INTRODUCTION TO TORIC VARIETIES

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Contents

Let T be a torus. This means that T is a connected abelian linear algebraic group, and $T \cong \mathbb{C}^{*r}$ for some r. We do not fix an isomorphism between T and $\mathbb{C}^{*r}$.

Let $M = X^*(T) = \mathrm{Hom}(T, \mathbb{C}^*)$. We denote the trivial character $\lambda(t) = 1$ for all $t \in T$ by $\lambda = 0$. M is an abelian group under pointwise multiplication, but we write the group structure additively.

Let $N = X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$. We denote the trivial cocharacter $\mu(z) = e$ for all $z \in \mathbb{C}^*$ by $\mu = 0$. N is an abelian group under pointwise multiplication, but we write the group structure additively.

If $T = \mathbb{C}^*$, let $\chi_n : T \to \mathbb{C}^*$ be given by $\chi_n(z) = z^n$. $X^*(T) = \{\chi_n : n \in \mathbf{Z}\}$. We also define $\mu_n : \mathbb{C}^* \to T$ by $\mu_n(z) = z^n$, and $X_*(T) = \{\mu_n : n \in \mathbf{Z}\}$.

If we fix an identification $T = \mathbb{C}^{*r}$, then given $(n_1, \dots, n_r) \in \mathbf{Z}^n$, there is an induced character $\chi_{n_1,\dots,n_r} \in X^*(T)$ given by

$$\chi_{n_1,\dots,n_r}(z_1, \dots, z_r) = \prod_{i=1}^{r} {z_i}^{n_i}$$

and $X^*(T) = \{\chi_{n_1,\dots,n_r} : (n_1, \dots, n_r) \in \mathbf{Z}^n\}$.

Given $(n_1, \dots, n_r) \in \mathbf{Z}^n$, we may consider $\mu_{n_1,\dots,n_r} : \mathbb{C}^* \to T$ given by

$$\mu_{n_1,\dots,n_r}(z) = (z^{n_1}, \dots, z^{n_r})$$

Then $X_*(T) = \{\mu_{n_1,\dots,n_r} : (n_1, \dots, n_r) \in \mathbf{Z}^n\}$.

We will not use a fixed identification $T = \mathbb{C}^{*r}$ in the sequel. Note that there is a pairing $< \cdot, \cdot >: M \times N \to \mathbf{Z}$ given as follows. If $\chi \in M$ and $\mu \in N$, then $\chi \circ \mu : \mathbb{C}^* \to \mathbb{C}^*$, so $\chi \circ \mu(z) = z^{n(\chi,\mu)}$ for some $n(\chi, \mu) \in \mathbf{Z}$. By definition, we set $< \chi, \mu >= n(\chi, \mu)$.

Let $M_{\mathbb{R}} = M \otimes_{\mathbf{Z}} \mathbb{R}$ and let $N_{\mathbb{R}} = N \otimes_{\mathbf{Z}} \mathbb{R}$. Then M and N are free abelian sugroups in the vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. We extend $< \cdot, \cdot >$ to a pairing $M_{\mathbb{R}} \otimes N_{\mathbb{R}} \to \mathbb{R}$ by the formula

$$< \lambda \otimes a, \mu \otimes b >= ab < \lambda, \mu >$$

This pairing identifies $M_{\mathbb{R}} = N_{\mathbb{R}}^*$, the dual vector space. This pairing has the property that:

$$N = \{v \in N_{\mathbb{R}} :< \lambda, v >\in \mathbf{Z}\ \forall \lambda \in M\}$$

and

$$M = \{f \in M_{\mathbb{R}} :< f, \mu >\in \mathbf{Z}\ \forall \mu \in N\}$$

These facts can be verified using an identification $T = \mathbb{C}^{*r}$, which identifies N and M as coordinate lattices $\mathbf{Z}^r \subset \mathbb{R}^r$. The choice of an identification corresponds to picking an ordered basis for the lattice N, and a corresponding dual basis for the lattice M.

In the sequel, we will not choose an identification. Making an identification of this nature is useful for doing calculations.

Definition 0.1. *A normal T-variety X is called a toric variety if there is $x \in X$ such that $T \cdot x$ is open in X and the stabilizer $T_x = \{e\}$.*

Recall that a smooth variety is normal.

Suppose X is a smooth projective toric variety. Let X^T be the fixed point set for the action of T on X. Then $X^T = \{x_1, \dots, x_n\}$ is a nonempty finite set.

T acts linearly on the tangent space $T_{x_j}(X)$ at x_j. Let $v_1, \dots, v_r$ be a basis for $T_{x_j}(X)$ consisting of weight vectors. Then $t \cdot v_i = \lambda_i(t)v_i$ for some $\lambda_i \in X^*(T)$, and the set $\lambda_1, \dots, \lambda_r$ consists of r distinct characters. We call this set the weights of T at x_j.

Let

$$\sigma_j\check{} = \{\sum_{i=1}^{r} a_i\lambda_i \in M_{\mathbb{R}} : a_i \geq 0, a_i \in \mathbb{R}\}$$

Then $\sigma_j\check{}$ is a cone in $M_{\mathbb{R}}$.

The pair $(\sigma_j\check{}, M)$ has the following properties:

(*1) $\sigma_j\check{}$ is a convex set containing 0

(*2) $\sigma_j\check{} = \{\sum_{i=1,\dots,s} a_i\eta_i : a_i \in \mathbb{R}^{\geq 0}\}$ for a finite set of vectors $\eta_1, \dots, \eta_s$ in M.

(*3) If $v \in \sigma_j\check{} - \{0\}$, then $t \cdot v \in \sigma_j\check{}$ if and only if $t \in \mathbb{R}^{\geq 0}$.

In (*2), we may take $s = r$ and $\eta_i = \lambda_i$.

Let

$$\sigma_j = \{v \in N_{\mathbb{R}} :< f, v >\geq 0, \forall f \in \sigma_j\check{}\}$$

Then the pair (σ_j, N) also satisfies (*1), (*2), and (*3). Let $\mu_1, \dots, \mu_r$ be a dual basis to the basis $\lambda_1, \dots, \lambda_r$. Then

$\sigma_j = \{\sum_{i=1}^r b_i\mu_i : b_i \in \mathbb{R}^{\geq 0}\}$

and $\sigma_j \cap N = \{\sum_{i=1}^r c_i\mu_i : c_i \in \mathbf{Z}^{\geq 0}\}$

Theorem 0.2. *(1) $N_{\mathbb{R}} = \cup_{j=1,\dots,n}\sigma_j$. Moreover, $\sigma_j \cap \sigma_k$ is contained in the boundary of σ_j.*

(2) For $i = 1, \cdots, n$, there is a T-stable open set $U_j \cong \mathbb{C}^r$ containing x_j such that $U_j \cong T_{x_j}(X)$ as a T-variety. Moreover, $X = \cup_{i=1\dots n}U_j$.

We make some remarks.

We can find the open set U_j as follows. Find a cocharacter μ in the interior of $\sigma_j\check{}$. Then $U_j = \{x \in X : \lim_{z \to 0} \mu(z) \cdot x = x_j\}$.

By definition, a face of a cone is either the cone itself, or is an intersection of the cone with a hyperplane that does not meet the interior of the cone. A face need not have codimension 1. Indeed, if σ is the first quadrant in $\mathbb{R}^2$, then its faces are $\{(x, 0) : x \geq 0\}$, $\{(0, y) : y \geq 0\}$, and $\{(0, 0)\}$.

Let

$$\Delta = \{\tau \in N_{\mathbb{R}} : \tau \text{ is a face of } \sigma_j, j \in \{1, \ldots, n\}\}.$$

Δ is called the fan of X.

The fan is completely determined by X.

In the theorem, (1) follows from the fact that X is complete.

For each cone σ_j in the fan, consider the set $\sigma_j \cap N$. This is a semigroup under addition. Smoothness of X implies that this semigroup is generated by a basis of the lattice N.

There are finitely many T-orbits on X and orbits correspond to faces of the cones.

Exercise 1: Let $X = \mathbb{P}(\mathbb{C}^3)$ and let $T = \{(1, t_1, t_2) : t_i \in \mathbb{C}^*\}$. Show that X is a smooth toric variety and show that its fan consists of three cones. The first cone is the cone bounded by the vectors $(1, 0)$ and $(0, 1)$. The second cone is the cone bounded by the vectors $(0, 1)$ and $(-1, -1)$. The third cone is the cone bounded by the vectors $(-1, -1)$ and $(1, 0)$. Show that the corresponding open subsets U_j give the usual open affine cover of X.

Exercise 2: Let $X = \overline{\psi(T)}$ (using language of wondeful compactification notes), and let $\mathfrak{t}^*_{\mathbb{R}}$ be the real span of the roots in $\mathfrak{t}^*$. Let $\mathfrak{t}_{\mathbb{R}}$ be the subset of $\mathfrak{t}$ on which all roots take real values. Define the lattice N so its dual lattice M is the span of the roots in $\mathfrak{t}^*_{\mathbb{R}}$. Show that the fan of the toric variety is given by the decomposition of $\mathfrak{t}_{\mathbb{R}}$ into Weyl chambers.

Exercise 3: Let $G = PSL(3, \mathbb{C})$ and let $X = \overline{\psi(T)}$ as in Exercise 2. Compute the fan using the identification $T = \mathbb{C}^{*2}$ from Exercise 1. Show each of the resulting cones is a subset of a cone in exercise 1.

Conversely, let $\Delta = \{\sigma_1, \ldots, \sigma_n\}$ be a finite collection of cones in N such that (N, σ_i) satisfies the properties (*1), (*2), (*3) above. Note that we do not assume that each σ_i is r-dimensional. Assume also that if $\sigma \in \Delta$, then every face of σ is in Δ. In addition, assume that if σ and η are in Δ, then the cone $\sigma \cap \eta$ is a face of both σ and τ. For a cone $\sigma \in \Delta$, the interior of σ is the complement of the union of proper cones τ in σ. Then $\cup \sigma_i$ is set-theoretically a disjoint union of the interiors of the σ_i.

Let $\sigma_i\check{} = \{f \in N_{\mathbb{R}} : f(v) \geq 0, \forall v \in \sigma_i\}$. Let $S_i = \sigma_i\check{} \cap M$. Then S_i is a semigroup in M. Let $\mathbb{C}[S_i]$ be the group algebra of this semigroup. This is the polynomial algebra on generators X_λ for $\lambda \in S_i$ subject to the relation, $X_{\lambda+\mu} = X_\lambda X_\mu$. S_i is a finitely generated semigroup so that $\mathbb{C}[S_i]$ is a finitely generated algebra over $\mathbb{C}$ with no nilpotents. It follows from the Hilbert nullstellensatz that $\mathbb{C}[S_i]$ is the ring of regular functions $R(U_i)$ on an affine variety U_i. Suppose $\tau = \sigma_i \cap \sigma_j$ is nonempty. Then performing the same construction on τ gives a semigroup $S_\tau \subset M$, and we may consider its group algebra $\mathbb{C}[S_\tau]$. Then $\mathbb{C}[S_\tau]$ is the ring of regular functions on a toric variety U_τ, and there is an induced open embedding U_τ into U_i and an induced open embedding U_τ into U_j. Then we may define a toric variety $X(\Delta) = \cup U_i$, with identification of the image of U_τ in U_i with the image of U_τ in U_j.

Then $X(\Delta)$ is complete if and only if $N_{\mathbb{R}} = \cup \sigma_i$.

$X(\Delta)$ is smooth if and only if we may take $s = r$ for each open cone in (*2) above.

Fixed points for the T-action correspond to interiors of r-dimensional cones of the fan.

More generally, orbits Q_i for T on $X(\Delta)$ correspond to the cones σ_i and the dimension of $Q_i = r - \dim(\sigma_i)$.

If X is a smooth projective toric variety with fan Δ, then $X(\Delta) = X$.

Some references:

William Fulton, "An Introduction to toric varieties", Princeton University Press, 1993.

Lecture Notes by David Cox at http://www.amherst.edu/ dacox/

V. Danilov, "The Geometry of Toric Varieties", Russian Math Surveys, vol. 33, 1978, pp. 97–154.

I have never read Danilov's article, but it is supposed to be very good.

The way I have presented this material is somewhat different from in Cox and Fulton. I emphasized deriving the fan from the geometry more.

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