\lambda} + d^{''}_{*,\lambda}$ and $\partial_\lambda = \partial^{'}_\lambda + \partial^{''}_\lambda$ are related by*

$$d^{'}_{*,\lambda} = -i\partial^{'}_\lambda \qquad \text{and} \qquad d^{''}_{*,\lambda} = i\partial^{''}_\lambda.$$

Introduce the Hodge Laplacian of d with respect to the Hermitian metric h_λ:

$$S_\lambda = dd_{*,\lambda} + d_{*,\lambda}d.$$

The following is a corollary of Proposition 5.10 and the fact that $d\partial_\lambda + \partial_\lambda d = 0$ on C:

Corollary 5.11 *On $C = \oplus_{p,q} C^{p,q}$, we have*

$$d^{'} d^{''}_{*,\lambda} + d^{''}_{*,\lambda} d^{'} = 0, \qquad d^{''} d^{'}_{*,\lambda} + d^{'}_{*,\lambda} d^{''} = 0,$$

and

$$S_\lambda = 2(d^{'} d^{'}_{*,\lambda} + d^{'}_{*,\lambda} d^{'}) = 2(d^{''} d^{''}_{*,\lambda} + d^{''}_{*,\lambda} d^{''})$$

has bi-degree $(0,0)$*. Consequently, the natural map*

$$\psi_{d,S_\lambda}:\ \mathrm{Ker}\,(S_\lambda) \longrightarrow H(C,d):\ \xi \longmapsto [\xi]_d$$

is an isomorphism of bi-degree $(0,0)$*.*

We also immediately have

Corollary 5.12 *A* K*-invariant (complex valued) differential form* ξ *on* K/T *of pure bi-degree is harmonic with respect to the Poisson structure* π_λ *(and a* K*-invariant volume form* μ_0*) if and only if it is* $(d, d_{*,\lambda})$*-harmonic, i.e., it is Hodge harmonic with respect to the Hermitian metric* h_λ *on* K/T*.*

Corollary 5.13 *Every class in the de Rham cohomology of* K/T *has a unique representative that is Poisson harmonic with respect to the Poisson structure* π_λ *(and a* K*-invariant volume form* μ_0 *on* K/T*).*

Remark 5.14 The answer to Question 2.8 is then "yes" for the Poisson structure π_λ and a K-invariant volume form μ_0 on K/T.

Remark 5.15 It is interesting to compare the identities in this section and the basic Kahler identities. See, for example, Corollary 4.10 in Chapter 5 of [Wl]. Compare also our operator $\partial_\lambda = J^{-1} d_{*,\lambda} J$ with the operator d_c^* on Page 191 in [Wl]. On the one hand, the Riemannian structure g_λ and the complex structure J we have here are not compatible to give a Kahler structure, but on the other hand, the symplectic structure π_λ can be made into a Kahler form by a result of Ginzburg and Weinstein [G-W]. It would be interesting to understand the connections here.

5.3 The Kostant operator ∂ as a limit of adjoint operators of d

Recall from Section 3.3 that ∂ is the Kostant operator on $C = \Omega(K/T, \mathbb{C})^K$ of degree -1 and that $S = d\partial + \partial d$ is called the "Laplacian" of d and ∂ in [Ko2]. Our next theorem says that ∂ is the limit of the operators $d_{*,\lambda}$ and that Kostant's Laplacian S is a limit of the Hodge Laplacians S_λ.

Theorem 5.16 *With respect to the vector space topology on* $C = \Omega(K/T, \mathbb{C})^K$*, we have*

$$\lim_{t\to+\infty} d_{*,\lambda+tH_\rho} = \partial; \qquad \lim_{t\to+\infty} S_{\lambda+tH_\rho} = S$$

for any $\lambda \in \mathfrak{h}^*$*.*

Proof. This follows directly from Theorem 5.8 and 5.4 and Theorem 3.10.

Q.E.D.

Remark 5.17 One can also prove the statements in Theorem 5.16 directly using special properties of the Hermitian metrics h_λ. We omit such a proof here due to the length of the paper.

5.4 Another proof of the disjointness of d and ∂

Recall that Theorem 4.5 in [Ko2] says that the two operators d and ∂ on $C = \Omega(K/T, \mathbb{C})^K$ are disjoint in the sense that $\mathrm{Im}(d) \cap \mathrm{Ker}(\partial) = \mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$. Note that the operator d is always disjoint from its adjoint operator with respect to any Hermitian metric. Although ∂ is in general not the adjoint operator of d with respect to any Hermitian metric, as we have seen from Example 3.20, Theorem 5.16 says that it is the limit of such operators with respect to the family of Hermitian metrics h_λ. In this section, we use this fact and some simple linear algebra arguments to give another proof of the disjointness of d and ∂.

Consider again the operator $S = d\partial + \partial d$ on C. Recall that $S = L + E$, where $L = \partial\delta + \delta\partial$ is the Hodge Laplacian of ∂ with respect to the Hermitian inner product h on C defined in Section 3.4 (see (28)), and that the operator $E_I := I^{-1}EI$ on $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$ has the explicit formula given in (26). We need two lemmas from linear algebra.

Lemma 5.18 *Let C be a finite dimensional complex vector space. Assume that L and E are two linear operators on C such that there exists a basis $\{e_1, e_2, ..., e_m\}$ of C in which L is diagonal and E strictly lower triangular. Let L_1 be Green's operator of L, i.e., $L_1(e_j) = 0$ if $L(e_j) = 0$ and $L_1(e_j) = \frac{1}{\lambda_j} e_j$ if $L(e_j) = \lambda_j e_j$ and $\lambda_j \neq 0$, for $j = 1, ..., m$. Set $S = L + E$. Then, we have two injective linear maps*

$$\begin{aligned} \phi : \quad & \mathrm{Ker}(S) \longrightarrow \mathrm{Ker}(L) : \ x \longmapsto (1 + L_1 E)x; \\ \psi : \quad & \mathrm{Im}(L) \longrightarrow \mathrm{Im}(S) : \ x \longmapsto (1 + EL_1)x. \end{aligned}$$

If, in addition, ϕ is an isomorphism, then so is ψ, and $\mathrm{Ker}(S) \cap \mathrm{Im}(S) = 0$.

Proof. For $x \in \mathrm{Ker}(S)$, let $y = (1 + L_1E)x$. Using $S(x) = 0$, we have $L(y) = L(x) + LL_1E(x) = -E(x) + LL_1E(x) \in \mathrm{Ker}(L)$. But $\mathrm{Ker}(L) \cap \mathrm{Im}(L) = 0$ since L is semi-simple. Thus $L(y) = 0$. Now for $x \in \mathrm{Im}(L)$, set $z = (1 + EL_1)x$. Then using $x = LL_1(x)$, we get $z = LL_1(x) + EL_1(x) = SL_1(x) \in \mathrm{Im}(S)$. Since L_1E and EL_1 are both nilpotent, the operators $1 + L_1E$ and $1 + EL_1$ on C are both invertible. Hence both ϕ and ψ are injective.

If ϕ is an isomorphism, then, by reason of dimensions, ψ is also an isomorphism. Suppose now $x \in \mathrm{Ker}(S) \cap \mathrm{Im}(S)$. We want to show that $x = 0$. Since $x \in \mathrm{Ker}(S)$, we have

$\phi(x) = (1 + L_1E)x \in \mathrm{Ker}(L)$. Since $x \in \mathrm{Im}(S)$ and since ψ is an isomorphism, there exists $y \in \mathrm{Im}(L)$ such that $x = \psi(y) = (1 + EL_1)y$. Thus

$$\phi(x) \;=\; (1 + L_1E)(1 + EL_1)y \;=\; y + L_1E(y) + EL_1(y) + L_1E^2L_1(y). \tag{45}$$

Suppose that $x \neq 0$. Then $y \neq 0$. Write $y = y_1e_1 + y_2e_2 + \cdots + y_me_m$ and let $z_0 := y_{i_0}e_{i_0}$ be the first non-zero term in this expression. Since $y \in \mathrm{Im}(L)$ and since L is diagonal in the $\{e_j\}$ basis, we know that $z_0 \in \mathrm{Im}(L)$. But since E is strictly lower triangular in the $\{e_j\}$ basis, we know from (45) that $\phi(x) = z_0 + z_1$ for some $z_1 \in \mathrm{span}\{e_j : j > i_0\}$. Since $\phi(x) \in \mathrm{Ker}(L)$, we must have $z_0 \in \mathrm{Ker}(L)$. Thus $z_0 = 0$ because $\mathrm{Ker}(L) \cap \mathrm{Im}(L) = 0$. This is a contradiction. Hence $x = 0$, and $\mathrm{Ker}(S) \cap \mathrm{Im}(S) = 0$.

Q.E.D.

The proof of the following lemma is trivial.

Lemma 5.19 *Suppose that V_1 and V_2 are two finite dimensional vector spaces and that $A_t \in \mathrm{Hom}(V_1, V_2)$, for $t > 0$, is a smooth family of linear operators from V_1 to V_2 such that*

$$\lim_{t\to+\infty} A_t \;=\; A \;\in\; \mathrm{Hom}(V_1, V_2).$$

If $\dim(\mathrm{Ker}(A_t)) = n_0$ *for some integer n_0 for all $t > 0$, then* $\dim(\mathrm{Ker}(A)) \geq n_0$.

Lemma 5.18 can be applied to the operators S, L and E on $C = \Omega(K/T, \mathbb{C})^K$ since all the assumptions are satisfied. Thus we have the linear maps $\phi : \mathrm{Ker}(S) \to \mathrm{Ker}(L)$ and $\psi : \mathrm{Im}(L) \to \mathrm{Im}(S)$ which are both injective. Green's operator L_1 for L, when considered as an operator on $\wedge(\mathfrak{n}_- \oplus \mathfrak{n}_+)^T$ via its identification with C by I, is the operator L_0 given in Section 3.4. We use m_0 to denote the common number

$$m_0 \;=\; \dim(\mathrm{Ker}(L)) \;=\; \dim H(C, \partial) \;=\; |W| \;=\; \dim(H^\bullet(K/T, \mathbb{C})),$$

where $|W|$ is the number of elements in the Weyl group W.

Claim 1. We have $\dim(\mathrm{Ker}(S)) = m_0$. Consequently, the map $\phi : \mathrm{Ker}(S) \to \mathrm{Ker}(L)$ is an isomorphism, and $\mathrm{Ker}(S) \cap \mathrm{Im}(S) = 0$.

Proof. Consider the family of operators $S_t := S_{tH_\rho} \in \mathrm{End}(C)$. By Theorem 5.16, we have $\lim_{t\to+\infty} S_t = S$. But we know from the usual Hodge theory that $\dim(\mathrm{Ker}(S_t)) = m_0$ for all $t > 0$. Thus by Lemma 5.19, we know that $\dim(\mathrm{Ker}(S)) \geq m_0$. We also know that $\dim(\mathrm{Ker}(S)) \leq m_0$ because $\phi : \mathrm{Ker}(S) \to \mathrm{Ker}(L)$ is injective. Thus $\dim(\mathrm{Ker}(S)) = m_0$. It follows from Lemma 5.18 that ϕ is an isomorphism and that $\mathrm{Ker}(S) \cap \mathrm{Im}(S) = 0$.

Q.E.D.

Claim 2. We have $\mathrm{Im}(S) = \mathrm{Im}(d) + \mathrm{Im}(\partial)$.

Proof. Clearly, $\mathrm{Im}(S) \subset \mathrm{Im}(d) + \mathrm{Im}(\partial)$. Let $m = \dim(C)$. Since $\dim(\mathrm{Ker}(S)) = m_0$, we have

$$\dim(\mathrm{Im}(S)) \;=\; m - m_0 \;=\; m - \dim(\mathrm{Ker}(d)) + \dim(\mathrm{Im}(d)) = 2\dim(\mathrm{Im}(d)).$$

On the other hand,

$$\dim(\mathrm{Im}(d) \;+\; \mathrm{Im}(\partial)) \;\le\; \dim(\mathrm{Im}(d)) \;+\; \dim(\mathrm{Im}(\partial)).$$

Consider now the family of operators d_{*,tH_ρ} on C for $t > 0$. We know that $\lim_{t\to+\infty} d_{*,tH_\rho} = \partial$ from Theorem 5.16. Thus by Lemma 5.19 we have

$$\dim(\mathrm{Im}(\partial)) \;\le\; \dim(\mathrm{Im}(d_{*,tH_\rho})) \;=\; \dim(\mathrm{Im}(d)),$$

and hence $\dim(\mathrm{Im}(d) + \mathrm{Im}(\partial)) \le 2\dim(\mathrm{Im}(d)) = \dim(\mathrm{Im}(S))$. Therefore $\mathrm{Im}(S) = \mathrm{Im}(d) + \mathrm{Im}(\partial)$.

Q.E.D.

We are now ready to prove the disjointness of d and ∂.

Claim 3. The two operators d and ∂ are disjoint.

Proof. By Claim 2, both $\mathrm{Im}(d) \cap \mathrm{Ker}(\partial)$ and $\mathrm{Im}(\partial) \cap \mathrm{Ker}(d)$ are subspaces of $\mathrm{Im}(S)$. They are clearly also subspaces of $\mathrm{Ker}(d) \cap \mathrm{Ker}(\partial) \subset \mathrm{Ker}(S)$. Since $\mathrm{Ker}(S) \cap \mathrm{Im}(S) = 0$ by Claim 1, we know that $\mathrm{Im}(d) \cap \mathrm{Ker}(\partial) = \mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$.

Q.E.D.

Remark 5.20 The fact that $\mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$ can be proved without using Theorem 5.16. Indeed, suppose that $d\partial x = 0$ for some $x \in C$. Then $\partial x \in \mathrm{Ker}(S)$. Thus $\phi(\partial x) = (1 + L_1 E)\partial x \in \mathrm{Ker}(L)$. Since ∂ commutes with $L_1 E$ (see Remark 3.19), we have $\partial(1 + L_1 E)x \in \mathrm{Ker}(L)$. But we know from the usual Hodge theory that $\mathrm{Im}(\partial) \cap \mathrm{Ker}(L) \subset \mathrm{Im}(L) \cap \mathrm{Ker}(L) = 0$. Thus $(1 + L_1 E)x = 0$, or, $x = 0$. Therefore $\mathrm{Im}(\partial) \cap \mathrm{Ker}(d) = 0$.

5.5 The Kostant harmonic forms as limits of Hodge harmonic forms

As we have seen from Example 3.20, the Kostant operator ∂ is not the adjoint of the de Rham d with respect to any Hermitian metric on K/T, so the Kostant harmonic forms $s^w, w \in W$, are not harmonic in the sense of Hodge. However, we will show by using Theorem 5.16 that they are limits of Hodge harmonic forms with respect to the family of Hermitian metrics h_λ.

Recall from Section 5.4 that $C = \mathrm{Ker}(S) + \mathrm{Im}(S)$ is a direct sum decomposition. Consider the linear map

$$\psi_{S,S_\lambda}: \ \mathrm{Ker}(S_\lambda) \longrightarrow \mathrm{Ker}(S): \ x \longmapsto x_1, \qquad x \in \mathrm{Ker}(S_\lambda)$$

if $x = x_1 + x_2$ with $x_1 \in \mathrm{Ker}(S)$ and $x_2 \in \mathrm{Im}(S)$. Since

$$\mathrm{Ker}(\psi_{S,S_\lambda}) \subset \mathrm{Ker}(S_\lambda) \cap \mathrm{Im}(S) \ = \ \mathrm{Ker}(S_\lambda) \cap \mathrm{Ker}(d) \cap \mathrm{Im}(S) \ \subset \ \mathrm{Ker}(S_\lambda) \cap \mathrm{Im}(d) \ = \ 0,$$

the map ψ_{S,S_λ} is injective. But since $\mathrm{Ker}(S)$ and $\mathrm{Ker}(S_\lambda)$ have the same dimension, the map ψ_{S,S_λ} is an isomorphism.

Definition 5.21 For $w \in W$, we define the Hodge harmonic form s^w_λ (with respect to the Hermitian metric h_λ) to be

$$s^w_\lambda \ = \ (\psi_{S,S_\lambda})^{-1} s^w \ \in \ \mathrm{Ker}(S_\lambda).$$

Theorem 5.22 *For each $w \in W$, we have*

$$\lim_{t \to +\infty} s^w_{\lambda + tH_\rho} \ = \ s^w.$$

Proof. Since $C = \mathrm{Ker}(S) + \mathrm{Im}(S)$ is a direct sum, we have Green's operator S_1 of S, namely $S_1 \in \mathrm{End}(C)$ is such that $S_1|_{\mathrm{Ker(S)}} = 0$ and $S_1|_{\mathrm{Im}(S)} = (S|_{\mathrm{Im}(S)})^{-1}$. Set $F_\lambda = S_\lambda - S$. Then, for each $w \in W$,

$$s^w \ = \ s^w_\lambda \ - \ S_1 S(s^w_\lambda) \ = \ s^w_\lambda \ + \ S_1 F_\lambda(s^w_\lambda) \ = \ (1 \ + \ S_1 F_\lambda)(s^w_\lambda).$$

Since $F_{\lambda + tH_\rho}$ goes to 0 as $t \to +\infty$, the operator $1 \ + \ S_1 F_{\lambda + tH_\rho}$ is invertible for t large enough, and thus

$$\lim_{t \to +\infty} s^w_{\lambda + tH_\rho} \ = \ \lim_{t \to +\infty} (1 \ + \ S_1 F_{\lambda + tH_\rho})^{-1} s^w \ = \ s^w.$$

Q.E.D.

6 Appendix: The Schouten bracket

Let P be a manifold of dimension n. The Schouten bracket $[\ ,\ ]$ is the graded Lie bracket on the space $\chi^\bullet(P)$ of multi-vector fields on P that is characterized as follows [Kz]:

1) for any smooth function $f \in \chi^0(P) = C^\infty(P)$ and any (1-)vector field $X \in \chi^1(P)$,

$$[X,\ f]\ =\ X(f) \in C^\infty(P);$$

2) for any (1-)vector fields X and Y, the Schouten bracket $[X,\ Y]$ is the usual Lie bracket between X and Y.

3) the bracket between two general multi-vector fields is obtained according to the following two rules:

$$[X, Y] = -(-1)^{(|X|-1)(|Y|-1)}[Y, X];$$
$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(|X|-1)|Y|} Y \wedge [X, Z].$$

It satisfies the graded Jacobi identity:

$$(-1)^{(|X|-1)(|Z|-1)}[X,\ [Y,\ Z]]\ +\ c.p.(X, Y, Z)\ =\ 0,$$

where $c.p.(X, Y, Z)$ means the cyclic permutation between X, Y and Z. Explicitly, if α is a form on P of degree $|X| + |Y| - 1$, we have,

$$(\alpha,\ [X, Y])\ =\ (-1)^{(|X|-1)(|Y|-1)} i_X d i_Y \alpha\ -\ i_Y d i_X \alpha\ +\ (-1)^{|X|} i_{X\wedge Y} d\alpha. \tag{46}$$

For $X \in \chi^{|X|}(P)$, introduce the operator

$$\partial_X\ =\ i_X d\ -\ (-1)^{|X|} d i_X :\ \Omega^k(P) \longrightarrow \Omega^{k-|X|+1}(P).$$

Then it follows from the definition of the Schouten bracket that

$$i_{X\wedge Y}\ =\ i_Y i_X, \tag{47}$$
$$i_{[X,Y]}\ =\ (-1)^{(|X|-1)(|Y|-1)}(\partial_X i_Y\ -\ (-1)^{(|X|-1)|Y|} i_Y \partial_X), \tag{48}$$
$$\partial_{[X,Y]}\ =\ (-1)^{(|X|-1)(|Y|-1)}(\partial_X \partial_Y\ -\ (-1)^{(|X|-1)(|Y|-1)} \partial_Y \partial_X). \tag{49}$$

We can think of the operators i_X and ∂_X as defining a "right" representation of the Gerstenhaber algebra $\chi(P)$ [KS] [X] on the graded vector space $\Omega(P)$.

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