CONCORDANCE CLASSES OF POSITIVE SCALAR CURVATURE METRICS

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§1. Introduction

Recently there has been some progress concerning the question which closed manifolds admits Riemannian metrics of positive scalar curvature. In fact, this problem has been solved completely for spin manifolds M of dimension ≥ 5 whose fundamental group is trivial, a cyclic group, or a quaternionic group. (cf. [St1], [BGS]). These results encourage us to attempt to classify positive scalar curvature metrics on a manifold M up to concordance: in analogy with the corresponding terminology for diffeomorphisms we call two positive scalar curvature metrics on M *concordant* if there is a positive scalar curvature metric on $M \times I$ (I is the unit interval) which is a product metric near the boundary and restricts to the given metrics on the boundary.

One perhaps surprising result of this paper is that the set of concordance classes of positive scalar curvature metrics on M – if non-empty – does not depend much on the topology on M. In fact, it depends only on the triple $\gamma(M) = (\pi, w, \hat{\pi})$, where π is the fundamental group of M, $w\colon \pi \longrightarrow \mathbb{Z}/2$ is the orientation character of M (which may be identified with the first Stiefel-Whitney class of M), and $\hat{\pi} \longrightarrow \pi$ is an extension of π by a group of order ≤ 2 which is determined by the second Stiefel-Whitney class of M (see Prop. 2.11). Geometrically, $\hat{\pi}$ is the fundamental group of $O(M)/r$, the quotient of the orthonormal frame bundle of M obtained by identifying an isometry $f\colon \mathbb{R}^n \longrightarrow T_xM$ with the isometry given by composing f with the reflection $r \in O(n)$ in the hyperplane perpendicular to the first coordinate axis. We will call such a tripel $\gamma = (\pi, w, \hat{\pi})$ a *supergroup* (the motivation for this terminology is discussed in §2).

This result can be made much more precise by introducing bordism groups $R_n(\gamma)$ depending functorially on the supergroup γ. Before defining these groups, we want

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to state the following result, which shows the central role of these groups for the existence of positive scalar curvature metrics as well as their classification up to concordance.

1.1 Theorem. *Let M be a smooth, connected manifold of dimension $n \geq 5$.*

(1) *A positive scalar curvature metric h on ∂M extends to a positive scalar curvature metric on M which is a product metric near the boundary if and only if an obstruction $\sigma(M,h) \in R_n(\gamma(M))$ vanishes.*
(2) *If h extends to a positive scalar curvature metric on M, then the group $R_{n+1}(\gamma(M))$ acts freely and transitively on the set of concordance classes of such metrics.*

The group $R_n(\gamma)$ is the bordism group of pairs (N,h), where N is an n-dimensional manifold with a γ-structure, and h is a positive scalar curvature metric on the (possibly empty) boundary ∂N. Here a 'γ-structure' is the reduction of the structure group of the tangent bundle of M to a Lie group $G(n,\gamma)$ associated to the tripel $\gamma = (\pi, w, \hat{\pi})$ (see Def. 2.6). If $\gamma = (\pi, 0, \pi)$ (resp. $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$), then a γ-structure on a manifold N amounts to specifying an orientation (resp. spin-structure) on N and a principal π-bundle over N (see Lemma 2.7). We remark that $\gamma(M) = (\pi, 0, \pi)$ if M is an orientable manifold with fundamental group π, whose universal covering does not admit a spin structure (Lemma 2.10(1)). If M admits a spin-structure, then the supergroup $\gamma(M)$ is isomorphic to $(\pi, 0, \pi \times \mathbb{Z}/2)$; in fact, the choice of a spin-structure on M determines such an isomorphism (Lemma 2.10(2)).

An important observation is that any manifold M admits a distinguished $\gamma(M)$-structure, which we refer to as the *canonical $\gamma(M)$-structure* (as discussed in remark 2.13, this claim does *not* lead to conclusions like the existence of prefered spin-structures). It allows us to define the obstruction $\sigma(M,h) \in R_n(\gamma(M))$ as the bordism class of the pair (M,h), where M is furnished with its canonical $\gamma(M)$-structure.

The groups $R_n(\gamma)$ in the 'spin case' $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ were introduced by Hajduk who used the notation $\Omega_n^{rel}(B\pi)$ [Ha, §6]. He also proved part (1) for closed spin manifolds (see [Ha, Thm. 7.1(i)]; however, as explained in Remark 5.7, his argument fails for some groups π), as well as part (2) in the special case where M is a spin manifold constructed as the boundary of a thickening of a 2-dimensional CW-complex [Ha, Thm. 6.2].

The main tool in the proof of Theorem 1.1 is the systematic use of results concerning surgery on positive scalar curvature metrics due to Gromov-Lawson [GL1], Schoen-Yau [SY] and Gajer [Ga].

It is interesting to compare Theorem 1.1 with the results of surgery theory concerning the existence and classification of smoothings of a Poincaré complex X. A central role in that theory is played by C.T.C. Wall's surgery obstruction group $L_n(\pi, w)$ which depends on the fundamental group $\pi = \pi_1(X)$ and the first Stiefel-Whitney class $w = w_1(X)$. The 'surgery exact sequence' shows that the set $\mathfrak{S}(X)$ of smoothings of X can be calculated in terms of the groups $L_*(\pi, w)$ and generalized cohomology groups $[X, G/O]_*$; in particular, $\mathfrak{S}(X)$ does not just depend on π and w. However, in the (hypothetical) situation $[X, G/O]_* = 0$ the analogue of

Theorem 1.1 holds if we replace 'positive scalar curvature metric' by 'smoothing' and the groups $R_*(\gamma)$ by $L_*(\pi, w)$ (cf. [RS3]).

Unlike the groups $L_n(\pi, w)$, which have a geometric description as bordism groups [Wa2, §9], as well as a purely algebraic description [Wa2, p. 49, 65], so far there is unfortunately so far no *algebraic* description of the groups $R_n(\gamma)$ (or any description not involving positive scalar curvature metrics). Probably as a result of this, so far the groups $R_n(\gamma)$ have not been determined for *any* pair (n, γ) with $n \geq 4$ (not even for $\gamma = (\{1\}, 0, \{1\})$ or $\gamma = (\{1\}, 0, \mathbb{Z}/2)$).

However, in many cases, a lower bound for the size of $R_n(\gamma)$ can be obtained by means of an "index homomorphism"

$$\theta\colon R_n(\gamma) \longrightarrow KO_n(C^*\gamma) \tag{1.2}$$

to the K-theory of a $\mathbb{Z}/2$-graded real C^*-algebra $C^*\gamma$ associated to $\gamma = (\pi, w, \hat{\pi})$. If $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ (resp. $\gamma = (\pi, 0, \pi)$), then $C^*\gamma$ is the (trivially graded) reduced group C^*-algebra $C^*\pi$ (resp. $C^*\gamma = 0$).

Given an n-dimensional manifold N with γ-structure and a positive scalar curvature metric h on ∂N, then $\theta(N, h) \in KO_n(C^*\gamma)$ is the KO-valued 'index' of a Dirac type operator on N (or, more accurately, on the complete manifold $\widehat{N} = N \cup_{\partial N} \partial N \times [0, \infty)$). This operator is determined by the γ-structure and a metric on N extending the given metric h on ∂N; we call it the *γ-Dirac operator.* If $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, then as mentioned above, a γ-structure on N is the same as a spin-structure on N together with a principal π-bundle $\widetilde{N} \longrightarrow N$. In this case, the γ-Dirac operator is the usual (Clifford-linear) Dirac operator twisted by the Miščenko-Fomenko line bundle $\widetilde{N} \times_\pi C^*\pi$. Rosenberg [Ro] shows for closed manifolds N that this operator has an 'index' in $KO_n(C^*\pi)$, whose vanishing is a necessary condition for the existence of a positive scalar curvature metric on N. Bunke constructs a $KO_n(C^*\pi)$-valued index $\theta(M, h)$ for manifolds M equipped with a positive scalar curvature metric h on their boundary [Bu]. Furthermore, he proves an additivity result for this index [Bu, Thm. 1.2], which implies that $\theta(M, h)$ only depends on the bordism class (M, h) represents in $R_n(\gamma)$.

If we denote by $\Omega_n(\gamma)$ the bordism group of n-dimensional closed manifolds with γ-structure, there is an obvious map $F\colon \Omega_n(\gamma) \longrightarrow R_n(\gamma)$. The Cartesian product $M_1 \times M_2$ of manifolds M_i furnished with a γ_i-structure carries a natural $\gamma_1 \widehat{\times} \gamma_2$-structure, where $\widehat{\times}$ is a suitably defined product for supergroups (cf. 2.). It is compatible with the construction of the C^*-algebra $C^*\gamma$ in the sense that $C^*(\gamma_1 \widehat{\times} \gamma_2)$ is isomorphic to the graded tensor poduct $C^*\gamma_1 \widehat{\otimes} C^*\gamma_2$.

1.3 Theorem. *There is an (external) product μ_R for the R_n-groups which makes the following diagram commutative.*

$$\begin{tikzcd}
\Omega_{n_1}(\gamma_1) \otimes \Omega_{n_2}(\gamma_2) \arrow[r, "\mu_\Omega"] \arrow[d, "F\otimes F"'] & \Omega_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2) \arrow[d, "F"] \\
R_{n_1}(\gamma_1) \otimes R_{n_2}(\gamma_2) \arrow[r, "\mu_R"] \arrow[d, "\theta\otimes\theta"'] & R_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2) \arrow[d, "\theta"] \\
KO_{n_1}(C^*\gamma_1) \otimes KO_{n_2}(C^*\gamma_2) \arrow[r, "\mu_{KO}"] & KO_{n_1+n_2}(C^*\gamma_1 \widehat{\otimes} C^*\gamma_2)
\end{tikzcd}$$

Here the pairing μ_Ω is induced by the Cartesian product of representing manifolds, while the pairing μ_{KO} is the (external) Kasparov product.

We remark that the existence of the map θ implies (in view of part (1) of Theorem 1.1) that the vanishing of $\theta(M,h)\in KO_n(C^*\gamma(M))$ is a necessary condition for extending h to a positive scalar curvature metric on M. This generalizes Rosenberg's result mentioned above. The converse (which is known as the 'Gromov-Lawson-Rosenberg Conjecture' in the case where M is spin and has no boundary) is *not true* as the recent counterexample of Schick [Sch] shows. In particular, the map θ is not injective.

Bott-periodicity for the range of the map θ implies that it factors through a map

$$\theta_{st}\colon R_n(\gamma)[B^{-1}]\to KO_n(C^*\gamma)$$

whose domain is a 'periodic' or 'stable' version of the R_n-group obtained by 'inverting the Bott manifold B'. Here B is any 8-dimensional spin manifold with $\widehat{A}(B)=1$; Cartesian product with B induces maps $R_n(\gamma)\to R_{n+8}(\gamma)\to R_{n+16}(\gamma)\to\dots$ (which do not depend on the choice of the manifold B); the direct limit of this sequence is $R_n(\gamma)[B^{-1}]$.

1.4 Conjecture. The map θ_{st} is an isomorphism.

We remark that for $\gamma=(\pi,w,\hat\pi)$ with $\hat\pi=\pi$ the Conjecture is true, but not so interesting, since in that case domain and range of θ_{st} are both trivial (cf. 4.). Conjecture 1.4 is related to the Baum-Connes Conjecture as follows. Let $T_n(\gamma)\subset\Omega_n(\gamma)$ be the subgroup represented by total spaces of fiber bundles whose fiber is the quaternionic projective plane $\mathbb{HP}^2$, and whose structure group is the isometry group of $\mathbb{HP}^2$. It is a result of Kreck and the author [KS] that $(\Omega_n(\gamma)/T_n(\gamma))[B^{-1}]$ for $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$ can be identified with $KO_n(B\pi)$. Work in progress by Joachim based on his thesis [Jo] shows that $(\Omega_n(\gamma)/T_n(\gamma))[B^{-1}]$ for $\hat\pi\neq\pi$ can be identified with a 'twisted' version of the n-th KO-homology of $B\pi$. Hence it seems appropriate to denote this group by $KO_n(\gamma)$. The composition

$$A\colon KO_n(\gamma)\overset{\text{def}}{=}(\Omega_n(\gamma)/T_n(\gamma))[B^{-1}]\to R_n(\gamma)[B^{-1}]\xrightarrow{\theta_{st}}KO_n(C^*\gamma),\tag{1.5}$$

(where the first map is induced by the forgetful map $F\colon\Omega_n(\gamma)\to R_n(\gamma)$) agrees in the 'spin case' $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$ with the so called 'assembly map'. According to the Baum-Connes Conjecture [BC], this map is (in the spin case) an isomorphism if π is torsion free. The factorization (1.5) shows that this implies that θ_{st} is surjective (in fact, it implies that $R_n(\gamma)[B^{-1}]$ contains $KO_n(C^*\gamma)$ as a direct summand). For finite groups A is not surjective; still Feher showed in his thesis [Fe] that θ_{st} is surjective if $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$ for a finite p-group π.

We remark that if the map θ_{st} is injective, then the vanishing of $\theta(M)\in KO_n(C^*\gamma(M))$ for a closed manifold M of dimension n implies that the product $M\times B\times\cdots\times B$ of M with k copies of the Bott manifold B represents zero in $R_{n+8k}(\gamma(M))$ for some k. In particular, by Theorem 1.1 the manifold M admits 'stably' a positive scalar curvature metric in the sense that the product of M with sufficiently many copies of B admits a positive scalar curvature metric. Hence Conjecture 1.4 implies the following 'stable' version of the Gromov-Lawson-Rosenberg Conjecture.

1.6 Stable Gromov-Lawson-Rosenberg Conjecture. A closed manifold of dimension n admits stably a positive scalar curvature metric if and only if the index obstruction $\theta(M) \in KO_n(\gamma(M))$ vanishes.

This conjecture has been proved in the spin case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ for finite groups π [RS2], and more recently for discrete groups for which the 'Baum-Connes map' (which the Baum-Connes Conjecture claims to be an isomorphism) is injective [St3]; this has been proved e.g. for discrete subgroups of Lie groups.

Contents

§2. γ-STRUCTURES

The goal of this section is twofold. First, we define what we mean by a "γ-structure" on a vector bundle $E \to X$, for a given supergroup γ; this notion will be basic for this paper. Secondly, we associate to any (pointed) vector bundle E a supergroup $\gamma(E)$. This construction can be viewed as a generalization of the fundamental group; in fact, Proposition 2.12 below shows that up to isomorphism, $\gamma(E)$ is determined by the fundamental group of the space over which E lives, together with the first and second Stiefel-Whitney class of E.

2.1 Definition. A *supergroup* consists of the following data:

(1) a group $\hat{\pi}$;
(2) a homomorphism $\hat{w}\colon \hat{\pi} \to \mathbb{Z}/2$;
(3) a distinguished central element c in the kernel of $\hat{w}$ with $c^2 = 1$.

The above definition of "supergroup" is a variant of the definition given by Dave Benson in [Be, §7]; the difference is that the above definition allows for the possibility that c is the trivial element. The author is grateful to George McNinch for showing me the reference [Be].

The following example of a supergroup is probably the origin of the terminology.

2.2 Example of a supergroup. Let $A = A^0 \oplus A^1$ be a $\mathbb{Z}/2$-graded algebra (also called 'super algebra') over the real numbers. Then the group of units $A^\times$ (consisting of the invertible, homogeneous elements of A) is a supergroup with w sending x to its degree, and $c = -1$. More generally, any subgroup of $A^\times$ containing -1 is again a supergroup. In particular, if A is the Clifford algebra $C\ell_n$ (the $\mathbb{R}$-algebra with unit generated by $v \in \mathbb{R}^n$ subject to the relation $v \cdot v = -|v|^2 \cdot 1$), then the subgroup $Pin(n) \subset C\ell_n^\times$ generated by unit vectors $v \in \mathbb{R}^n$ is a supergroup.

Sending a unit vector v to the element of the orthogonal group given by reflection in the hyperplane perpendicular to v defines a homomorphism $\sigma\colon Pin(n) \longrightarrow O(n)$ which is a double covering.

2.3 Remark on notation. It is suggestive (as Benson does) to use the same notation for the supergroup and its underlying group. Unfortunately, it can also lead to misunderstandings, for example in section 8, where we associate to a supergroup γ a C^*-algebra $C^*\gamma$, which is *not* the C^*-algebra of its underlying group; rather it agrees in special cases with the group C^*-algebra $C^*\pi$, where π is the group obtained by quotiening out the central element c. This suggests to think of a supergroup as a triple $(\pi, w, \hat{\pi})$, where π is a group, $w\colon \pi \longrightarrow \mathbb{Z}/2$ is a homomorphism, and $\hat{\pi} \twoheadrightarrow \pi$ is an extension of π such that the order of $\ker(\hat{\pi} \longrightarrow \pi)$ is less or equal to 2. From such a triple, we obtain a supergroup in the sense of Definition 2.1 by taking the group to be $\hat{\pi}$, the distingished element to be the (possibly trivial) generator of $\ker(\hat{\pi} \longrightarrow \pi)$, and the homomorphism to be the composition $\hat{\pi} \longrightarrow \pi \xrightarrow{\hat{w}} \mathbb{Z}/2$.

We will switch freely between these two points of view on supergroups; in particular, we will call a tripel $(\pi, w, \hat{\pi})$ as above a 'supergroup'. Also, from now on we will use the letter w for all grading homomorphisms to $\mathbb{Z}/2$ (like $\hat{w}$). We will use the superscript $(\quad)^{ev}$ to denote the kernel of a grading homomorphism.

2.4 Product of supergroups. If γ and δ are supergroups, their (super) product $\gamma\hat{\times}\delta$ is a supergroup defined as follows. As a set, $\gamma\hat{\times}\delta$ is the quotient of $\gamma \times \delta$ by the equivalence relation $(ac, b) \sim (a, cb)$, where $a \in \gamma$, $b \in \delta$, and c is the distinguished element of γ respectively δ. Multiplication is given by

$$[a_1, b_1][a_2, b_2] = c^{w(b_1)w(a_2)}[a_1a_2, b_1b_2],$$

where we write $[a, b] \in \gamma\hat{\times}\delta$ for the element represented by $(a, b) \in \gamma \times \delta$, and c is the distinguished element $[c, 1] = [1, c]$. The grading homomorphism $w\colon \gamma\hat{\times}\delta \longrightarrow \mathbb{Z}/2$ is given by $w([a, b]) = w(a) + w(b)$.

2.5 Example. The above product for supergroups is motivated by the $\mathbb{Z}/2$-graded tensor product of $\mathbb{Z}/2$-graded algebras: More precisely, if A, B are $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras, then there is a morphism of supergroups

$$A^\times\hat{\times}B^\times \longrightarrow (A\hat{\otimes}B)^\times \qquad \text{given by} \qquad [a, b] \mapsto a\hat{\otimes}b.$$

2.6 Definition of γ-structure. Let $\gamma = (\pi, w, \hat{\pi})$ be a supergroup, and let E be an n-dimensional vector bundle equipped with a Riemannian metric. A *γ-structure* on E is a reduction of the structure group of E through the homomorphism

$$G(n, \gamma) \overset{\text{def}}{=} (Pin(n)\hat{\times}\hat{\pi})^{ev} \xrightarrow{\rho_1} O(n) \qquad [a, b] \mapsto \sigma(a).$$

That is, a γ- structure consists of a principal $G(n, \gamma)$-bundle $P \longrightarrow X$, together with a $G(n, \gamma)$-equivariant map $\xi\colon P \longrightarrow O(E)$, where $G(n, \gamma)$ acts on $O(E)$ via ρ_1. More precisely, a γ-structure is an equivalence class of such pairs, where (P, ξ), is equivalent to (P', ξ'), if there there is an isomorphism $f\colon P \longrightarrow P'$ of principal bundles such that $\xi' \circ f = \xi$.

A γ-structure on a manifold M is by definition a γ-structure on the tangent bundle of M. We will refer to a manifold equipped with a γ-structure as *γ-manifold.*

2.7 Lemma. *Let $E \longrightarrow X$ be a vector bundle furnished with a Riemannian metric, and let $\gamma = (\pi, w, \hat{\pi})$ be a supergroup.*

(1) *A γ-structure on E determines a principal π-bundle $q: \widetilde{X} \longrightarrow X$.*
(2) *If $w = 0$, then a γ-structure on E determines an orientation on E.*
(3) *If $w = 0$, $\hat{\pi} \neq \pi$, and the extension $\hat{\pi} \longrightarrow \pi$ splits, then a splitting map $s: \pi \longrightarrow \hat{\pi}$ and a γ-structure on E determine a spin structure on E.*
(4) *If $w = 0$, and $\hat{\pi} = \pi$, then the above construction gives a bijection between γ-structures and pairs consisting of an orientation on E together with a principal π-bundle over X.*
(5) *If $w = 0$ and $\hat{\pi} \neq \pi$, then a splitting map $s: \pi \longrightarrow \hat{\pi}$ gives via the above construction a bijection between γ-structures on E and pairs consisting of a spin structure on E together with a principal π-bundle over X.*

Proof. To prove part (1), we define an epimorphism

$$\rho_2: (Pin(n) \widehat{\times} \hat{\pi})^{ev} \longrightarrow \pi \quad \text{by} \quad [a, b] \mapsto [b],$$

where $[b]$ is the image of $b \in \hat{\pi}$ under the projection map $\hat{\pi} \longrightarrow \pi$. Then $\widetilde{X} \stackrel{\text{def}}{=} P/\ker \rho_2 \longrightarrow X$ is the desired principal π-bundle.

Concerning the second statement, we note that for $w = 0$ the image of the map $\rho_1: (Pin(n) \widehat{\times} \hat{\pi})^{ev} \longrightarrow O(n)$ defined in 2.6 is the special orthogonal group $SO(n)$. It follows that $P/\ker \rho_1 \longrightarrow X$ is a principal $SO(n)$-bundle, and hence the $SO(n)$-equivariant map $P/\ker \rho_1 \longrightarrow O(E)$ induced by ξ can be interpreted as an orientation on E.

If $w = 0$ and $\hat{\pi} \neq \pi$, then a splitting map $s: \pi \longrightarrow \hat{\pi}$ allows us to define a homomorphism

$$\hat{\rho}_1: (Pin(n) \widehat{\times} \hat{\pi})^{ev} \longrightarrow Spin(n)$$

by $\hat{\rho}_1([a, b]) = a$ for b in the image of s (we note that every element of $(Pin(n) \widehat{\times} \hat{\pi})^{ev}$ has a unique representative (a, b) with b in the image of s). Then $P/\ker \hat{\rho}_1 \longrightarrow X$ is a principal $Spin(n)$-bundle, and hence the $Spin(n)$-equivariant map $P/\ker \hat{\rho}_1 \longrightarrow O(E)$ induced by ξ is the sought after spin-structure on E.

If $w = 0$, and $\hat{\pi} = \pi$, then

$$(Pin(n) \widehat{\times} \hat{\pi})^{ev} \xrightarrow{\rho_1 \times \rho_2} SO(n) \times \pi$$

is an isomorphism, which implies part (4).

If $w = 0$, $\hat{\pi} \neq \pi$, and $s: \pi \longrightarrow \hat{\pi}$ is a splitting map, then the homormorphism

$$(Pin(n) \widehat{\times} \hat{\pi})^{ev} \xrightarrow{\hat{\rho}_1 \times \rho_2} Spin(n) \times \pi$$

is an isomorphism, which implies part (5). $\square$

In the next section we will define the groups $R_n(\gamma)$ as bordism groups of γ-manifolds with extra structure; this requires being able to restrict a γ-structure on a manifold M to a γ-structure on its boundary ∂M. Clearly, a γ-structure on the tangent bundle TM restricts to a γ-structure on $TM_{|\partial M} = T\partial M \oplus \underline{\mathbb{R}}$ (under this identification the outward pointing normal vector field corresponds to the canonical section of the trivial line bundle $\underline{\mathbb{R}}$ with). The following lemma shows that a γ-structure on $T\partial M \oplus \underline{\mathbb{R}}$ indeed induces a γ-structure on $T\partial M$.

2.8 Lemma. *There is a natural bijection between γ-structures on a vector bundle E and γ-structures on $E \oplus \underline{\mathbb{R}}$.*

Proof. We note that $G(n,\gamma)$ can be identified with the subgroup of $G(n+1,\gamma)$ which under the projection map $\rho_1 \colon G(n+1,\gamma) \to O(n+1)$ maps to $O(n) \subset O(n+1)$. Hence if (P,ξ) is a γ-structure for $E \oplus \underline{\mathbb{R}}$, then

$$P' \stackrel{\text{def}}{=} \{p \in P \mid \xi(p) \in O(E) \subset O(E \oplus \underline{\mathbb{R}})\}$$

is a principal $G(n,\gamma)$-bundle, and $\xi' \stackrel{\text{def}}{=} \xi_{|P'} \colon P' \to O(E)$ is a $G(n,\gamma)$-equivariant map. Hence the pair (P',ξ') is a γ-structure for E.

Conversely, if (P',ξ') is a γ-structure for E, then $P \stackrel{\text{def}}{=} P' \times_{G(n,\gamma)} G(n+1,\gamma)$ is a principal $G(n+1,\gamma)$-bundle, which together with the map

$$\xi \colon P' \times_{G(n,\gamma)} G(n+1,\gamma) \longrightarrow O(E) \times_{O(n)} O(n+1) = O(E \oplus \underline{\mathbb{R}})$$

induced by $\xi' \times \rho_1$ represents a γ-structure for $E \oplus \underline{\mathbb{R}}$. It is easy to check that these two constructions are inverse to each other. $\square$

2.9 The supergroup $\gamma(E)$ associated to a vector bundle E. Let E be an n-dimensional vector bundle over a connected space X equipped with a Riemannian metric. Let $p \colon O(E) \to X$ be the orthonormal frame bundle of E; i.e., the principal $O(n)$-bundle over X, whose fiber over $x \in X$ is the space of isometries $\mathbb{R}^n \to E_x$. We call the vector bundle E *pointed* if $O(E)$ is equipped with a base point y_0. We denote by $r \in O(n)$ the reflection in the hyperplane perpendicular to the vector $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$, and by $O(E)/r$ the quotient obtained by identifying an isometry $f \colon \mathbb{R}^n \to E_x$ with $f \circ r$. We observe that $O(E)/r$ is a bundle over X whose fibers are homeomorphic to $SO(n)$; however, $O(E)/r$ is *not* a principal bundle, since the order two subgroup of $O(n)$ generated by r is not normal in $O(n)$.

We associate to the pointed vector bundle E a supergroup $\gamma(E) = (\pi, w, \hat{\pi})$ as follows:

(1) π is the fundamental group $\pi_1(X)$.
(2) $w \colon \pi \to \mathbb{Z}/2$ is the orientation character of E (also called first Stiefel-Whitney class) which sends an element represented by a loop $b \colon S^1 \to X$ to the non-trivial element of $\mathbb{Z}/2$ if and only if b^*E is the non-trivial vector bundle over S^1.
(3) The extension $\hat{\pi} \to \pi$ is the map induced by the projection map $O(E)/r \to X$ on fundamental groups if $\dim E \geq 3$; otherwise we replace E by the Whitney sum of E with a trivial line bundle $\underline{\mathbb{R}}^k$ such that $\dim(E \oplus \underline{\mathbb{R}}^k) \geq 3$.

The author is grateful to Laszlo Feher for suggesting the above description of $\hat{\pi}$ replacing a more complicated one found by the author. If M is a Riemmanian manifold (furnished with a base point x_0 and an isometry $\mathbb{R}^n \cong T_{x_0}M$), we denote by $\gamma(M)$ the supergroup associated to the tangent bundle of M.

We remark that the construction $E \mapsto \gamma(E)$ is multiplicative in the sense that $\gamma(E_1 \times E_2)$ is the product $\gamma(E_1) \widehat{\times} \gamma(E_2)$.

2.10 Lemma. *Let E be a vector bundle over a connected space X and let $\gamma(E) = (\pi, w, \hat{\pi})$ be the associated supergroup. Let $q: \widetilde{X} \longrightarrow X$ be the universal covering.*

(1) *q^*E admits a spin structure if and only if $\ker(\hat{\pi} \to \pi) = \mathbb{Z}/2$.*
(2) *A spin structure on E induces a splitting map $s: \pi \longrightarrow \widehat{\pi}$ for the extension $\hat{\pi} \longrightarrow \pi$.*

Proof. Replacing E by $E \oplus \underline{\mathbb{R}}^k$ if necessary, we may assume $n = \dim E \geq 3$. The homotopy exact sequence of the fiber bundle $SO(n) \xrightarrow{i} O(E)/r \xrightarrow{p} X$ shows that $\ker(\widehat{\pi} \longrightarrow \pi) = \ker(p_*: \pi_1(O(E)/r) \longrightarrow \pi_1(X))$ is the image of $i_*: \pi_1(SO(n)) \longrightarrow \pi_1(O(E)/r))$. Since i factors as $SO(n) \to O(q^*E)/r \to O(E)/r$, where the second map is a covering map and hence induces an injective map on π_1, we can identify the image of i_* with the image of $\pi_1(SO(n)) \longrightarrow \pi_1(O(q^*E)/r) = \pi_1(SO(q^*E))$. Interpreted in terms of covering spaces, non-triviality of this image is equivalent to the existence of a double covering of $SO(q^*E)$ restricting to the non-trivial double covering on each fiber $SO(n) \subset SO(q^*E)$. Such a double covering is just a spin structure on q^*E, which proves part (1).

To prove the second part, we note that a spin structure on E is given by a principal $Spin(n)$-bundle $Spin(E)$ and a $Spin(n)$-equivariant map from $Spin(E)$ to the orthonormal frame bundle $O(E)$. The homotopy exact sequence of the fiber bundle $\hat{p}: Spin(E) \longrightarrow X$ shows that $\hat{p}$ induces an isomorphism on π_1. Hence the composition $Spin(E) \to O(E) \to O(E)/r$ induces on π_1 the desired splitting map. □

The next result is a refinement of the previous lemma. It shows that the extension $\hat{\pi} \longrightarrow \pi$ is determined (up to isomorphism) by the second Stiefel-Whitney class of E. In particular, up to isomorphism the supergroup $\gamma(E)$ is determined by the fundamental group of the base space X, and the Stiefel-Whitney classes $w_i(E) \in H^i(X; \mathbb{Z}/2)$ for $i = 1, 2$.

2.11 Proposition. *Let E be a pointed vector bundle over a connected space X and let $\gamma(E) = (\pi, w, \widehat{\pi})$ be the associated supergroup.*

(1) *If the second Stiefel-Whitney class $w_2(E) \in H^2(X; \mathbb{Z}/2)$ is not in the image of the homomorphism u^* induced by the the classifying map $u: X \longrightarrow B\pi$ of the universal covering of X, then $\hat{\pi} \longrightarrow \pi$ is an isomorphism.*
(2) *If $w_2(E) = u^*(e)$ for $e \in H^2(B\pi; \mathbb{Z}/2)$, then $\ker(\hat{\pi} \longrightarrow \pi) = \mathbb{Z}/2$ and the cohomology class associated to the extension $\mathbb{Z}/2 \longrightarrow \hat{\pi} \longrightarrow \pi$ (cf. 3.1) is e.*

The proof of this proposition is somewhat technical and is defered to the next section (cf. 3.8).

2.12 Proposition. *Let E be a pointed vector bundle of dimension n over a connected space X. Suppose that E is equipped with a Riemannian metric.*

(1) *E has a distinguished $\gamma(E)$-structure, which we refer to as canonical.*
(2) *For $n \geq 3$ the map $c: X \longrightarrow BG(n, \gamma(E))$ classifying the principal $G(n, \gamma)$-bundle determined by the canonical $\gamma(E)$-structure is a 2-equivalence.*

2.13 Remarks.

(1) For readers worried that the existence of a canonical γ-structure, $\gamma = \gamma(E)$, might be too strong of a claim, we would like to emphasized that the existence of a canonical γ-structure e.g. does *not* imply the existence of a canonical spin structure for a bundle E admitting a spin structure: If E admits a spin structure, then by Lemma 2.7(3) the canonical γ-structure determines a spin-structure on E only *after choosing a splitting map* $s\colon \pi \longrightarrow \hat{\pi}$. In fact, there is a bijection between spin-structures on M and splitting maps for the extension $\hat{\pi} \longrightarrow \pi$.

(2) The composition $X \xrightarrow{c} BG(n,\gamma) \xrightarrow{B\rho_1} BO(n)$ is the classifying map of E. Since c is a 2-equivalence and $B\rho_1$ induces an isomorphism on the homotopy group π_i for $i \geq 3$ and an injection for $i = 2$ (follows from the homotopy exact sequence of the fiber bundle $B\rho_1$ with fiber $B\hat{\pi}^{ev}$), this composition can be interpreted as a stage in the Postnikov decomposition of the map classifying E.

(3) Let $VECT_n$ be the homotopy category of pointed n-dimensional vector bundles over connected spaces. The functor $F\colon VECT_n \longrightarrow \Gamma$ to the category Γ of supergroups which sends E to $\gamma(E)$ factors through the category obtained from $VECT_n$ by inverting morphisms $(\widehat{f}\colon E \longrightarrow E', f\colon X \longrightarrow X')$ for which f is a 2-equivalence (this follows directly from the construction of $\gamma(E)$). Proposition 2.12 implies that for $n \geq 3$ this category is actually equivalent to Γ. To see this, we note that mapping a supergroup γ to the vector bundle $EG(n,\gamma) \times_{G(n,\gamma)} \mathbb{R}^n$ over the classifying space $BG(n,\gamma)$ gives a functor $G\colon \Gamma \longrightarrow VECT_n$ such that $F \circ G$ is the identity functor on Γ. The functor $G \circ F$ is not the identity; rather, it sends a vector bundle E to $EG(n,\gamma(E)) \times_{G(n,\gamma(E))} \mathbb{R}^n$. The canonical $\gamma(E)$-structure on E provides us with a morphism between these objects, which is a 2-equivalence on space level, and hence becomes an *isomorphism* upon inverting 2-equivalences.

Proof of Proposition 2.12. By Lemma 2.8 and 2.9(3) we may assume for the proof of part (1) that $n = \dim E \geq 3$. We will construct a topological group $G(E)$, a continuous homomorphism $\eta_1\colon G(E) \longrightarrow O(n)$, a principal $G(E)$-bundle $P(E)$ over X, and a $G(E)$-equivariant map $\xi\colon P(E) \longrightarrow O(E)$ (here $G(E)$ acts on $O(E)$ via η_1). In Lemma 2.15 below, we will show that there is a canonical group isomorphism $\Psi\colon G(n,\gamma(E)) \longrightarrow G(E)$ such that $\eta_1 \circ \Psi = \rho_1$. This shows that we can interpret $(P(E),\xi)$ as a $\gamma(E)$-structure on E and proves part (1).

To define $G(E)$, let $p\colon O(E) \longrightarrow X$ be the orthonormal frame bundle of E. The base point $y_0 \in O(E)$ allows us to identify the fiber $O(E)_{x_0}$ over $x_0 = p(y_0)$ with $O(n)$ via $O(n) \ni g \mapsto y_0 g$. We define

$$G(E) \overset{\text{def}}{=} \{b\colon [0,1] \longrightarrow O(E) \mid b(0) = y_0,\ b(1) \in O(E)_{x_0}\}/\sim,$$

where we identify two paths if they have the same endpoint and are homotopic relative endpoints. A group structure on $G(E)$ is given by

$$[a] \cdot [b] = [ab(1) * b]; \tag{2.14}$$

were $ab(1)$ is the path obtained by multiplying the path a by $b(1) \in O(E)_{x_0} = O(n)$, and $ab(1) * b$ is the concatenation of the paths $ab(1)$ and b. Here concatenated paths are to be read like functions from right to left; that is, the path $ab(1) * b$ runs first through b and then through $ab(1)$. We define homomorphisms

$$O(n) \xleftarrow{\eta_1} G(E) \xrightarrow{\eta_2} \pi$$

by $\eta_1([b]) = b(1)$ and $\eta_2([b]) = [p \circ b] \in \pi_1(X) = \pi$. The compact open topology on the path space induces a topology on the quotient space $G(E)$ which makes $G(E)$ a topological group, and η_1, η_2 continuous homomorphisms.

Let

$$P(E) \overset{\text{def}}{=} \{a\colon [0,1] \to O(E) \mid a(0) = y_0\}/\sim,$$

where again we identify paths if they are homotopic relative endpoints. The compact open topology on the path space induces a topology on P, which makes the projection map $P \to X$ defined by $[a] \mapsto p(a(1))$ continuous. The group $G(E)$ acts on P (from the right) by formula (2.14). This gives P the structure of a principal $G(E)$-bundle. It is clear that the map

$$\xi\colon P(E) \to O(E) \qquad [a] \mapsto a(1)$$

is $G(E)$-equivariant, where $G(E)$ acts on $O(E)$ via η_1 and the $O(n)$-action on $O(E)$.

To prove the second part of Proposition 2.12, we note that P is the standard construction of the universal covering of $O(E)$; in particular, P is simply connected. Hence the exact sequence of homotopy groups for the homotopy fibration $P \to X \to BG(n, \gamma(E))$ implies the second part of 2.12. $\square$

2.15 Lemma. *Let E be a vector bundle of dimension $n \geq 3$. Then there is an isomorphism $\Psi\colon G(n, \gamma(E)) \to G(E)$ making the following diagram commutative.*

$$\begin{tikzcd}
O(n) \arrow[d, equal] & G(n,\gamma(E)) \arrow[l, "\rho_1"'] \arrow[r, "\rho_2"] \arrow[d, "\Psi"] & \pi \arrow[d, equal] \\
O(n) & G(E) \arrow[l, "\eta_1"'] \arrow[r, "\eta_2"] & \pi
\end{tikzcd}$$

This lemma is proved in the next section after we have obtained a geometric description of $Pin(n)$ in terms of (homotopy classes of) paths in $O(n)$ (see Lemma 3.5).

§3. Central extensions

The purpose of this section is to provide the proofs for Proposition 2.11 and Lemma 2.15.

3.1 The cohomology class associated to a $\mathbb{Z}/2$-extension. It is well-known that a $\mathbb{Z}/2$-extension

$$\mathbb{Z}/2 \rightarrowtail \widehat{\pi} \twoheadrightarrow \pi \tag{3.2}$$

of a discrete group π is classified up to isomorphism of such extensions by a cohomology class $e \in H^2(B\pi;\mathbb{Z}/2)$ (cf. [Br, Ch. IV, Thm. 3.12]). The cohomology class e can be described in two ways:

(1) Pick a (set theoretic) cross section $s\colon \pi \longrightarrow \widehat{\pi}$. Then the function

$$u\colon \pi \times \pi \longrightarrow \mathbb{Z}/2 \subset \widehat{\pi} \qquad (g,h) \mapsto s(g)s(h)s(gh)^{-1},$$

which measures the deviation of s from being a group homomorphism, is a 2-cocycle which represents e in group cohomology.

(2) Apply the classsifying space functor to the extension to get a fibration $B\widehat{\pi} \longrightarrow B\pi$ with fiber $B\mathbb{Z}/2$. The multiplication map $\mathbb{Z}/2 \times \widehat{\pi} \longrightarrow \widehat{\pi}$ is a homomorphism since $\mathbb{Z}/2$ is central in $\widehat{\pi}$, and hence induces a map $B\mathbb{Z}/2 \times B\widehat{\pi} \longrightarrow B\widehat{\pi}$. This shows that the fibration is principal and hence classified by a map $B\pi \longrightarrow K(\mathbb{Z}/2, 2)$. This map corresponds to a cohomology class in $H^2(B\pi;\mathbb{Z}/2)$.

The first construction shows that if we can choose s to be a homomorphism (i.e., if the extension (3.2) *splits*), then the cocycle u and hence the cohomology class e vanish. Conversely, if e vanishes, then the extension splits (see e.g. [Br, Chap. IV, §3)].

For the proof of Proposition 2.11, we want to interpret the second Stiefel-Whitney class $w_2 \in H^2(BO(n);\mathbb{Z}/2)$ as the cohomology class associated to a $\mathbb{Z}/2$-extension of $O(n)$ (the second construction above shows that we can construct e in the more general situation where π is a topological group).

3.3 Lemma. *Let $\widehat{O}(n)$ be a topological group which is a double covering of $O(n)$ and restricts to the universal covering of $SO(n)$. Let $e \in H^2(BO(n);\mathbb{Z}/2)$ be the cohomology class associated to the extension $\widehat{O}(n) \longrightarrow O(n)$. Let $r \in O(n)$ be the reflection in the hyperplane perpendicular to $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$, and let $\widehat{r} \in \widehat{O}(n)$ be a preimage of r under the projection map. Let $w_i \in H^i(BO(n);\mathbb{Z}/2)$ be the i-th Stiefel-Whitney class.*

(1) *If $\widehat{r}$ has order 2, then $e = w_2$.*
(2) *If $\widehat{r}$ has order 4, then $e = w_2 + w_1^2$.*

Proof. The cohomology class classifying the extension is some linear combination $aw_2 + bw_1^2 \in H^2(BO(n);\mathbb{Z}/2)$, $a, b \in \mathbb{Z}/2$.

Restriction to the subgroup $SO(n) \subset O(n)$ shows $a \neq 0$, since the pullback of e to $BSO(n)$ must be non-trivial due to our assumption that $\widehat{O}(n) \longrightarrow O(n)$ restricts to the universal covering of $SO(n)$ and the fact that the induced map $H^*(BO(n);\mathbb{Z}/2) \longrightarrow H^*(BSO(n);\mathbb{Z}/2)$ maps w_1 to zero.

Restricting the extension $\widehat{O}(n) \longrightarrow O(n)$ to the subgroup $O(1) \subset O(n)$ generated by r leads to the extension $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow O(1) = \mathbb{Z}/2$ if $\hat{r}$ has order 2, and to the extension $\mathbb{Z}/4 \longrightarrow O(1) = \mathbb{Z}/2$ if $\hat{r}$ has order 4. It follows that $b = 0$ in the former case, and $b \neq 0$ in the latter case, since the induced map $H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\dots,w_n] \longrightarrow H^*(BO(1);\mathbb{Z}/2) = \mathbb{Z}/2[w_1]$ is the projection map. $\square$

3.4 Examples.

(1) The extension $\sigma: Pin(n) \longrightarrow O(n)$ (see Example 2.2) has $e = w_2 + w_1^2$, since the element $e_1 = (1, 0, \dots, 0) \in S^{n-1} \subset Pin(n)$ maps to r under $\sigma: Pin(n) \to O(n)$ and $e_1^2 = -|e_1|^2 = -1$.

(2) If $\hat{\pi}$ is any supergroup, we can define a new supergroup $\hat{\pi}'$, which as a set is just $\hat{\pi}$ (furnished with the same distiguished element c and the same map w), but equipped with a new group structure given by $(a, b) \mapsto c^{w(a)w(b)}ab$. Applying this to the supergroup $Pin(n)$, we obtain a new supergroup $Pin'(n)$. Put differently, we have constructed a new covering group $\sigma: Pin'(n) \longrightarrow O(n)$ by modifying the multiplication on the total space of the covering $\sigma: Pin(n) \longrightarrow O(n)$. With respect to the multiplication in $Pin'(n)$ we have $e_1^2 = 1$, and hence the cohomology class associated to $Pin'(n)$ is $e = w_2$.

3.5 Lemma. *Let $\widehat{O}(n)$ be the set of homotopy classes relative endpoint of paths $a: [0, 1] \longrightarrow O(n)$ with $a(0) \in \{1, r\}$. Define a group structure on $\widehat{O}(n)$ by*

$$[a] \cdot [a'] \stackrel{\text{def}}{=} [aa'(1) * r^{w(a)}a'], \tag{3.6}$$

where $*$ denotes concatenation of paths (to be read from right to left as in §2), and $w(a') \in \mathbb{Z}/2$ is characterized by $a'(0) = r^{w(a')}$. Then the double covering $\widehat{O}(n) \longrightarrow O(n)$ given by $[a] \mapsto a(1)$ is isomorphic to $Pin'(n) \longrightarrow O(n)$. The double covering $\widehat{O}(n)' \longrightarrow O(n)$ (obtained by modifying the multiplication of $\widehat{O}(n)$ as in Example 3.4(2)) is isomorphic to $Pin(n) \longrightarrow O(n)$.

Proof. To see that the product (3.6) is in fact associative and defines a group structure on $\widehat{O}(n)$, we note that the map $\{(s, t) \mid s, t \in [0, 1],\ s \leq t\} \longrightarrow O(n)$ given by $(s, t) \mapsto a(s)a'(t)$ can be interpreted as a homotopy relative endpoints between the paths $aa'(1) * r^{w(a)}a'$ and $t \mapsto a(t)a'(t)$. This shows that the product (3.6) can alternatively be described as induced by the pointwise product of paths, which clearly defines a group structure on $\widehat{O}(n)$. It is also evident that evaluation at the endpoint defines a double covering whose kernel identifies with $\pi_1(SO(n)) = \mathbb{Z}/2$.

To show that the double covering $\widehat{O}(n) \longrightarrow O(n)$ is isomorphic to $Pin'(n) \longrightarrow O(n)$, choose a preimage $\hat{r} \in Pin'(n)$ of $r \in O(n)$. Since $\hat{r}$ has order 2, the set $\hat{\pi}$ of homotopy classes relative endpoints of paths $a: [0, 1] \longrightarrow Pin'(n)$ with $a(0) \in \{1, \hat{r}\}$ is a group with multiplication induced by the pointwise product of paths. We note that the evaluation map $ev: \hat{\pi} \longrightarrow Pin'(n)$ given by $[a] \mapsto a(1)$ and the homomorphism $q_*: \hat{\pi} \longrightarrow \widehat{O}(n)$ induced by the the covering $q: Pin'(n) \longrightarrow O(n)$ are both compatible with projections to $O(n)$. Furthermore both maps are isomorphisms. The evaluation map is an isomorphism, since both components of $Pin'(n)$ are simply connected; the lifting property for the double covering q shows that q_* is an isomorphism.

The isomorphism $\widehat{O}(n) \cong Pin'(n)$ is obviously compatible with the natural 'supergroup structure'. Hence the supergroup $\widehat{O}(n)'$ is isomorphic to $(Pin'(n))' \cong Pin(n)$. □

3.7 Proof of Lemma 2.15. Let $\gamma(E) = (\pi, w, \hat{\pi})$. Using the double covering $O(E) \longrightarrow O(E)/r$, we identify $\hat{\pi} = \pi_1(O(E)/r)$ with homotopy classes relative endpoints of paths $b\colon [0,1] \longrightarrow O(E)$ with $b(0) = y_0$ and $b(1) = y_0 r^{w(b)}$; then multiplication is given by $[b][b'] = [br^{w(b')} * b']$. Likewise, using the geometric description of $Pin(n)$ provided by Lemma 3.5, we define

$$\Psi\colon G(n,\gamma(E)) = (Pin(n)\widehat{\times}\hat{\pi})^{ev} \longrightarrow G(E) \qquad \text{by} \qquad [[a],[b]] \mapsto [a*b].$$

We note that the concatenation $a*b$ makes sense, since $b(1) = r^{w(b)} = r^{w(a)} = a(0)$ (here we identify $O(n)$ and the fiber $O(E)_{x_0}$).

It is clear that Ψ maps the distiguished element $[c,1] \in (Pin(n)\widehat{\times}\hat{\pi})^{ev}$ (where c is the generator of $\pi_1(SO(n)) \subset Pin(n)$) to the distinguished element of $G(E)$ given by the image of the non-trivial element of $\pi_1(SO(n))$ under the homomorphism $i_*\colon \pi_1(SO(n)) \longrightarrow \pi_1(O(E)/r)$ induced by the inclusion of a fiber. It is also clear that Ψ makes the diagram of Lemma 2.15 commutative. What needs to be checked is that Ψ is in fact a homomorphism.

$$\Psi([[a],[b]]\cdot[[a'],[b']]) = \Psi(c^{w(a')w(b)}[c^{w(a)w(a')}[aa'(1) * r^{w(a)}a'],[br^{w(b')} * b']) =$$
$$\Psi([aa'(1) * r^{w(a)}a'],[br^{w(b')} * b']) = [aa'(1) * r^{w(a)}a' * br^{w(b')} * b']$$

Here the factor $c^{w(a')w(b)}$ comes from the multiplication in the product supergroup $Pin(n)\widehat{\times}\hat{\pi}$ (cf. Def. 2.4), while the factor $c^{w(a)w(a')}$ comes from the geometric description of $Pin(n)$ as the 'primed' version of a supergroup of paths in $O(n)$ provided by Lemma 3.5. We note that these factors cancel, since $w(a) = w(b)$, due to the assumption that $[[a],[b]]$ is in the even part of $Pin(n)\widehat{\times}\hat{\pi}$.

$$\Psi([[a],[b]])\cdot\Psi([[a'],[b']]) = [a*b][a'*b'] = [aa'(1) * ba'(1) * a' * b']$$

This shows that Ψ is a homomorphism, provided the path $ba'(1) * a'$ is homotopic (relative endpoints) to $r^{w(a)}a' * br^{w(b')}$. The desired homotopy is given by the map $[0,1]^2 \longrightarrow O(E)$, $(s,t) \mapsto b(s)\cdot a'(t)$. $\square$

3.7 Proof of Prop. 2.11. Let $\gamma = \gamma(E) = (\pi, w, \hat{\pi})$, and assume that the canonical γ-structure on E provided by Prop. 2.12 (1) is represented by the principal $G(n,\gamma)$-bundle $P \longrightarrow X$ and the $G(n,\gamma)$-equivariant map $\xi\colon P \longrightarrow O(E)$ (where $G(n,\gamma) = (Pin(n)\widehat{\times}\hat{\pi})^{ev}$ acts on $O(E)$ via $\rho_1\colon (Pin(n)\widehat{\times}\hat{\pi})^{ev} \longrightarrow O(n)$, $[a,b] \mapsto \sigma(a)$). In terms of classifying maps this means that the map $c_E\colon X \longrightarrow BO(n)$ classifying E is homotopic to $B\rho_1 \circ c$, where $c\colon X \longrightarrow BG(n,\gamma)$ is the classifying map of P (cf. Remark 2.13(2)).

Similarly, if $\rho_2\colon (Pin(n)\widehat{\times}\hat{\pi})^{ev} \longrightarrow \pi$ is given by $[a,b] \mapsto [b]$ (as in the proof of Lemma 2.7(1)), then $\widetilde{X} = P/\ker\rho_2 \longrightarrow X$ is a principal π-bundle, which can be identified with the universal covering of X. Hence the map $u\colon X \longrightarrow B\pi$ classifying the universal covering is homotopic to $B\rho_2 \circ c$. Since c is a 2-equivalence by Prop. 2.12(2), the induced map $c_P^*\colon H^2(BG(n,\gamma);\mathbb{Z}/2) \longrightarrow H^2(X;\mathbb{Z}/2)$ is injective. It follows that $u^*(e) = w_2(M)$ is equivalent to the statement $B\rho_1^*(w_2) = B\rho_2^*(e)$.

If $\hat{\pi} = \pi$, then $G(n,\gamma)$ can be identified with the group $(O(n)\times\pi)^{ev}$; clearly, in this case $B\rho_1^*(w_2)$ is not in the image of $B\rho_2^*$, and hence $w_2(M)$ is not in the image of u^*.

If $\hat{\pi} \neq \pi$, we claim that $B\rho_1^*(e) = B\rho_2^*(w_2)$. In terms of extensions, we need to show that the pull-back of $Pin'(n) \longrightarrow O(n)$ (representing w_2) via ρ_1 agrees with the pull-back of $\hat{\pi} \longrightarrow \pi$ (representing e) via ρ_2. In other words, we need to show that there is a homomorphism $\hat{\rho}_2: \rho_1^*(Pin'(n)) \longrightarrow \gamma$ making the diagram

$$\begin{array}{ccc} \rho_1^*(Pin'(n)) & \xrightarrow{\hat{\rho}_2} & \hat{\pi} \\ \downarrow & & \downarrow \\ (Pin(n)\widehat{\times}\hat{\pi})^{ev} & \xrightarrow{\rho_2} & \pi \end{array}$$

commutative. We note that every element in

$$\rho_1^*(Pin'(n)) \subset Pin'(n) \times (Pin(n)\widehat{\times}\hat{\pi})^{ev}$$

has a representative of the form $(a, [a, b])$, allowing us to define $\hat{\rho}_2$ by mapping $(a, [a, b])$ to $b \in \hat{\pi}$. This is in fact a homomorphism, since

$$\begin{aligned}(a, [a, b])(a', [a', b']) &= (c^{w(a)w(a')}aa', c^{w(b)w(a')}[aa', bb']) \\ &= (c^{w(a)w(a')}aa', [c^{w(a)w(a')}aa', bb']).\end{aligned}$$

Here the last equality holds, since $[a, b]$ is an even element of $Pin(n)\widehat{\times}\hat{\pi}$. □

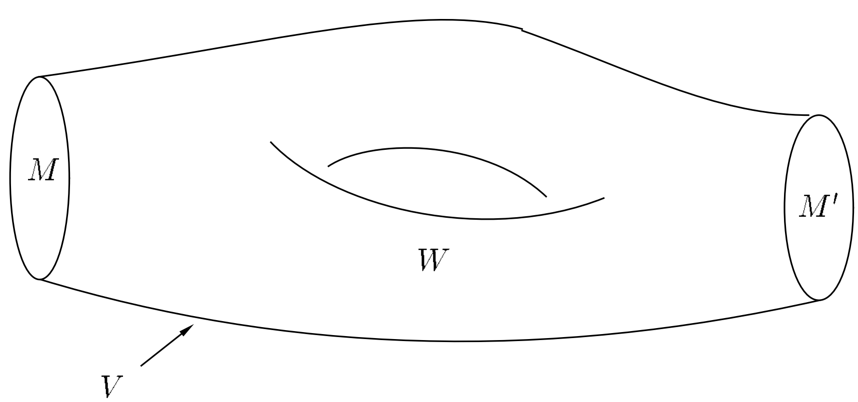
§4. Definition of the groups $R_n(\gamma)$

In this section we define the groups $R_n(\gamma)$ for a supergroup γ and discuss their elementary properties.

4.1 Definition. Given a supergroup γ, we define $R_n(\gamma)$ to be the bordism group of n-dimensional γ-manifolds (cf. Def. 2.6) with positive scalar curvature metrics on the boundary, i.e., the objects of of this bordism group are pairs (M, h), where M is an n-dimensional γ-manifold, and h is a positive scalar curvature metric on ∂M (which may be empty). Two pairs (M, h), (M', h') are bordant if

(1) There is a manifold V whose boundary is the disjoint union of ∂M and $\partial M'$ and a positive scalar curvature metric on V which restricts to h resp. h' on the boundary.
(2) There is a manifold W with boundary $M \cup_{\partial M} V \cup_{\partial M'} M'$.
(3) There is a γ-structure on W which restricts to the given γ-structure on $M \subset \partial W$ and to the 'opposite' γ-structure (explained below) on M'.

The following picture shows such a bordism.



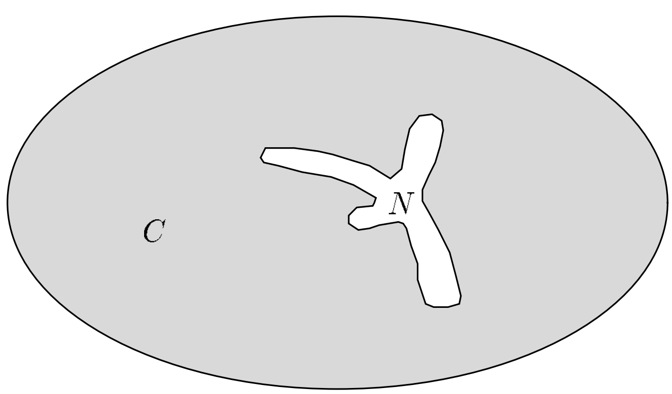
We note that the bijection provided by Lemma 2.8 between γ-structures on a vector bundle E and γ-structures on the sum $E \oplus \underline{\mathbb{R}}$ of E with a trivial line bundle makes it possible to restrict a γ-structure on M to a γ-structure on ∂M. To define the 'opposite' of a γ-structure we note that a γ-structure on a vector bundle E and a (metric preserving) vector bundle isomorphism $E \cong G$ induces a γ-structure on G. In particular, the isomorphism $id \oplus -id\colon E \oplus \underline{\mathbb{R}} \to E \oplus \underline{\mathbb{R}}$ induces an involution on the set of γ-structures on $E \oplus \underline{\mathbb{R}}$ and hence (via the above bijection) on the set of γ-structures on E. The *opposite* of a γ-structure is its image under this involution.

The disjoint union gives $R_n(\gamma)$ the structure of a semigroup; its unit is represented by the empty manifold. We claim that $R_n(\gamma)$ is in fact a group and that the inverse of the bordism class $[M, h]$ represented by (M, h) is given by $[\overline{M}, h]$, where $\overline{M}$ denotes the manifold M equipped with its opposite γ-structure. To see this, consider $M \times I$ with the γ-structure induced by the identification $T(M \times I) = p_1^*(TM) \oplus \underline{\mathbb{R}}$. Its restriction to $M \times \{0\}$ is the given γ-structure on M, while its restriction to $M \times \{1\}$ is the opposite. Hence $M \times I$ can be interpreted as a bordism between the disjoint union of (M, h) and $(\overline{M}, h)$ and the empty set, which shows $[\overline{M}, h] = -[M, h]$.

4.2 Remark. We recall from Lemma 2.7 that for $\gamma = (\pi, 0, \pi)$ (resp. $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$) a γ-structure on a manifold M amounts to an orientation (resp. spin-structure) on M and a principal π-bundle $\widetilde{M} \to M$, or, equivalently, a map $M \to B\pi$. In particular, $R_n(\gamma)$ for $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ agrees with the bordism group $\Omega_n^{rel}(B\pi)$ introduced by Hajduk in [Ha, §6].

4.3 Lemma. *Let M be a γ-manifold and let h be a positive scalar curvature metric on ∂M. Assume that h extends to a positive scalar curvature metric H on the manifold C obtained from M by removing the interior of a codimension zero submanifold $N \subset M$ (here we assume that N is contained in the interior of M and that H is a product metric near the boundary of C). Then (M, h) and $(N, H_{|\partial N})$ represent the same bordism class in $R_n(\gamma)$. In particular, if h extends to a positive scalar curvature metric on all of M, then $[M, h] = 0$.*

The following picture illustrates the situation; the shaded region is C over which h extends to the positive scalar curvature metric H.



Proof. A bordism is between (M, h) and $(N, H_{|\partial N})$ is given by $W = M \times I$ (after rounding the corners) equipped with the γ-structure which restricts to the given

γ-structure on $M \times \{0\}$. Its boundary decomposes as $\partial W = M \times \{0\} \cup V \cup N \times \{1\}$, where $V = \partial M \times I \cup C \times \{1\}$. The required positive scalar curvature metric on V is the product metric $h \times s$ on $\partial M \times I$ (s is the standard metric on I), and the metric H on $C \times \{1\}$. $\square$

It is not hard to show that the 'relative' bordism group $R_n(\gamma)$ fits into a long exact sequence

$$(4.4) \qquad \longrightarrow R_{n+1}(\gamma) \xrightarrow{\partial} \mathrm{Pos}_n(\gamma) \longrightarrow \Omega_n(\gamma) \longrightarrow R_n(\gamma) \xrightarrow{\partial} \mathrm{Pos}_{n-1}(\gamma) \longrightarrow .$$

Here $\Omega_n(\gamma)$ is the bordism group of n-dimensional (closed) manifolds with γ-structure, and $\mathrm{Pos}_n(\gamma)$ is the bordism group of pairs (M, g), where M is a (closed) n-dimensional γ-manifold, and g is a positive scalar curvature metric on M. The unlabeled maps in the sequence (4.4) are the obvious maps, while ∂ takes $[M, h]$ to $[\partial M, h]$ (h is a positive scalar curvature metric on the γ-manifold ∂M).

4.5 Remark. We note that $\Omega_n(\gamma)$ for $\gamma = (\pi, 0, \pi)$ (resp. $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$) can be identified with the oriented bordism group $\Omega_n^{SO}(B\pi)$ (resp. the spin bordism group $\Omega_n^{spin}(B\pi)$) (cf. Remark 4.2). For $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ the sequence (4.4) is discussed in [Ha, §7].

Given a supergroup $\gamma = (\pi, w, \hat{\pi})$, let $T_n(\gamma) \subset \Omega_n(\gamma)$ be the subgroup represented by total spaces of '$\mathbb{HP}^2$-bundles', *i.e.*, fiber bundles whose fiber is the quaternion projective plane $\mathbb{HP}^2$ and whose structure group is the isometry group of $\mathbb{HP}^2$ with its standard metric induced from $S^{11} \subset H^3$. Total spaces of $\mathbb{HP}^2$-bundles admit metrics of positive scalar curvature and hence $T_n(\gamma)$ is in the kernel of the map $\Omega_n(\gamma) \longrightarrow R_n(\gamma)$. This implies the following result.

4.6 Corollary. *The map $\Omega_n(\gamma) \longrightarrow R_n(\gamma)$ in the exact sequence (4.4) factors through a homomorphism $\Omega_n(\gamma)/T_n(\gamma) \longrightarrow R_n(\gamma)$. Its image consist of those bordism classes in $R_n(\gamma)$ that can be represented by closed γ-manifolds.*

We remark that in the 'spin case' $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ the quotient $\Omega_n(\gamma)/T_n(\gamma)$ is 2-locally isomorphic to the connective KO-homology group $ko_n(B\pi)$ by [St2, Thm. B]. At odd primes there is a geometric description of $ko_*(B\pi)$ in terms of manifolds with singularities in the sense of Baas-Sullivan. Using arguments due to Jung [Ju] this can be used to construct a homomorphism $ko_*(B\pi) \longrightarrow R_n(\gamma)$ at odd primes which is compatible with the 2-local map constructed above and hence gives a map without localizing.

Our next goal is to define various (external) products on bordism groups which are induced by the Cartesian product on representing manifolds. So our first task is to show that a γ_1-structure on a manifold M_1 and a γ_2-structure on a manifold M_2 induce a $\gamma_1 \widehat{\times} \gamma_2$-structure on the Cartesian product $M_1 \times M_2$.

4.7 Construction of the $\gamma_1 \widehat{\times} \gamma_2$-product structure. Let γ_1, γ_2 be supergroups, $\gamma_i = (\pi_i, w, \hat{\pi}_i)$, and let $\gamma = \gamma_1 \widehat{\times} \gamma_2 = (\pi, w, \hat{\pi})$ be their product (see 2.4). If $E_i \longrightarrow X_i$ is a vector bundle equipped with a γ_i-structure, we define a γ-structure on $E = E_1 \times E_2 \longrightarrow X = X_1 \times X_2$ as follows.

We recall that a γ-structure on an n-dimensional vector bundle E consists of a principal $G(n, \gamma)$-bundle $P \longrightarrow X$, $G(n, \gamma) = (Pin(n) \widehat{\times} \hat{\pi})^{ev}$, and a $G(n, \gamma)$-equivariant map $\xi \colon P \longrightarrow O(E)$ (cf. Definition 2.6). We note that for $n = n_1 + n_2$ there is a commutative diagram

$$\begin{array}{ccc} G(n_1,\gamma_1)\times G(n_2,\gamma_2) & \xrightarrow{\hat{i}} & G(n,\gamma) \\ {\scriptstyle \rho_1\times\rho_1}\big\downarrow & & \big\downarrow{\scriptstyle \rho_1} \\ O(n_1)\times O(n_2) & \xrightarrow{i} & O(n), \end{array}$$

where i is the obvious inclusion, and $\hat{i}$ is given by

$$\begin{aligned} (Pin(n_1)\widehat{\times}\hat{\pi}_1)^{ev}\times(Pin(n_2)\widehat{\times}\hat{\pi}_2)^{ev} &\xrightarrow{\hat{i}} (Pin(n)\widehat{\times}\hat{\pi})^{ev} \\ ([a_1,b_1],[a_2,b_2]) &\mapsto [(-1)^{w(b_1)w(a_2)}a_1a_2,[b_1,b_2]] \end{aligned}$$

(here we identify $Pin(n_i)$ with subgroups of $Pin(n)$). Hence if (P_i,ξ_i) represents the γ_i-structure on E_i, then

$$P \stackrel{\text{def}}{=} (P_1\times P_2)\times_{G(n_1,\gamma_1)\times G(n_2,\gamma_2)} G(n,\gamma)$$

is a principal $G(n,\gamma)$-bundle over X, and $\xi_1\times\xi_2\times\rho_1$ induces a $G(n,\gamma)$-equivariant map

$$\xi\colon P\longrightarrow (O(E_1)\times O(E_2))\times_{O(n_1)\times O(n_2)} O(n) = O(E_1\times E_2).$$

The pair (P,ξ) is the *product γ-structure* on $E_1\times E_2$.

4.8 Construction of external products. Let γ_1, γ_2 be supergroups, and let $\gamma_1\widehat{\times}\gamma_2$ be their product. Then we define a pairing

$$(4.9)\quad \mu_\Omega\colon\Omega_{n_1}(\gamma_1)\otimes\Omega_{n_2}(\gamma_2)\longrightarrow\Omega_{n_1+n_2}(\gamma_1\widehat{\times}\gamma_2)\quad\text{by}\quad [M_1]\otimes[M_2]\mapsto[M_1\times M_2],$$

where $M_1\times M_2$ is equipped with the product $\gamma_1\widehat{\times}\gamma_2$-structure. More generally, we can define a pairing

$$(4.10)\qquad \begin{aligned} \Omega_{n_1}(\gamma_1)\otimes R_{n_2}(\gamma_2) &\longrightarrow R_{n_1+n_2}(\gamma_1\widehat{\times}\gamma_2) \\ \text{by}\qquad [M_1]\otimes[M_2,h] &\mapsto [M_1\times M_2, g+th], \end{aligned}$$

where g is any Riemannian metric on M_1, the metric $g\times th$ is the product metric on $M_1\times\partial M_2=\partial(M_1\times M_2)$, and $t>0$ is any number sufficiently small such that the scalar curvature of $g\times th$ is positive. It is clear that the bordism class of $(M_1\times M_2, g\times th)$ is independent of the choice of t, and that it only depends on the bordism classes $[M_1]\in\Omega_{n_1}(\gamma_1)$ and $[M_2,h]\in R_{n_2}(\gamma_2)$.

Let $\Omega_n^+(\gamma)$ denote the kernel of the map $\Omega_n(\gamma)\longrightarrow R_n(\gamma)$; it consists of those bordism classes which can be represented by γ-manifolds M which admit metrics of positive scalar curvature. We note that if $[M_1]$ belongs to the subgroup $\Omega_{n_1}^+(\gamma)\subset\Omega_{n_1}(\gamma)$, we can assume that it admits a positive scalar curvature metric g. Then for any $t>0$ the scalar curvature of $g\times th$ is positive, and for sufficently large t the metric $g\times tH$ has positive scalar curvature where H is any metric on M_2 (its scalar curvature is not required to be positive) extending h. Hence by Lemma 4.3,

$[M_1 \times M_2, g \times th]$ is the trivial element of $R_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2)$. This shows that the pairing 4.10 factors through a pairing

$$(4.11) \qquad \left(\Omega_{n_1}(\gamma_1)/\Omega^+_{n_1}(\gamma_1)\right) \otimes R_{n_2}(\gamma_2) \longrightarrow R_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2).$$

Via the homomorphism $\Omega_n(\gamma) \longrightarrow R_n(\gamma)$ we can identify $\Omega_n(\gamma)/\Omega^+_n(\gamma)$ with a subgroup of $R_n(\gamma)$ (the subgroup consisting of those bordism classes which can be represented by *closed* γ-manifolds). In section 6 we will show that the product (4.11) extends to a product $R_{n_1}(\gamma_1) \otimes R_{n_2}(\gamma_2) \longrightarrow R_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2)$.

Now we specialize the pairing (4.11) to the case where the supergroup γ_1 is $\epsilon = (\{1\}, 0, \mathbb{Z}/2)$ or $\epsilon^{SO} = (\{1\}, 0, \{1\})$. We recall $\Omega_n(\epsilon) = \Omega^{spin}_n$ and $\Omega_n(\epsilon^{SO}) = \Omega^{SO}_n$. Gromov and Lawson showed in [GL1] that the ideal $\Omega^{SO,+}_* \subset \Omega^{SO}_*$ is the augmentation ideal; in other words, the quotient $\Omega^{SO}_n/\Omega^{SO,+}_n$ is $\mathbb{Z}$ for $n = 0$ and trivial otherwise. The author proved in [St1] that $\Omega^{spin}_n/\Omega^{spin,+}_n$ can be identified with ko_*, the connective KO-homology of a point. We recall that the graded ring ko_* has generators η, ω and μ of degrees 1, 4 and 8 respectively, and relations $2\eta = 0$, $\eta^3 = 0$, $\eta\omega = 0$ and $\omega^2 = 4\mu$.

We observe that ϵ is a unit for supergroups in the sense that for any supergroup γ the product $\epsilon \widehat{\times} \gamma$ can be identified with γ. Similarly, if $\gamma = (\pi, w, \hat{\pi})$ with $\hat{\pi} = \pi$, then $\epsilon^{SO} \widehat{\times} \gamma$ can be identified with γ. Hence the pairing (4.11) and the discussion above implies the following result.

4.12 Proposition. *For any supergroup $\gamma = (\pi, w, \hat{\pi})$, the $\mathbb{Z}$-graded abelian group $R_*(\gamma) = \bigoplus_{n \geq 0} R_n(\gamma)$ is a $\mathbb{Z}$-graded module over ko_*. If $\hat{\pi} = \pi$, then ko_n acts trivially for $n > 0$.*

We recall from the introduction that a 'Bott-manifold' is any spin manifold B of dimension 8 with $\widehat{A}(B) = 1$. This does not determine the spin bordism class of B since $\Omega^{spin}_8 \cong \mathbb{Z} \oplus \mathbb{Z}$, where the isomorphism is given by sending the bordism class of a manifold M to $(\widehat{A}(M), \text{signature}(M))$. However, it does determine the class of B in $ko_8 \cong \Omega^{spin}_8/\Omega^{spin,+}_8 \cong \mathbb{Z}$ (the quaternion projective plane $\mathbb{HP}^2$ is an 8-dimensional spin manifold with signature one which admits a positive scalar curvature metric).

Multiplication by $B \in ko_8$ gives a map $R_n(\gamma) \xrightarrow{\times B} R_{n+8}(\gamma)$ which in general is *not* injective as the following example shows.

4.13 Example. Let M be the connected sum $M = T^6 \#(\mathbb{CP}^2 \times S^2)$ of the 6-dimensional torus with $\mathbb{CP}^2 \times S^2$. The minimal hypersurface method of Schoen-Yau [SY] shows that the connected sum M does not admit a positive scalar curvature metric [GL2, p. 186]. This implies by Theorem 1.1 that M represents a non-trivial element in $R_6(\gamma(M))$. Lemma 2.10 shows that $\gamma(M) = (\mathbb{Z}^6, 0, \mathbb{Z}^6)$ since M is orientable, and its universal covering is non-spin. Hence Proposition 4.12 implies that the product of $[M]$ with B vanishes.

We recall from the introduction that we can define a 'periodic' or 'stable' version of the R_n-groups by 'inverting' B. More precisely, we define

$$(4.14) \qquad R_n(\gamma)[B^{-1}] \overset{\text{def}}{=} \varinjlim \left(R_n(\gamma) \xrightarrow{\times B} R_{n+8}(\gamma) \xrightarrow{\times B} R_{n+16}(\gamma) \xrightarrow{\times B} \dots \right).$$

Then Proposition 4.12 implies:

4.15 Corollary. *If $\gamma = (\pi, w, \hat{\pi})$ is a supergroup with $\hat{\pi} = \pi$, then $R_n(\gamma)[B^{-1}] = 0$.*

We observe that the homomorphism $F\colon \Omega_n(\gamma)/T_n(\gamma) \to R_n(\gamma)$ of Corollary 4.6 induces a homomorphism

$$F\colon KO_n(\gamma) \overset{\text{def}}{=} (\Omega_n(\gamma)/T_n(\gamma))\,[B^{-1}] \to R_n(\gamma)[B^{-1}].$$

We recall $\Omega_n(\epsilon) = \Omega_n^{spin}$, where ϵ is the supergroup $(\{1\}, 0, \mathbb{Z}/2)$ (cf. Remark 4.5), and that $ko_n = \Omega_n(\epsilon)/\Omega_n^+(\epsilon) = \Omega_n(\epsilon)/T_n(\epsilon)$ by [St1, Thm. B]. It follows that Cartesian product of manifolds makes $\Omega_*(\gamma)/T_*(\gamma)$ a ko_*-module; in particular, we can multiply by $B \in ko_8$ and define similarly as in (4.14) the group $(\Omega_n(\gamma)/T_n(\gamma))\,[B^{-1}]$ as the direct limit of homomorphisms given by multiplication by B. The notation $KO_n(\gamma)$ for this group is motivated by the fact that in the 'spin case' $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ it is isomorphic to $KO_n(B\pi)$ by [KS, Thm. C].

§5. Proof of the existence and classification theorem

In this section we prove Theorem 1.1 of the introduction. We begin by reminding the reader of our convention from the introduction.

5.1 Convention. All metrics considered on a manifold with boundary are product metrics near the boundary.

If M is a manifold we denote by $\mathfrak{R}^+(M)$ the space of positive scalar curvature metrics on M. If h is a positive scalar curvature metric on the boundary ∂M, we denote by $\mathfrak{R}^+(M \operatorname{rel} h)$ the space of positive scalar curvature metrics on M restricting to h on the boundary. Furthermore, we write $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ for the concordance classes of such metrics. As defined in the introduction two metrics $g_0, g_1 \in \mathfrak{R}^+(M \operatorname{rel} h)$ are *concordant* if there is a positive scalar curvature metric G on $M \times I$ 'restricting to the given metrics on the boundary'. To make this precise if M has a boundary, it is best to think of $M \times I$ as a manifold with corners (we *could* regard it as a manifold with boundary by smoothing the corners, but that turns out to be less convenient). We extend our convention (5.1) to such manifolds:

5.2 Convention. All metrics considered on a manifold M with corners are product metrics near the boundary and near the corners. This means that a neighbourhood of the set C of corner points is isometric to $C \times [0, \epsilon)^2$ for some $\epsilon > 0$, and that a neigbourhood of $\partial M \setminus C$ is isometric to $(\partial M \setminus C) \times [0, \epsilon)$.

5.3 Definition. Two metrics $g_0, g_1 \in \mathfrak{R}^+(M \operatorname{rel} h)$ are *concordant* if there is a positive scalar curvature metric G on $M \times I$ which restricts to the metric

$$g_0 \cup h \times s \cup g_1 \quad \text{on} \quad (M \times \{0\}) \cup (\partial M \times I) \cup (M \times \{1\}) = \partial(M \times I).$$

5.4 Theorem. *Let π be a finitely presented group, and let $\gamma = (\pi, w, \hat{\pi})$ be a supergroup. Let M be a γ-manifold of dimension $n \geq 5$ such that the map $c\colon M \to BG(n, \gamma)$ determined by the γ-structure is a 2-equivalence (i.e., the induced map on π_i is an isomorphism for $i < 2$, and surjective for $i = 2$). Let h be a positive scalar curvature metric on the boundary ∂M.*

(1) *The metric h extends to a positive scalar curvature on M if and only if $[M, h]$ is the trivial element of $R_n(\gamma)$.*

(2) *If $\mathfrak{R}^+(M \operatorname{rel} h)$ is not empty then the group $R_{n+1}(\gamma)$ acts freely and transitively on $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$.*

This result implies Theorem 1.1 since by Prop. 2.12 the map c determined by the *canonical* $\gamma(M)$-structure is a 2-equivalence; also, if $\gamma(M) = (\pi, w, \hat{\pi})$, then π is the fundamental group of M and hence finitely generated.

The main tool in the proof of Theorem 5.4 is the following result.

5.5 Extension Theorem. *Let W be a manifold, whose boundary decomposes in the form $\partial W = M \cup_{\partial M} V$, where M, V are manifolds of dimension $n \geq 5$ with the same boundary. If the inclusion $M \to W$ is a 2-equivalence, then any positive scalar curvature metric on V extends to a positive scalar curvature metric on W.*

This follows quite directly from a refinement of results of Gromov-Lawson [GL1] and Schoen-Yau [SY] due to Gajer [Ga]. With the additional assumption $\partial M = \partial V = \emptyset$ this is proved in [St3, Thm. 3.3]. The same proof works in the more general situation.

For the proof of Theorem 5.4 we also need the next Lemma which is proved by the standard 'surgery below the middle dimension' argument. We will use the following terminology. If M and M' are γ-manifolds with $\partial M = \partial M'$, they are *γ-bordant relative boundary* if the closed γ-manifold $M \cup_{\partial M} \overline{M}'$ (where $\overline{M}'$ is the manifold M' with its opposite γ-structure; see Def. 4.1) is the boundary of some γ-manifold.

5.6 Lemma. *Let $\gamma = (\pi, w, \hat{\pi})$ be a supergroup with π finitely presented, and let M be a γ-manifold of dimension n.*

(1) *For $n \geq 5$ the γ-manifold M is γ-bordant relative boundary to a γ-manifold M' such that the map $c_{M'}\colon N' \to BG(n,\gamma)$ determined by the γ-structure is a 2-equivalence;*
(2) *If $n \geq 6$, and N is a codimension zero submanifold of ∂M such that $c_N\colon N \to BG(n,\gamma)$ is a 2-equivalence, then M is γ-bordant relative boundary to a γ-manifold M' such that the inclusion $N \to M'$ is a 2-equivalence.*

5.7 Remark. We note that a stronger form of part (1), namely the statement that $c_{M'}$ induces an isomorphism on π_i for $i = 0, 1, 2$ would imply part (2). This line of reasoning is used by Hajduk [Ha] whose Lemma 3.1 claims that any class of $\Omega_n^{spin}(B\pi)$ can be represented by a connected spin manifold M and a map $f\colon M \to B\pi$ such that f induces an isomorphism on π_1, and $\pi_2(M) = 0$ (this is the stronger form of part (1) in the special case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ and $\partial M = \emptyset$).

However, this statement is not correct in general; if true, we could produce the classifying space $B\pi$ by attaching cells of dimension ≥ 4 to M in order to kill the homotopy groups $\pi_i(M)$ for $i \geq 3$. In particular, we would have constructed a CW complex with *finite* 3-skeleton representing $B\pi$ implying that $H_3(B\pi; \mathbb{Z})$ is finitely generated. However, an example of Stallings [Sta] shows that there are finitely presented groups π, whose 3-dimensional integral homology is not finitely generated.

What goes wrong in Hajduk's argument is that although every element in $\pi_2(M)$ can be represented by an embedded 2-sphere in M and may hence by killed surgery on this 2-sphere, we need to make sure that $\pi_2(M)$ is finitely generated in order to kill *all* of $\pi_2(M)$.

Proof. The above Lemma is essentially a special case of a result of Wall [Wa, Thm. 1.2] after interpreting γ-structures as normal maps into a suitable space X. For the convenience of the reader, we state Wall's result and deduce the Lemma.

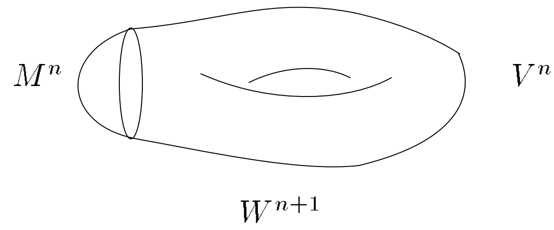
Let ν be a vector bundle over a space X. We recall that a map c from a manifold M to X, together with a (stable) trivialization of $TM \oplus c^*\nu$ is called a *normal map*. According to Wall's result a finite sequence of surgeries in the interior of M (on embedded spheres of dimension $< k$) will produce a manifold M' with a normal map $c': M' \longrightarrow X$ which is a k-equivalence, provided $n = \dim M \geq 2k$, and X is a finite complex. In fact, the proof shows that the number of r-surgeries (*i.e.* surgeries on embedded spheres of dimension r) corresponds to the number of $r+1$-cells needed to construct the mapping cylinder of ϕ by attaching cells to W. Hence the assumption that X is finite may be weakened to the assumption that X has a CW complex with finite k-skeleton.

To apply Wall's Theorem, let X be the $(n+1)$-skeleton of $BG(n,\gamma)$ and let ν be an inverse to the vector bundle $\alpha = EG(n,\gamma) \times \rho_1\mathbb{R}^n$ associated to the representation $\rho_1: G(n,\gamma) \longrightarrow O(n)$ (see Def. 2.6). We note that a γ-structure (P,ξ) on M gives a map $c: M \longrightarrow X \subset BG(n,\gamma)$ (the classifying map of P) and a vector bundle isomorphism $TM \cong c^*\alpha$ (induced by the $G(n,\gamma)$-equivariant map $\xi: P \longrightarrow O(M)$). Hence a γ-structure on M determines a 'normal map' to (X,ν); in fact, this construction gives a bijection between γ-structures and 'normal maps'.

Our assumption that π is finitely presented implies that $BG(n,\gamma)$ has a CW decomposition with finite 2-skeleton (the homomorphism $\rho_2: G(n,\gamma) \longrightarrow \pi$ defined in the proof of Lemma 2.7 leads to a fiber bundle $B\rho_2: BG(n,\gamma) \longrightarrow \pi$, whose fiber is $BSpin(n)$ for $\hat{\pi} \neq \pi$, and $BSO(n)$ for $\hat{\pi} = \pi$). Hence we may apply Wall's Theorem to obtain after finitely many surgeries a normal map $c': M' \longrightarrow X$ which is a 2-equivalence. Interpreting the normal maps on M' and on the trace of the surgeries as γ-structures then proves part (1) of the Lemma.

To prove part (2) consider M' as a CW complex obtained from N by attaching cells of dimension ≥ 2 (no 0-cells or 1-cells are needed since $N \longrightarrow M'$ induces an isomorphism on π_1, and both are connected). In particular, the 2-skeleton of M' is obtained from $N^{(2)}$ by attaching finitely many 2-cells. We note that the attaching maps are (homotopically) trivial, since the inclusion map $N \longrightarrow M'$ induces an isomorphism on π_1. Hence the 2-skeleton of M' is homotopy equivalent to the one-point union of $N^{(2)}$ and finitely many copies of S^2. It follows that the cokernel of the homomorphism $\pi_2(N) \longrightarrow \pi_2(M')$ is finitely generated. Surgery on embedded 2-spheres $S^2 \subset M'$ representing these generators then leads to a manifold M" bordant to M' relative boundary, for which the inclusion map $N \longrightarrow M$" is a 2-equivalence. This proves part (2). $\square$

Proof of Theorem 5.4(1). We recall from Lemma 4.3 that if h extends to a positive scalar curvature metric on M then (M,h) represents zero in $R_n(\gamma)$. Conversely, let (W,H) be a zero bordism for (M,h); *i.e.*, W is an $(n+1)$-dimensional γ-manifold whose boundary decomposes in the form $\partial W = M \cup_{\partial M} V$, and H is a positive scalar curvature metric on V restricting to h on $\partial V = \partial M$ (cf. picture).



Applying Lemma 5.6(2) to the γ-manifold W and the submanifold $M \subset \partial W$ we may assume that the inclusion map $M \longrightarrow W$ is a 2-equivalence.Then the Extension Theorem 5.5 implies that H extends to a positive scalar curvature metric over all of W. The restriction of this metric to M is then the desired positive scalar curvature metric extending h.

For the proof of part (2) of Theorem 5.4 we will need the following result which is similar to Theorem 6.2 in in Hajduk's paper [Ha].

5.8 Proposition. *Let N be a manifold with boundary of dimension $n \geq 6$ which is simply homotopy equivalent to a finite 2-complex. Assume that N is furnished with a γ-structure such that the map $c_N \colon N \longrightarrow BG(n,\gamma)$ is a 2-equivalence. Then every element of $R_n(\gamma)$ can be represented by (N, q) for some positive scalar curvature metric q on ∂N.*

Proof. Suppose an element of $R_n(\gamma)$ is represented by an n-dimensional γ-manifold M and a positive scalar curvature metric h on ∂M. By Lemma 5.6(1) we can assume without loss of generality that $c_M \colon M \longrightarrow BG(n,\gamma)$ is a 2-equivalence. Let K be a finite 2-dimensional CW-complex and let $\alpha \colon K \longrightarrow N$ be a simple homotopy equivalence. We construct a homotopy commutative diagram

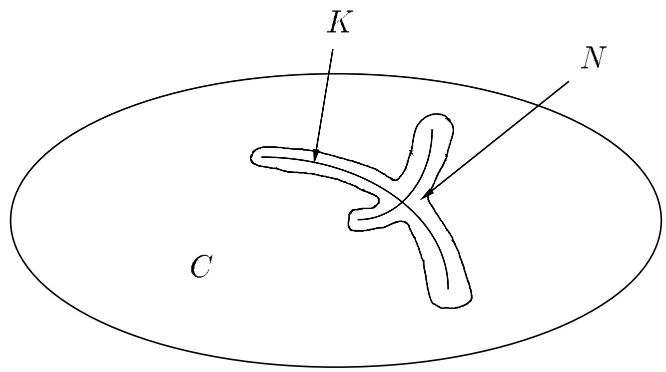
$$(5.9)\qquad \begin{tikzcd} K \arrow[r, equal] \arrow[d, "\alpha'"'] & K \arrow[r, "\alpha"] \arrow[d, "j"'] & N \arrow[d, "c_N"] \\ N' \arrow[r, "i"] & M \arrow[r, "c_M"] & BG(n,\gamma) \end{tikzcd}$$

as follows. By obstruction theory the composition $K \xrightarrow{\alpha} N \xrightarrow{c_N} BG(n,\gamma)$ factors up to homotopy through c_M; i.e., there is a map $j \colon K \longrightarrow M$ making the right hand square homotopy commutative. We observe that j is a 2-equivalence since c_M induces an isomorphism on π_i for $i = 0, 1, 2$, and the composition $c_M \circ j = c_N \circ \alpha$ is a 2-equivalence.

By the Embedding Theorem of Wall [Wa1, §2] the map j factors up to homotopy in the form $K \xrightarrow{\alpha'} N' \xrightarrow{i} M$, where α' is a simple homotopy equivalence, N' is an n-dimensional manifold with $\pi_1(\partial N') = \pi_1(N')$, and i is an embedding of N' as a codimension zero submanifold of M.

We note that both, $K \xrightarrow{\alpha} N$ and $K \xrightarrow{\alpha'} N'$ are "n-thickenings" of K in the sense of Wall [Wa1, §1]. Moreover, these are "stable" thickenings in the sense that $n \geq 2k + 1$ where k is the dimension of K. Wall shows that stable thickenings $K \xrightarrow{\alpha} N$ are classified by the pull back $\alpha^*(TN)$ of the tangent bundle [Wa1, Proposition 5.1]. The homotopy commutative diagram 5.9 implies that $\alpha^*(TN)$ and $(\alpha')^*(TN')$ are isomorphic since TN and TN' are both pullbacks of the bundle $EG(n,\gamma)\times_{G(n,\gamma)}\mathbb{R}^n$ via c_N and $c_M\circ i$, respectively. This implies that there is a diffeomorphism $d\colon N\longrightarrow N'$ such that $d\circ\alpha$ is homotopic to α'.

Identifying N' with N via this diffeomorphism, we decompose M as $M=N\cup_{\partial N} C$, where C is the complement of the interior of N in M.



The decomposition $M=N\cup_{\partial N} C$

We note that the inclusion $N\hookrightarrow M$ is a 2-equivalence since the composition $N\hookrightarrow M\xrightarrow{c_M} BG(n,\gamma)$ is a 2-equivalence and c_M induces an isomorphism on π_i for $i=0,1,2$. Then the pushout diagram

$$\begin{array}{ccc} \partial N & \longrightarrow & C \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

implies that $\partial N\longrightarrow C$ is a 2-equivalence. By the Extension Theorem 5.5 the metric h on ∂M extends to a positive scalar curvature metric H on C. In particular, $(N,H_{|\partial N})$ represents the same element in $R_n(\gamma)$ as (M,h) by Lemma 4.3. $\square$

Proof of 5.4(2). For the proof of part (2) we note that the existence of a free and transitive action of $R_{n+1}(\gamma)$ on $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ is equivalent to the existence of a pairing

$$i\colon \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)\times\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)\longrightarrow R_{n+1}(\gamma) \tag{5.10}$$

with the following properties.

(i) $i(g,g)=0$ and $i(g,g')+i(g',g'')=i(g,g'')$ for $g,g',g''\in\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$.

(ii) The map $i_g\colon\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)\longrightarrow R_{n+1}(\gamma)$, which sends $g'\in\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ to $i(g,g')$ is bijective for any $g\in\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$.

In fact, if i is such a pairing we can define the a product of elements $x\in R_{n+1}(\gamma)$ and $g\in\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ by $x\cdot g=i_g^{-1}(x)$. Property (i) ensures that this defines an action of $R_{n+1}(\gamma)$ on $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ and (ii) implies that this action is free and transitive.

To define a pairing i with properties (i) and (ii) consider the γ-manifold $M\times I$. Given two elements $g,g'\in\mathfrak{R}^+(M \operatorname{rel} h)$ we get a positive scalar curvature metric

$$g\cup h\times s\cup g' \quad\text{on}\quad (M\times\{0\})\cup(\partial M\times I)\cup(M\times\{1\})=\partial(M\times I).$$

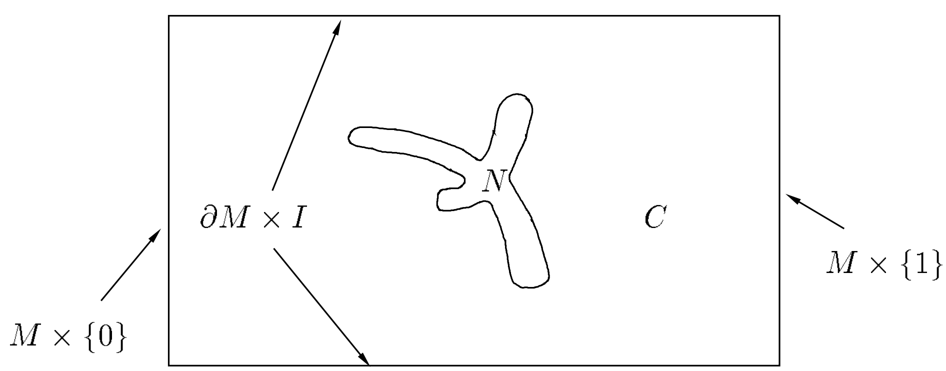
Here s is the standard metric on I and $h \times s$ is the product metric on $\partial M \times I$. We define a pairing i in (5.10) by

$$i(g, g') = [M \times I, g \cup h \times s \cup g'] \in R_{n+1}(\gamma). \tag{5.11}$$

It is easy to check that this pairing has property (i): $g \cup h \times s \cup g$ extends over $M \times I$ and hence $i(g, g) = 0$. The sum $i(g, g') + i(g', g'')$ is represented by two disjoint copies of $(M \times I)$ with the metric $g \cup h \times s \cup g'$ on the boundary of the first copy and $g' \cup h \times s \cup g''$ on the boundary of the second copy. As usual, this disjoint union is bordant to the manifold obtained by identifying $M \times \{1\}$ of the first copy of $M \times I$ with $M \times \{0\}$ of the second copy. Note that the metrics and the γ-structures match and that the resulting manifold is diffeomorphic to $M \times I$ with positive scalar curvature metric $g \cup h \times s \cup g''$ on its boundary. This proves $i(g, g') + i(g', g'') = i(g, g'')$.

Concerning property (ii) we note that injectivity of i_g is a direct consequence of part (1) of Theorem 5.4 as follows. If g' and g'' map to the same element under i_g, then $i(g', g'') = i(g', g) + i(g, g'') = -i_g(g') + i_g(g'') = 0$ (the second equation follows from property (i)). This implies by part (1) that $g' \cup h \times s \cup g''$ extends to a positive scalar curvature metric over $M \times I$, which shows that g', g'' are concordant relative h.

To prove surjectivity of i_g, let $N \subset \mathrm{int}(M \times I)$ (the interior of $M \times I$) be a thickening of the 2-skeleton of M. By Poposition 5.8 any element of $R_{n+1}(\gamma)$ can be represented by (N, q) for some positive scalar curvature metric q on ∂N. We decompose $M \times I$ in the form $M \times I = N \cup_{\partial N} C$, where C is the complement of the interior of N.



The decomposition $M \times I = N \cup_{\partial N} C$

We wish to extend the positive scalar curvature metric

$$g \cup h \times s \cup q \quad \text{on} \quad (M \times \{0\}) \cup (\partial M \times I) \cup \partial N$$

to a positive scalar curvature metric G on all of C. By the Extension Theorem 5.5 this can be done provided the inclusion $M \times \{1\} \hookrightarrow C$ is a 2-equivalence. Assuming this for now, the pairs $(M \times I, g \cup h \times s \cup g')$ with $g' = G_{|M \times \{1\}}$ and (N, q) represent the same element in $R_{n+1}(\gamma)$ by Lemma 4.3. This proves surjectivity of i_g.

To prove that $M\times\{1\}\hookrightarrow C$ is a 2-equivalence we note that by transversality the pair $(N,\partial N)\sim(N,N\setminus M^{(2)})$ ($M^{(2)}$ is the 2-skeleton of M) is $(n-2)$-connected. Hence with our assumption $n\geq 5$ the map $\partial N\to N$ is a 3-equivalence. Then the pushout diagram

$$\begin{array}{ccc} \partial N & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ N & \longrightarrow & M\times I \end{array}$$

implies that $C\to M\times I$ is also a 3-equivalence. It follows that the inclusion $M\times\{1\}\to C$ is a 2-equivalence (since the composition $M\times\{1\}\to C\to I\times M$ is a homotopy equivalence). $\square$

§6. An external product for the R_n-groups

The goal of this section is to prove the following result.

6.1 Theorem. *There is an (external) product*

$$\mu_R\colon R_{n_1}(\gamma_1)\otimes R_{n_2}(\gamma_2)\xrightarrow{\mu_R} R_{n_1+n_2}(\gamma_1\widehat{\times}\gamma_2)$$

which on $\left(\Omega_{n_1}(\gamma_1)/\Omega^+_{n_1}(\gamma_1)\right)\otimes R_{n_2}(\gamma_2)\subset R_{n_1}(\gamma_1)\otimes R_{n_2}(\gamma_2)$ *restricts to the product (4.11).*

Roughly speaking, the product of $[M_1,h_1]\in R_{n_1}(\gamma_1)$ and $[M_2,h_2]\in R_{n_2}(\gamma_2)$ is represented by the Cartesian product $M_1\times M_2$ together with a suitable positive scalar curvature metric on its boundary manufactured from h_1 and h_2. We encounter two difficulties:

(1) The Cartesian product $M_1\times M_2$ is in general a manifold with corners.
(2) We need to extend the positive scalar curvature metric h_i on ∂M_i to a metric g_i on M_i (whose scalar curvature is not necessarily positive). Then $h_1\times g_2\cup g_1\times h_2$ is a metric on $\partial M_1\times M_2\cup M_1\times\partial M_2=\partial(M_1\times M_2)$. This will be a positive scalar curvature metric only if we assume suitable lower bounds for the scalar curvature of the metrics h_1,g_1,h_2,g_2.

We will avoid the first difficulty by considering a version of the R_n-groups where the objects are complete manifolds without boundary, whose scalar curvature is positive outside a compact set (attaching an infinite cylinder to the boundary gives a map between these two versions of the R_n-groups). The second difficulty is overcome by putting suitable restrictions on the scalar curvature function of these complete manifolds which will guarantee that the scalar curvature of the Cartesian product is positive outside a compact set. It turns out that a convenient class of metrics to work with is the following.

6.2 Definition of (a,b)-adapted metrics. Let N be a manifold and let K be a subset of N. Given real numbers a,b we say that a riemannian metric g is (K,a,b)-*adapted* if g is complete, $s(x)>a$ for all $x\in N$ and $s(x)>b$ for $x\in N\setminus K$. A metric g is called (a,b)-*adapted* if there is a compact subset $K\subset N$ such that

(1) g is (K,a,b)-adapted and

(2) there is a compact submanifold $M \subset N$ of codimension zero with $K \subset M$ and a homotopy g^t of (K, a, b)-adapted metrics on N such that $g^0 = g$ and g^t agrees with g outside a compact neigbourhood $U(\partial M) \subset N \setminus K$ of ∂M. Furthermore, g^1 is a product metric in a neighbourhood of ∂M.

We recall here our convention (5.1) that all riemannian metrics considered in this paper are product metrics near the boundary. In particular, the metric g^1 in (ii) above is a product metric near ∂N as well as near ∂M. We require that these product structures are compatible in the sense that a neighbourhood of $\partial M \cap \partial N$ is isometric to the product metric on $\partial M \cap \partial N \times (-\epsilon, \epsilon) \times [0, \epsilon)$.

Condition (2) above is a technical condition that will be necessary in our proof that the bordism group $R_n(\gamma; a, b)$ defined below is isomorphic to the group $R_n(\gamma)$. It seems to the author that condition (2) might be a consequence of condition (1).

6.3 Definition of $R_n(\gamma; a, b)$. Let γ be a supergroup. We define $R_n(\gamma; a, b)$ to be the bordism group of n-dimensional γ-manifolds equipped with (a, b)-adapted metrics; i.e., $R_n(\gamma; a, b)$ consists of equivalence classes of pairs (N, g), where N is an n-dimensional manifold without boundary (in general non compact), equipped with a γ-structure, and g is an (a, b)-adapted complete riemannian metric on M. Two pairs (N, g), (N', g') are *bordant* if

(1) There is a complete manifold (W, G) with G (a, b)-adapted whose boundary is the disjoint union of (N, g) and (N', g').
(2) There is a γ-structure on W which restricts to the given γ-structure on $N \subset \partial W$ and to the opposite γ-structure on N'.

6.4 Lemma. *For $a < 0 < b$ there is a canonical isomorphism $R_n(\gamma; a, b) \cong R_n(\gamma)$.*

Proof. Let g be a (a, b)-adapted metric on a γ-manifold N. Let $M \subset N$ and g^t be as in Definition (6.2). Then

$$R_n(\gamma; a, b) \longrightarrow R_n(\gamma) \qquad [N, g] \mapsto [M, g^1_{|\partial M}]$$

is a well-defined homomorphism. An inverse to this map is given by

$$R_n(\gamma) \longrightarrow R_n(\gamma; a, b) \qquad [M, h] \mapsto [\widehat{M}, g].$$

Here $\widehat{M}$ is the manifold obtained by attaching a cylinder $\partial M \times [0, \infty)$ along the boundary; that is

$$\widehat{M} = M \cup_{\partial M} \partial M \times [0, \infty).$$

To construct the metric g on $\widehat{M}$, let H be any metric on M with $H_{|\partial M} = h$, and define

$$\begin{aligned} g_{|M} &= C_1\, H \\ g_{|\partial M \times [C_3, \infty)} &= C_2\, h + dt^2 \\ g_{|\partial M \times [0, C_3]} &= \phi(t/C_3) h + dt^2. \end{aligned}$$

Here $C_1 > 0$ is a constant chosen large enough so that the scalar curvature of the rescaled metric $C_1\, H$ is $> a$ (a is negative!) for all $x \in M$, and $C_2 > 0$ is a constant chosen small enough so that the scalar curvature of the rescaled metric $C_2\, h$ is $> b$

(b is positive!) for all $x \in \partial M$. The smooth function $\phi\colon [0,1] \longrightarrow \mathbb{R}^+$ is chosen such that $\phi(t) = C_1$ for $t \in [0, \epsilon)$, $\phi(t) = C_2$ for $t \in (1-\epsilon, 1]$; for sufficiently large C_3, the scalar curvature of g is $> a$ for $x \in \partial M \times [0, C_3]$. Hence g is a (a,b)-adapted metric. $\square$

6.5 Lemma. *If Riemannian metrics g_i, $i = 1, 2$, are (a_i, b_i)-adapted, then the product metric $g = g_1 \times g_2$ is (a, b)-adapted, where $a = a_1 + a_2$, and $b = \min\{a_1 + b_2, b_1 + a_2\}$.*

Proof. If s (resp. s_i) is the scalar curvature functions of g (resp. g_i), then

$$s(x_1, x_2) = s_1(x_1) + s_2(x_2) \qquad \text{for } x_i \in N_i.$$

In particular, $s(x_1, x_2) > a = a_1 + a_2$ for all $(x_1, x_2) \in N = N_1 \times N_2$; moreover, if $s_i(x_i) \geq b_i$ for $x_i \in N_i \setminus K_i$ for a compact set $K_i \subset N_i$, then $s(x_1, x_2) \geq b$ for $(x_1, x_2) \in N \setminus (K_1 \times K_2)$. This shows that g satisfies the condition (1) for (a, b)-admissibility.

For the proof of the lemma it remains to check condition (2). We note that every compact set $K \subset N$ is contained in $K_1 \times K_2$ for compact subsets $K_i \subset N_i$. Furthermore, without loss of generality we can assume

1. $s(g_i, x_i) > b_i$ for $x_i \in N_i \setminus K_i$,
2. $K_i \subset M_i \subset N_i$ where M_i is a submanifold of codimension zero,
3. a neighbourhood of ∂M_i is isometric to $(\partial M_i \times [-1, 1], g_{i|\partial M_i} \times C\, dt^2)$ for any given $C > 0$.

We note that all requirements of condition (2) are satisfied if we define $M = M_1 \times M_2$ and $g^t = g$ for all $t \in [0,1]$ except that $M_1 \times M_2$ is not a smooth submanifold due to the corner points $\partial M_1 \times \partial M_2$. A neighbourhood U of $\partial M_1 \times \partial M_2 \subset N$ is isometric to $\partial M_1 \times \partial M_2 \times [-1,1]^2$. The intersection of $M_1 \times M_2$ with U corresponds to $\partial M_1 \times \partial M_2 \times [-1, 0]^2$. Hence we obtain a *smooth* submanifold $M \subset N_1 \times N_2$ if we smooth the corners of $M_1 \times M_2$ by replacing its intersection with U by $\partial M_1 \times \partial M_2 \times \Omega$, where $\Omega \subset [-1,1]^2$ is a submanifold of codimension zero, which agrees with $[-1,0]^2$ near the boundary of $[-1,1]^2$. Moreover, the following claim shows that we can deform the metric g into a product metric near ∂M while keeping it (K, a, b)-adapted.

> Claim: For each $\epsilon > 0$ there is a $C > 0$, and a continuous family of metrics k^s on $[-1,1]^2$ which agree with the metric $C(dt_1^2 + dt_2^2)$ near the boundary such that $|s(k^s, t)| \leq \epsilon$ for all $s \in [0,1]$, $t = (t_1, t_2) \in [-1,1]^2$ and such that k^1 is a product metric near $\partial\Omega$.

To prove the claim, choose any metric on $[-1,1]^2$ which agrees with the standard metric $dt_1^2 + dt_2^2$ near the boundary of $[-1,1]^2$ and is a product metric near $\partial\Omega$. Choose any homotopy from this metric to the standard metric. Multiplying this family of metrics by some large constant C results in a family of metrics k^t with arbitrarily small scalar curvature. $\square$

Proof of Theorem 6.1. Lemma 6.5 shows that there is a well-defined pairing

$$\begin{aligned} R_{n_1}(\gamma_1; a_1, b_1) \otimes R_{n_2}(\gamma_2; a_2, b_2) &\longrightarrow R_{n_1+n_2}(\gamma_1 \widehat{\times} \gamma_2; a, b) \\ [M_1, g_1] \otimes [M_2, g_2] &\mapsto [M_1 \times M_2, g_1 \times g_2], \end{aligned}$$

where $a = a_1 + a_2 < 0$ and $b = \min\{a_1 + b_2, b_1 + a_2\}$. We note that for suitable choices of $a_i < 0 < b_i$, $i = 1, 2$, we have $a < 0$ and $b > 0$. Hence via the isomorphism of Lemma 6.4, the above pairing leads to the desired external product. $\square$

6.6 Remark. Let ϵ (resp. ϵ^{SO}) be the supergroup $\epsilon = (\{1\}, 0, \mathbb{Z}/2)$ (resp. $\epsilon^{SO} = (\{1\}, 0, \{1\})$. We note that by Theorem 1.1 and Cor. 2.10 $R_n(\epsilon)$ (resp. $R_n^{SO}(\epsilon)$) is the group relevant for the existence and concordance classification of positive scalar curvature metrics on simply connected manifolds which are spin (resp. non-spin).

We note that for any supergroup γ the product $\epsilon \widehat{\times} \gamma$ can be identified with γ. This shows that the product (6.1) gives $R_* \overset{\text{def}}{=} R_*(\epsilon)$ the structure of a graded ring, and makes $R_*(\gamma)$ a graded module over R_*. Similarly, if the supergroup $\gamma = (\pi, w, \hat{\pi})$ has $\hat{\pi} = \pi$, then $\epsilon^{SO} \widehat{\times} \gamma$ identifies with γ. This gives $R_*^{SO} \overset{\text{def}}{=} R_*(\epsilon^{SO})$ the structure of a graded ring, and makes $R_*(\gamma)$ a graded module over R_*^{SO}.

§7. Dirac bundles and Dirac operators

This section, which contains no new results, is a warmup for the constructions in the following two sections. In this section we describe the Atiyah-Singer operator on a spin manifold twisted by the Miščenko-Fomenko line bundle. This is a special case of the Dirac operator associated to a γ-structure that we will construct in §8.

For the description of these operators we find it convenient to use the notion of "Dirac bundle" developed e.g. in the book of Lawson-Michelsohn [LM, II.5.2] with the aim to unify the treatment of the Euler characteristic operator, the signature operator, and the Atiyah-Singer operator. We want to extend that definition in order to incorporate additional structure in form of an action of a C^*-algebra A which makes each fiber of the Dirac bundle a Hilbert A-module.

7.1 Hilbert A-modules (see [Co, II. A]). Let A be a $\mathbb{Z}/2$-graded C^*-algebra. A *pre-Hilbert A-module* is a $\mathbb{Z}/2$-graded right A-module furnished with an A-valued inner product $\langle \cdot, \cdot \rangle : H \times H \to A$ such that

(1) $\langle x, x \rangle$ is a positive element of A for all $x \in H$,
(2) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in H$, and
(3) $\langle x, y \cdot a \rangle = \langle x, y \rangle \cdot a$ for all $x, y \in H$, $a \in A$.

If H is complete with respect to the seminorm defined by $||x|| = ||\langle x, x \rangle||^{1/2}$, then H is a *Hilbert A-module*.

7.2 Definition of A-Dirac bundle. Let M be an n-dimensional Riemannian manifold, and let $C\ell(M) \to M$ be the bundle of graded Clifford algebras whose fiber over a point $x \in M$ consists of the Clifford algebra $C\ell(T_x M)$ generated by the tangent space $T_x M$ furnished with the quadratic form provided by the Riemannian metric (cf. 2.2). Given a graded C^*-algebra A, an *A-Dirac bundle* over M (or *$A - C^*$-Clifford bundle* in the terminology of [Bu]) is a $\mathbb{Z}/2$-graded vector bundle $S = S^0 \oplus S^1 \to M$ (whose fibers are possibly infinite dimensional as real vector spaces) furnished with the following structures.

(1) A connection ∇ on S preserving the grading.
(2) A vector bundle homomorphism $C\ell(M) \otimes S \to S$ refered to as "Clifford multiplication" which gives the fiber S_x over $x \in M$ the structure of a graded left module over $C\ell(T_x M)$.

(3) A vector bundle homomorphism $S \otimes \underline{A} \to S$, where $\underline{A}$ denotes the trivial bundle over M with fiber A, which gives the fiber S_x the structure of a finitely generated, projective, graded right A-modules.
(4) A grading preserving vector bundle homomorphism $\langle \cdot, \cdot \rangle : S \otimes S \to \underline{A}$ such that we have $\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle^*$ and $\langle \sigma_1, \sigma_1 \rangle \geq 0$ for $\sigma_1, \sigma_2 \in S_x$ (after identifying $\underline{A}_x$ with A).

Any two of these structures are required to be compatible with each other in the following sense.

(1-2) Clifford multiplication is parallel.
(1-3) The vector bundle homomorphism $S \otimes \underline{A} \to S$ is parallel.
(1-4) $\langle \cdot, \cdot \rangle : S \otimes S \to \underline{A}$ is parallel.
(2-3) $(v\sigma)a = v(\sigma a)$ for $v \in C\ell(T_xM)$, $\sigma \in S_x$, $a \in A$.
(2-4) $\langle v \cdot \sigma_1, \sigma_2 \rangle = \langle \sigma_1, v^* \cdot \sigma_2 \rangle$ for $v \in C\ell(T_xM)$, $\sigma_1, \sigma_2 \in S_x$.
(3-4) $\langle \sigma_1, \sigma_2 a \rangle = \langle \sigma_1, \sigma_2 \rangle a$ for $\sigma_1, \sigma_2 \in S_x$, $a \in A$.

Here "parallel" in the statements (1-2) – (1-4) refers to the obvious connections on the bundles involved; *i.e.*, the connection ∇ on S, the connection induced by the Levi-Civita connection on $C\ell(M)$, the trivial connection on $\underline{A}$, and the product connection on any tensor product of bundles.

We note that an A-Dirac bundle for $A = \mathbb{R}$ is just a "Dirac bundle" in the terminology of Lawson and Michelsohn [LM, II.5.2]. We also observe that the structures (3), (4) and the correspondending compatibility condition (3-4) make the fiber S_x a finitely generated, projective $\mathbb{Z}/2$-graded pre-Hilbert A-module. Hence the notion of A-Dirac bundle results naturally when giving the fibers of a Dirac bundle the structure of Hilbert A-modules. The reader should be warned that a $C\ell_n$-Dirac bundle in the sense of Lawson and Michelsohn [LM, II.7.2] does *not* involve a $C\ell_n$-valued inner product; it consists only of the structures (1), (2) and (3) in Definition 7.2, and the corresponding compatibility conditions. For finite dimensional C^*-algebras A such as $C\ell_n$, we could in fact do without an A-valued inner product on the fibers of S; however, for infinite dimensional A's this seems to be an important piece of structure in order to extract an index in $KO(A)$.

7.3 The A-linear Dirac operator associated to an A-Dirac bundle. Let S be an A-Dirac bundle over a (not necessarily compact) manifold M. Let $C_c^\infty(S)$ be the vector space of smooth sections of S with compact support, and define

$$D: C_c^\infty(S) \to C_c^\infty(S) \qquad \text{by} \qquad (D\psi)(x) = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi,$$

where $\{e_i\}$ is an orthonormal basis for T_xM.

We note that the A-module structure on the fibers of S induces an A-module structure on $C_c^\infty(S)$. Moreover, the A-valued inner product on the fibers of S induces by integration an A-valued inner product on $C_c^\infty(S)$:

$$\langle \phi, \psi \rangle = \int_M \langle \phi(x), \psi(x) \rangle \, dvol(x)$$

(we hope that the context will distinguish which of the two A-valued inner products denoted $\langle \cdot, \cdot \rangle$ we are talking about; using a different notation like $(\cdot, \cdot)$ for one of them tends to make formulas difficult to decipher). The compatibility condition (3-4) guarantees that $C_c^\infty(S)$ becomes a $\mathbb{Z}/2$-graded pre-Hilbert A-module.

The Dirac operator has the following properties:

(1) D is A-linear.
(2) D is odd with respect to the $\mathbb{Z}/2$-grading, i.e., D anticommutes with ϵ.
(3) D is symmetric; i.e., $\langle D\phi, \psi\rangle = \langle \phi, D\psi\rangle$ for $\phi, \psi \in C_c^\infty(S)$.

Here (1) and (2) follow directly from the relevant compatibility properties; concerning (3) the usual proof for $A = \mathbb{R}$ (e.g. [LM], chap. II, Prop. 5.3) works equally well for general C^*-algebras.

Completing $C_c^\infty(S)$ with respect to the norm defined by $||\phi|| = ||\langle\phi,\phi\rangle||^{1/2}$ is a Hilbert A-module denoted $L^2(S)$. We will then consider D as an (unbounded) operator on $L^2(S)$. We will refer to D as the *A-linear Dirac operator* associated to S.

7.4 The $C\ell_n$-linear Atiyah-Singer operator (see [LM, II.§7]**).** The basic example of an A-linear Dirac operator is the $C\ell_n$-linear Atiyah-Singer operator on a n-dimensional spin manifold M. It is the operator associated to the $C\ell_n$-Dirac bundle

$$S \overset{\text{def}}{=} Spin(M) \times_{Spin(n)} C\ell_n.$$

Here $Spin(M)$ is the principal $Spin(n)$-bundle determined by the spin structure, and $Spin(n) \subset C\ell_n^\times$ acts on $C\ell_n$ by left multiplication. We won't describe here the structures 7.2(1)-(4) of this Dirac bundle; this will be done in the next section (see example 8.8 and Prop. 8.10) in a more general setting.

7.5 The (reduced) C^*-algebra of a discrete group. Let π be a discrete group, and let

$$\mathbb{R}\pi = \left\{ \sum_{g\in\pi} r_g g \;\middle|\; g\in\pi, r_g \in \mathbb{R}, r_g = 0 \text{ for all but finitely many r} \right\}$$

be its real group algebra. A group element $g \in \pi$ acts on the set π by left multiplication, and hence on the Hilbert space $\ell^2(\pi)$ spanned by π. This allows us to embed $\mathbb{R}\pi$ as a subalgebra of the algebra of bounded operators on $\ell^2(\pi)$. The C^*-algebra obtained by completing with respect to the operator norm is the *reduced group C^*-algebra* which we will denote by $C^*\pi$, since in this paper we don't consider other C^*-algebras associated to groups.

We note that a group element g acts on $\ell^2(\pi)$ by an orthogonal transformation; in particular, π is a subgroup of the orthogonal elements of $C^*\pi$ (an element a of a real C^*-algebra is *orthogonal* if $aa^* = a^*a = 1$).

7.6 The Miščenko-Fomenko line bundle. If π is a discrete group, and $\widetilde{M} \to M$ a principal π-bundle over a manifold M, then the associated vector bundle

$$\widetilde{M} \times_\pi C^*\pi \to M,$$

where π acts on $C^*\pi$ by left multiplication is called the *Miščenko-Fomenko line bundle.* This is a flat real vector bundle over X whose fiber dimension is the order of π. It is referred to as a 'line bundle' since $C^*\pi$ acts on $\widetilde{M} \times_\pi C^*\pi$ by right multiplication which makes the fibers free rank 1 modules over $C^*\pi$.

7.7 The Atiyah-Singer operator twisted by the Miščenko-Fomenko line bundle. If M is an n-dimensional spin manifold, S is the $C\ell_n$-spinor bundle (cf. 7.4), then the tensor product $S\otimes(\widetilde{M}\times_\pi C^*\pi)$ has the structure of a $C\ell_n\otimes C^*\pi$-Dirac bundle (as Proposition 8.10 will show). The corresponding $C\ell_n\otimes C^*\pi$-linear Dirac operator is called the *Atiyah-Singer operator twisted by the Miščenko-Fomenko line bundle.*

7.8 The Weitzenböck formula. Let S be an A-Dirac bundle, and let D be the associated A-linear Dirac operator (see Def. 7.3). The Weitzen böck formula (also known as Bochner– or Lichnerowicz formula) expresses the square of D in the form

$$D^2 = \nabla^*\nabla + \mathcal{R}.$$

Here $\nabla^*\nabla$ is the "connection Laplacian" associated to the connection ∇ on the Dirac bundle S; it satisfies $\langle\nabla^*\nabla\phi,\psi\rangle = \langle\nabla\phi,\nabla\psi\rangle$ for $\phi,\psi\in C_c^\infty(S)$ (cf. [LM], chap. II, Prop. 8.1; the proof given there generalizes directly to our case of an A-valued inner product). $\mathcal{R}\colon S\longrightarrow S$ is a vector bundle endomorphism whose restriction $\mathcal{R}_x\colon S_x\longrightarrow S_x$ to the fiber S_x is determined by the curvature of ∇ at x (see [LM], chap. II, formula (8.8)). If D is the Atiyah-Singer operator (cf. 7.4), or more generally the Atiyah-Singer operator twisted with a *flat* bundle like the Miščenko-Fomenko line bundle (cf. 7.7), then $\mathcal{R}_x$ is multiplication by $scal(x)/4$, where $scal(x)$ denotes the scalar curvature at the point x.

§8. The γ-Dirac operator

In this section we describe two constructions:

(1) We associate to a supergroup γ a $\mathbb{Z}/2$-graded real C^*-algebra $C^*\gamma$, which agrees with $C^*\pi$ in the 'spin' case $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$ (see Lemma 8.2(2)).
(2) We construct on any n-dimensional γ-manifold M an A-Dirac bundle, where A is the C^*-algebra $C\ell_n\widehat{\otimes}C^*\gamma$. We call this bundle the *γ-Dirac bundle*, and the associated Dirac operator the *γ-Dirac operator*. It agrees with the $C\ell_n$-linear Atiyah-Singer operator twisted by the Miščenko-Fomenko line bundle in the spin case (see 7.7 and Lemma 8.11).

8.1 Definition of $C^*\gamma$. Let γ be a supergroup; here we wish to think of γ as a triple $(\hat\pi,w,c)$, where $w\colon\hat\pi\longrightarrow\mathbb{Z}/2$ is a group homomorphism and $c\in\hat\pi$ is a central element in the kernel of w with $c^2=1$ (cf. Definition 2.1 and Remark 2.3). We define $C^*\gamma$ to be the (-1)-eigenspace of the involution $C^*\hat\pi\longrightarrow C^*\hat\pi$ given by multiplication by c. This ideal in $C^*\hat\pi$ is a C^*-algebra with unit $e\overset{\text{def}}{=}(1-c)/2$. The homomorphism $w\colon\hat\pi\longrightarrow\mathbb{Z}/2$ defines a $\mathbb{Z}/2$-grading on $C^*\hat\pi$ by declaring that an element $\hat g$ of $\hat\pi\subset C^*\hat\pi$ has degree $w(\hat g)$. This $\mathbb{Z}/2$-grading restricts to a $\mathbb{Z}/2$-grading on $C^*\gamma\subset C^*\hat\pi$.

8.2 Lemma. *Let $\gamma=(\pi,w,\hat\pi)$ be a supergroup.*

(1) *If $\hat\pi=\pi$, then $C^*\gamma=0$.*
(2) *If $\hat\pi\neq\pi$, then a splitting map $s\colon\pi\longrightarrow\hat\pi$ determines an isomorphism between $C^*\gamma$ and $C^*\pi$, if we furnish $C^*\pi$ with the $\mathbb{Z}/2$-grading determined by w.*

Proof. If $\hat{\pi} = \pi$, then the generator c of $\ker(\hat{\pi} \longrightarrow \pi)$ is the trivial element. Hence $C^*\gamma$ – the (-1)-eigenspace of c acting on $C^*\hat{\pi}$ – is trivial.

If $\hat{\pi} \neq \pi$, and $s\colon \pi \longrightarrow \hat{\pi}$ is a splitting map, then an isomorphism $C^*\pi \cong C^*\gamma$ is given by sending $g \in \pi$ to $s(g)e \in C^*\gamma$, where $e = (1-c)/2$ is the idempotent corresponding to the projection to $C^*\gamma \subset C^*\hat{\pi}$. □

The following result will not be used in this paper. It is included to illustrate that for a finite group π the decomposition of the graded algebra $C^*\gamma$ into simple factors can be determined from the character table of the group $\hat{\pi}$ very much the same way as the decomposition of $\mathbb{R}\pi$ can be determined from the character table of π.

8.3 Proposition. *Let $\gamma = (\pi, w, \hat{\pi})$ be a supergroup with $\pi = \mathbb{Z}/2^r$ and $\ker(\hat{\pi} \longrightarrow \pi) = \mathbb{Z}/2$. The following table shows the C^*-algebra $C^*\gamma$ in terms of $w\colon \pi \longrightarrow \mathbb{Z}/2$ and the cohomology class $e \in H^2(B\pi; \mathbb{Z}/2) = \mathbb{Z}/2$ classifying the extension $\mathbb{Z}/2 \longrightarrow \hat{\pi} \longrightarrow \pi$.*

w	e	$C^*\gamma$ for $r = 1$	$C^*\gamma$ for $r > 1$
0	0	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R} \oplus \mathbb{R} \oplus (2^{r-1} - 1)\mathbb{C}$
$\neq 0$	0	$C\ell_{0,1}$	$C\ell_{0,1} \oplus C\ell_1 \oplus (2^{r-2} - 1)\mathbb{C}\ell_1$
0	$\neq 0$	$\mathbb{C}$	$2^{r-1}\mathbb{C}$
$\neq 0$	$\neq 0$	$C\ell_1$	$2^{r-2}\mathbb{C}\ell_1$

Here $C\ell_{0,1}$ is the (graded) Clifford algebra generated by an odd element whose square is the identity, $\mathbb{C}\ell_n$ is the complex Clifford algebra $\mathbb{C}\ell_n = C\ell_n \otimes \mathbb{C}$, and kA stands for the direct sum $A \oplus \cdots \oplus A$ of k copies of A.

Proof. It is well-known that the real group algebra of $\pi = \mathbb{Z}/2^r$ decomposes in the form

$$\mathbb{R}\mathbb{Z}/2^r = \mathbb{R} \oplus \mathbb{R} \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{2^{r-1}-1}.$$

This proves the claim in the case $w = 0$, $e = 0$. The summands of the above decomposition can be parametrized by the 2^r-th roots of unity modulo complex conjugation. A 2^r-th root of unity ζ corresponds to a $\mathbb{C}$-summand if $\zeta \neq \pm 1$, while $\zeta = \pm 1$ corresponds to the two $\mathbb{R}$-summands. The idempotent $e_\zeta \in \mathbb{R}\mathbb{Z}/2^r$ whose range is the summand corresponding to ζ is given by the following formulas (which can be read off from the character table of $\mathbb{Z}/2^r$).

$$e_\zeta = \frac{1}{2^r} \sum_{j=0}^{2^j - 1} \zeta^j \lambda^j \quad \text{for } \zeta = 1, -1 \quad \text{and} \quad e_\zeta = \frac{1}{2^{r-1}} \sum_{j=0}^{2^j - 1} \mathrm{Re}(\zeta^j)\lambda^j \quad \text{for } \zeta \neq 1, -1,$$

Here λ is a generator of the group π which we write multiplicatively, and $\mathrm{Re}(\zeta^j)$ denotes the real part of ζ^j. These formulas show that for $w \neq 0$ the grading involution $\epsilon\colon \mathbb{R}\pi \longrightarrow \mathbb{R}\pi$, $\sum r_g g \mapsto \sum (-1)^{w(g)} r_g g$ sends e_ζ to $e_{-\zeta}$. It follows that for $\zeta \neq i$ the element $e_\zeta - e_{-\zeta}$ is an odd element whose square is the idempotent $e_\zeta + e_{-\zeta}$. This shows that if we consider $\mathbb{R}\pi$ as $\mathbb{Z}/2$-graded algebra, then the two copies of $\mathbb{R}$ combine to form the Clifford algebra $C\ell_{0,1}$; moreover all copies of $\mathbb{C}$ except the one corresponding to $\zeta = i$ pair up and each pair contributs a copy of $\mathbb{C}\ell_1$. The copy of $\mathbb{C}$ corresponding to $\zeta = i$ is invariant under the grading involution ϵ; hence it is isomorphic to the Clifford algebra $C\ell_1$ as a *graded* algebra. Using Lemma 8.2(2), this verifies the second row of the table.

If e is non-trivial, then $C^*\gamma$ is a direct summand of the $\mathbb{Z}/2$-graded algebra $\mathbb{R}\hat{\pi}$, $\hat{\pi} = \mathbb{Z}/2^{r+1}$, namely the (-1)-eigenspace of the involution given by multiplication by the generator c of $\ker(\hat{\pi} \to \pi)$. We note that the $(+1)$-eigenspace can be identified (as graded algebra) with $\mathbb{R}\pi$; hence the summands of $C^*\gamma$ are the summands of $\mathbb{R}\hat{\pi}$ minus those summands that occur in $\mathbb{R}\pi$. $\square$

8.4 Lemma. *If γ_1, γ_2 are supergroups, then the C^*-algebra $C^*(\gamma_1 \widehat{\times} \gamma_2)$ associated to the product supergroup $\gamma_1 \widehat{\times} \gamma_2$ is canonically isomorphic to the $\mathbb{Z}/2$-graded tensor product $C^*\gamma_1 \widehat{\otimes} C^*\gamma_2$ (here we use the minimal tensor product of C^*-algebras).*

Proof. If $\gamma = (\pi, w, \hat{\pi})$ is a supergroup, let $\mathbb{R}\gamma$ be the (-1)-eigenspace of $c \in \hat{\pi} \subset \mathbb{R}\hat{\pi}$ acting on $\mathbb{R}\hat{\pi}$; in other words, $\mathbb{R}\gamma$ is the range of the idempotent $e = (1-c)/2 \in \mathbb{R}\hat{\pi}$. It is straight forward to check that the map

$$\mathbb{R}(\gamma_1 \widehat{\times} \gamma_2) \longrightarrow \mathbb{R}\gamma_1 \widehat{\otimes} \mathbb{R}\gamma_2 \quad \text{given by} \quad [a,b]e \mapsto ae \widehat{\otimes} be \qquad \text{for } a \in \hat{\pi}_1,\ b \in \hat{\pi}_2$$

is an isomorphism of $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras (in the notation we don't distinguish whether the supergroup $\gamma = (\pi, w, \hat{\pi})$ is γ_1, γ_2 or the product $\gamma_1 \widehat{\times} \gamma_2$). It is easy to see that the operatornorm of $[a,b]e$ acting on the Hilbert space $\ell^2(\hat{\pi}_1 \widehat{\times} \hat{\pi}_2)$ is the same as the operatornorm of $ae \widehat{\otimes} be$ acting on $\ell^2(\hat{\pi}_1) \widehat{\otimes} \ell^2(\hat{\pi}_2)$. Hence by passing to norm completions, we obtain the desired isomorphism of C^*-algebras. $\square$

8.5 Construction of the γ-spinor bundle. Let M be an n-dimensional manifold furnished with a γ-structure. We recall from Definition 2.6 that for a supergroup $\gamma = (\pi, w, \hat{\pi})$ a γ-structure is given by a principal bundle $P \to M$ with stucture group $G(n,\gamma) = (\mathrm{Pin}(n) \widehat{\times} \hat{\pi})^{ev}$, and a $G(n,\gamma)$-equivariant map $\xi\colon P \to O(M)$ to the frame bundle of M. We define the *γ-spinor bundle* $S_M \to M$ to be the vector bundle

$$S_M \overset{\text{def}}{=} P \times_{G(n,\gamma)} \left(C\ell_n \widehat{\otimes} C^*\gamma \right),$$

where $[a,b] \in (\mathrm{Pin}(n) \widehat{\times} \hat{\pi})^{ev} = G(n,\gamma)$ acts on $C\ell_n \widehat{\otimes} C^*\gamma$ by left multiplication by $a \widehat{\otimes} be$ (we recall from Def. 8.1 that $e \in C^*\hat{\pi}$ is the idempotent whose range is $C^*\gamma \subset C^*\hat{\pi}$; in particular, the element be lives in $C^*\gamma$).

8.6 Proposition. *The γ-spinor bundle determined by a γ-structure on a Riemannian manifold M of dimension n is a $C\ell_n \widehat{\otimes} C^*\gamma$-Dirac bundle (cf. Definition 7.2).*

It will be convenient to derive Proposition 8.6 as a corollary of a more general result (Prop. 8.10 below), also needed to identify S_M in the 'spin case' (Lemma 8.11) and for expressing S_M for $M = M_1 \times M_2$ as the external tensor product of the γ-spinor bundles of the factors (Lemma 8.13). To state the more general result, we need the following terminology.

8.7 Definition. An *algebra bundle with connection* is a $\mathbb{Z}/2$-graded vector bundle $C = C^0 \oplus C^1$ (possibly infinite dimensional) over a manifold equipped with

(1) a grading preserving vector bundle homomorphism $C \widehat{\otimes} C \to C$ which gives each fiber C_x the structure of a $\mathbb{Z}/2$-graded algebra;
(2) a connection on C compatible with the grading.

These two structures are required to be compatible in the sense that the vector bundle homomorphism $C\widehat{\otimes}C \to C$ is parallel.

If C is an algebra bundle with connection, and A is a $\mathbb{Z}/2$-graded algebra, then a *$C-A$-bimodule bundle with connection* is a possibly infinite dimensional $\mathbb{Z}/2$-graded vector bundle $S = S^0 \oplus S^1$ equipped with the following structure:

(1) a connection on S compatible with the grading
(2) a grading preserving vector bundle homomorphism $C\widehat{\otimes}S \to S$ which gives each fiber S_x the structure of a graded left C_x-module;
(3) a grading preserving vector bundle homomorphism $S \otimes \underline{A} \to S$, where $\underline{A}$ denotes the trivial bundle over M with fiber A, which gives each fiber S_x the structure of a finitely generated, projective right graded A-module;
(4) a grading preserving vector bundle homomorphism $\langle\cdot,\cdot\rangle: S \otimes S \to \underline{A}$ such that we have $\langle\sigma_1, \sigma_2\rangle = \langle\sigma_1, \sigma_2\rangle^*$ and $\langle\sigma_1, \sigma_1\rangle \geq 0$ for $\sigma_1, \sigma_2 \in S_x$ (after identifying $\underline{A}_x$ with A).

The structures (1)–(4) are required to satisfy the same compatibility conditions as in Definition 7.2; just replace $C\ell(M)$ by C everywhere.

8.8 Examples.

(1) An example of an algebra bundle with connection is the Clifford algebra bundle $C\ell(M)$ equipped with the connection induced from the Levi-Civita connection on TM. We note that a '$C\ell(M) - A$-bimodule bundle with connection' is the same thing as an 'A-Dirac bundle'.
(2) If M is a spin manifold, then the Cl_n-linear spinor bundle $Spin(M)\times_{Spin(n)} C\ell_n$ (cf. 7.4) is a $C\ell(M) - C\ell_n$-bimodule bundle with connection (this is a special case of Proposition 8.10(3) below).
(3) The Miščenko-Fomenko line bundle $\widetilde{M} \times_\pi C^*\pi$ (cf. 7.7) is a $C^*\pi$-module bundle, (*i.e.*, a $\underline{\mathbb{R}} - C^*\pi$-bimodule bundle where $\underline{\mathbb{R}}$ is the trivial real line bundle; again this follows from Prop. 8.10(3)).
(4) If C_1, C_2 are algebra bundles with connections over M, then the (graded) tensor product $C_1\widehat{\otimes}C_2$ (with its multiplication and connection induced from those of the factors) is again an algebra bundle with connection. Moreover, if S_i is a $C_i{-}A_i$-bimodule bundle for $i = 1, 2$, then the graded tensor product $S_1\widehat{\otimes}S_2$ becomes a $C_1\widehat{\otimes}C_2 - A_1\widehat{\otimes}A_2$-bimodule bundle. In particular, from (2) and (3) above we see that $S\widehat{\otimes}\nu_M$ is a $C\ell(M) - C\ell_n\widehat{\otimes}C^*\pi$-bimodule bundle; in other words, $S\widehat{\otimes}\nu_M$ is a $C\ell_n\widehat{\otimes}C^*\pi$-Dirac bundle.

8.9 Definition. An element $a \in A$ of a real $\mathbb{Z}/2$-graded C^*-algebra with unit is called *orthonormal* if $a^*a = aa^* = 1$ and a is homogeneous. We denote by $O(A)$ the group of all orthogonal elements, by $O^{ev}(A)$ the subgroup of elements of degree 0, and by $PO(A)$ the corresponding projective group $O(A)/\pm 1$.

All $C - A$-bimodule bundles we are interested in (like the γ-spinor bundle S_M) will be associated to some principal G-bundle by means of a homomorphism $\hat{\rho}: G \to O^{ev}(A)$. Such a homomorphism also gives rise to an associated algebra bundle with connection by letting $g \in G$ act on A by conjugation by $\hat{\rho}(g)$. However, important examples of algebra bundles with connection are *not* of this form (like the Clifford bundle $C\ell(M)$ of a non-spin manifold M); rather, they are associated to a homomorphism $\rho\colon G \to PO(A)$ by letting $PO(A)$ act on A via the *twisted adjoint representation*

$$\widetilde{Ad}\colon PO(A) \to \mathrm{Aut}(A)$$

which sends the element represented by $a \in O(A)$ to the automorphism $b \mapsto (-1)^{w(a)}aba^{-1} = (-1)^{w(a)}aba^*$.

8.10 Proposition. *Let $P \to M$ be a principal G-bundle equipped with a connection.*

(1) *If $\rho\colon G \to PO(A)$ is a homomorphism, then the vector bundle $C(\rho) \stackrel{\text{def}}{=} P \times_{\widetilde{Ad}\circ\rho} A$ is an algebra bundle with connection.*

(2) *If $A = A_1\widehat{\otimes}A_2$ and ρ factors as $G \xrightarrow{\rho_1\times\rho_2} PO(A_1)\times PO(A_2) \subset PO(A)$, then $C(\rho)$ is isomorphic (as algebra bundle with connection) to $C(\rho_1)\widehat{\otimes}C(\rho_2)$.*

(3) *If ρ factors in the form $G \xrightarrow{\hat\rho} O^{ev}(A) \to PO(A)$, then the vector bundle $S(\hat\rho) \stackrel{\text{def}}{=} P\times_{\hat\rho} A$ (defined by letting $g \in G$ act on A by left multiplication by $\hat\rho(g)$) is a $C(\rho) - A$-bimodule bundle with connection.*

(4) *If $A = A_1\widehat{\otimes}A_2$, and $\hat\rho$ factors in the form $G \xrightarrow{\hat\rho_1\times\hat\rho_2} O^{ev}(A_1)\times O^{ev}(A_2) \to O^{ev}(A)$, then $S(\hat\rho)$ is isomorphic (as $C(\rho) - A$-bimodule bundle with connection) to $S(\hat\rho_1)\widehat{\otimes}S(\hat\rho_2)$.*

Proof. To prove part (1) we observe that the G-action on A via $\widetilde{Ad}\circ\rho$ preserves the $\mathbb{Z}/2$-grading of A; and hence the associated vector bundle inherits an induced $\mathbb{Z}/2$-grading. Similarly, the multiplication map

$$\mu\colon A\widehat{\otimes}A \to A \qquad a\widehat{\otimes}b \mapsto ab$$

is G-equivariant, and hence it induces a vector bundle homomorphism

$$C(\rho)\widehat{\otimes}C(\rho) \to C(\rho).$$

Like any vector bundle homomorphism induced by an equivariant map of vector spaces, it is parallel with respect to the connection induced by the principal connection on P.

Concerning part (2), it is clear that

$$C(\rho_1\times\rho_2) \to C(\rho_1)\widehat{\otimes}C(\rho_2) \qquad [p, a_1\widehat{\otimes}a_2] \mapsto [p,a_1]\widehat{\otimes}[p,a_2]$$

is a vector bundle isomorphism; it is not hard to check that it is compatible with the multiplication and the connection on these two bundles.

To prove the third statement that $S(\hat\rho)$ is a $C(\rho){-}S(\hat\rho)$-bimodule bundle, we need to construct the required vector bundle homomorphisms. To do so, we note that all three bundles involved – $C(\rho)$, $S(\hat\rho)$, and the trivial bundle $\underline{A}$ – are associated to the principal G-bundle P by letting $g \in G$ act on A by $\widetilde{Ad}(\rho(g))$ (resp. left-multiplication by $\rho(g)$ resp. trivially). This allows to describe these vector bundle homomorphisms as induced by maps of vector spaces with appropriate equivariance conditions as follows:

(1) The left action of $C(\rho)$ on $S(\hat\rho)$ is induced by the multiplication map $\mu\colon A\widehat{\otimes}A \to A$ (which is $O^{ev}(A)$-equivariant if we let $O^{ev}(A)$ act on the

first copy of A by the twisted adjoint action, and the second and third copy by left multiplication).

(2) The right action of A on $S(\hat\rho)$ is also induced by μ (letting $O^{ev}(A)$ act on the first and third copy of A via left-multiplication, and trivially on the second copy).

(3) The A-valued inner product on $S(\hat\rho)$ is induced by the map $A\widehat\otimes A \longrightarrow A$, $a\widehat\otimes b \mapsto a^*b$, which is $O^{ev}(A)$-equivariant if we let $O^{ev}(A)$ act on the first two copies by left multiplication and trivially on the third.

It is straightforward to check that the required compatiblity conditions hold.

The isomorphism of bimodule bundles $S(\hat\rho_1\widehat\otimes\hat\rho_2)\cong S(\hat\rho_1)\widehat\otimes C(\hat\rho_2)$ claimed in part (4) is given by $[p,a_1\widehat\otimes a_2]\mapsto [p,a_1]\widehat\otimes[p,a_2]$. It is easy to check that this is in fact an isomorphism of bimodule bundles. $\square$

Proof of Prop. 8.6. Using the notation of Proposition 8.10, the γ-spinor bundle is $S(\hat\rho)$, where $\hat\rho\colon(\mathrm{Pin}(n)\widehat\times\hat\pi)^{ev}\longrightarrow O^{ev}(C\ell_n\widehat\otimes C^*\gamma)$ is given by $[a,b]\mapsto a\widehat\otimes be$. In particular, by Proposition 8.10 the γ-spinor bundle is a $C(\rho)-A$-bimodule bundle with connection, where $A=C\ell_n\widehat\otimes C^*\gamma$, and $\rho\colon G(n,\gamma)\longrightarrow PO(A)$ is the composition of $\hat\rho$ with the natural map $O^{ev}(A)\longrightarrow PO(A)$. We note that ρ factors in the form

$$G(n,\gamma)=(\mathrm{Pin}(n)\widehat\times\hat\pi)^{ev}\xrightarrow{\rho_1\times\rho_2} PO(C\ell_n)\times PO(C^*\gamma)\subset PO(A),$$

with $\rho_1([a,b])=[a]$ and $\rho_2([a,b])=[be]$. Hence by part (2) of Proposition 8.10, $C(\rho)$ is isomorphic to $C(\rho_1)\widehat\otimes C(\rho_2)$. Identifying $O(n)$ with the subgroup of $PO(C\ell_n)$ consisting of those elements whose twisted adjoint action on $C\ell_n$ preserves the subspace $\mathbb{R}^n\subset C\ell_n$, the map ρ_1 agrees with the map ρ_1 of Definition 2.6. Thus the map

$$C(\rho_1)=P\times_{\widetilde{Ad}\,\circ\,\rho_1}C\ell_n\longrightarrow O(M)\times_{\widetilde{Ad}}C\ell_n=C\ell(M)\quad\text{given by}\quad [p,a]\mapsto[\xi(p),a]$$

is an isomorphism of algebra bundles with connection. In particular, restricting the left action of $C(\rho)=C(\rho_1)\widehat\otimes C(\rho_2)=C\ell(M)\widehat\otimes C(\rho_2)$ on $S_M=S(\hat\rho)$ to $C\ell(M)$, the bundle S_M is a $C\ell(M)-A$-module bundle with connection, or, put differently, an A-Dirac bundle. $\square$

8.11 Lemma. *If $\gamma=(\pi,w,\hat\pi)$ with $w=0$, $\hat\pi\neq\pi$, and there is a splitting map $s\colon\pi\longrightarrow\hat\pi$, then the γ-Dirac operator on a γ-manifold M is the Atiyah-Singer operator on M (associated to the spin-structure on M determined by the γ-structure and the splitting map s; cf. 7.4 and Lemma 2.7(3)) twisted by the Miščenko-Fomenko line bundle (associated to the principal π-bundle determined by the γ-structure; cf. 7.6 and Lemma 2.7(1)).*

Proof. We note that the homomorphism $\hat\rho\colon(Pin(n)\widehat\times\hat\pi)^{ev}\longrightarrow O^{ev}(C\ell_n\widehat\otimes C^*\gamma)$ given by $[a,b]\mapsto a\widehat\otimes be$ factors in the form

$$(Pin(n)\widehat\times\hat\pi)^{ev}\xrightarrow{\hat\rho_1\times\hat\rho_2}O^{ev}(C\ell_n)\times O^{ev}(C^*\gamma)\longrightarrow O^{ev}(C\ell_n\widehat\otimes C^*\gamma),$$

where $\hat\rho_1([a,b])=a$ and $\hat\rho_2([a,b])=b$ for b in the image of s (note that any element has a unique representative (a,b) with b in the image of s).

Proposition 8.10(4) implies that $S(\hat\rho)$ is isomorphic to (as $C(\rho)-A$-bimodule bundle with connection) to $S(\rho_1)\widehat{\otimes}S(\rho_2)$. We recall from the proof of Lemma 2.7(3) that $P/\ker\hat\rho_1 \longrightarrow M$ is the principal $Spin(n)$-bundle $Spin(M)\longrightarrow M$ corresponding to the spin structure determined by the γ-structure and s. Hence $S(\hat\rho_1) = P\times_{\hat\rho_1} C\ell_n = Spin(M)\times_{Spin(n)} C\ell_n$ is the $C\ell_n$-spinor bundle on M.

To identify $S(\hat\rho_2)$, we note that $\hat\rho_2$ is the composition of $\rho_2\colon(\mathrm{Pin}(n)\widehat{\times}\hat\pi)^{ev}\longrightarrow O^{ev}(C^*\pi)$ given by $[a,b]\mapsto[b]$ and the isomorphism $O^{ev}(C^*\pi)\longrightarrow O^{ev}(C^*\gamma)$ given by $[b]\mapsto s([b])e$. Hence we may identify $S(\hat\rho_2)$ with $S(\rho_2)=P\times_{\rho_2}C^*\pi=\widetilde M\times_\pi C^*\pi$, the Miščenko-Fomenko line bundle. $\square$

Corollary 8.12. *Let $\gamma=(\pi,w,\hat\pi)$ be a supergroup with $\hat\pi\neq\pi$. Let D be the γ-Dirac operator on a γ-manifold M. Then the curvature endomorphism $\mathcal{R}_x$ in the Weitzenböck formula for D^2 (see 7.8) is given by multiplication by $scal(x)/4$.*

Proof. The statement in the 'spin case' $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$ is an immediate consequence of the Lemma and of the Weitzenböck formula for the Atiyah-Singer operator twisted by the flat Miščenko-Fomenko line bundle (see 7.8). Since the Weitzenböck formula is *local*, it suffices to show the claim for the pull back of the γ-Dirac operator to the covering space $q\colon\widetilde M=P/\ker\rho_2\longrightarrow M$ determined by the γ-structure (see proof of Lemma 2.7). This pullback operator is the Dirac operator corresponding to the Dirac bundle $q^*S_M=(q^*P)\times_{G(n,\gamma)}C\ell_n\widehat{\otimes}C^*\gamma$. We note that the structure group of $q^*P\longrightarrow\widetilde M=P/\ker\rho_2$ reduces to $\rho_2=Spin(n)$.

Restricted to $Spin(n)=\ker\rho_2\subset G(n,\gamma)$, the map $\hat\rho\colon G(n,\gamma)\longrightarrow O^{ev}(C\ell_n\widehat{\otimes}C^*\gamma)$ (cf. Proof of 8.6) factors through $O^{ev}(C\ell_n)\subset O^{ev}(C\ell_n\widehat{\otimes}C^*\gamma)$, and hence by Prop. 8.10 the correponding Dirac operator can be interpreted as the $C\ell_n$-linear Atiyah-Singer operator twisted by some flat vector bundle. It follows that the curvature endomorphism in the Weitzenböck formula for the pullback operator is just $scal(x)/4$. $\square$

8.13 Lemma. *Let M_1, M_2 be manifolds of dimension n_1 (resp. n_2) equipped with a γ_1-structure (resp. γ_2-structure). Let γ be the product supergroup $\gamma_1\widehat{\times}\gamma_2$, and let $M=M_1\times M_2$ be equipped with the product γ-structure. Then the γ-spinor bundle on M is isomorphic (as Dirac bundle over the C^*-algebra $C\ell_n\widehat{\otimes}C^*\gamma$ with $n=n_1+n_2$) to the tensor product $pr_1^*S_1\widehat{\otimes}pr_2^*S_2$, where S_i denotes the γ_i-spinor bundle over M_i, and $pr_i\colon M\longrightarrow M_i$ is the projection map.*

Proof. The definition of the product γ-structure on M (cf. 4.7) shows that the γ-spinor bundle on M is associated to the principal bundle $P_1\times P_2\longrightarrow M_1\times M_2$ (with structure group $G(n_1,\gamma_1)\times G(n_2,\gamma_2)$) by means of the homomorphism

$$G(n_1,\gamma_1)\times G(n_2,\gamma_2)\xrightarrow{\hat i}G(n,\gamma)\xrightarrow{\hat\rho}O^{ev}(C\ell_n\widehat{\otimes}C^*\gamma).$$

Identifying $O^{ev}(C\ell_n\widehat{\otimes}C^*\gamma)$ with $O^{ev}(C\ell_{n_1}\widehat{\otimes}C^*\gamma_1\widehat{\otimes}C\ell_{n_2}\widehat{\otimes}C^*\gamma_2)$, the above composition is equal to

$$\begin{aligned}G(n_1,\gamma_1)\times G(n_2,\gamma_2)\xrightarrow{\hat\rho_1\times\hat\rho_2}\;&O^{ev}(C\ell_{n_1}\widehat{\otimes}C^*\gamma_1)\times O^{ev}(C\ell_{n_2}\widehat{\otimes}C^*\gamma_2)\\&\longrightarrow O^{ev}(C\ell_{n_1}\widehat{\otimes}C^*\gamma_1\widehat{\otimes}C\ell_{n_2}\widehat{\otimes}C^*\gamma_2),\end{aligned}$$

where $\hat\rho_i$ sends $([a_1,b_1],[a_2,b_2])$ to $a_i\widehat{\otimes}b_ie \in O^{ev}(C\ell_{n_i}\widehat{\otimes}C^*\gamma_i)$ (these are *not* the $\hat\rho_i$'s of 8.10 and 8.11). Hence by Proposition 8.10(4), the γ-spinor bundle on M is isomorphic as bimodule bundle with connection (and consequently as Dirac bundle over $C^*\gamma$) to the tensor product of of the two bundles $(P_1\times P_2)\times_{\hat\rho_i}(C\ell_{n_i}\widehat{\otimes}C^*\gamma_i) = pr_i^*S_i$, $i=1,2$. □

§9. The index map $\theta\colon R_n(\gamma)\longrightarrow KO_n(C^*\gamma)$

The goal of this section is to define the homomorphism $\theta\colon R_n(\gamma)\longrightarrow KO_n(C^*\gamma)$ (see Definition 9.3) and to prove its multiplicative properties (see Corollary 9.21). The map θ agrees with the homomorphism constructed by Ulrich Bunke in [Bu, Theorem 1.17] in the 'spin case' $\gamma=(\pi,0,\pi\times\mathbb{Z}/2)$. In fact, using the the γ-Dirac operator constructed in the last section (see 8.5 and Prop. 8.6) and Bunke's K-theoretic relative index theorem [Bu, Thm. 1.2] it is straightforward to define the homomorphism θ for a general supergroup γ. However, the proof of the multiplicative property of θ forces us to go into the analytical details of the definition of the K-theoretic index. In fact, the key of the proof – which was suggested by Ulrich Bunke – is to construct this index in a *different* way than in Bunke's paper.

To define θ, it is convenient to replace $R_n(\gamma)$ by the group $R_n(\gamma;a,b)$ with $a,b\in\mathbb{R}$ (see Definition 6.3), which is isomorphic to $R_n(\gamma)$ for $a<0<b$ (see Lemma 6.4). We recall that elements of $R_n(\gamma;a,b)$ are equivalence classes of pairs (N,g), where N is an n-dimensional manifold without boundary (not necessarily compact), equipped with a γ-structure (cf. (3.1)), and g is a (a,b)-adapted metric on N (see Def. 6.2); this implies in particular $scal(x)>a$ for all $x\in N$, and $scal(x)>b$ for all x outside some compact subset of N.

Let $D(N,g)$ be the γ-Dirac operator on a γ-manifold N determined by the Riemannian metric g (cf. Prop. 8.6). We note that by Corollary 8.12 the curvature endomorphism $\mathcal{R}_x\colon S_x\longrightarrow S_x$ in the Weitzenböck formula for $D^2(N,g)$ is given by multiplication by $scal(x)/4$, where $scal(x)$ is the scalar curvature at the point x. In particular, if the metric g is (a,b)-adapted, then the Dirac operator $D(N,g)$ is (a,b)-adapted in the sense of the following definition.

9.1 Definition. Let A be a $\mathbb{Z}/2$-graded C^*-algebra. An *A-linear Dirac operator* is the Dirac operator associated to an A-Dirac bundle (see Definition 7.2 and 7.3). We call an A-linear Dirac operator D on a complete manifold N *(a,b)-adapted* if the curvature endomorphism $\mathcal{R}_x\colon S_x\longrightarrow S_x$ in the Weitzenböck formula for D^2 satisfies

(1) $\mathcal{R}_x>a/4$ for all $x\in N$.
(2) $\mathcal{R}_x>b/4$ outside a compact subset of N.

9.2 Theorem. *Let A be a $\mathbb{Z}/2$-graded real C^*-algebra, and let D be an (a,b)-adapted A-linear Dirac operator on a complete manifold N without boundary. If $b>0$, then D has a well-defined index* $\mathrm{ind}(D)\in KO(A)$. *If $a>0$, then* $\mathrm{ind}(D)=0$.

9.3 Definition of θ. For $a<0<b$ we define the homomorphism

$$\theta\colon R_n(\gamma;a,b)\longrightarrow KO_n(C^*\gamma)$$

by mapping $[N,g]$ to $\mathrm{ind}(D(N,g))\in KO(C\ell_n\widehat{\otimes}C^*\gamma)=KO_n(C^*\gamma)$. Abusing notation we also write θ for the composition

$$R_n(\gamma)\xrightarrow{\cong}R_n(\gamma;a,b)\xrightarrow{\theta}KO_n(C^*\gamma),$$

where the first map is the isomorphism of Lemma 6.4. We recall that it is given by sending $[M,h]$ to $[\widehat{M},g]$, where $\widehat{M}=M\cup_{\partial M}\partial M\times[0,\infty)$, and g is a complete metric on $\widehat{M}$, obtained by extending the positive scalar curvature metric h on ∂M (after rescaling) in a suitable way to $\widehat{M}$ (cf. 3.).

As in Bunke's paper, one can use his K-Theoretic Relative Index Theorem [Bu, Thm. 1.2] to show that θ is well-defined [Bu, Thm. 1.17]. We would like to point out that for $n\geq 5$, and if π is finitely presented, then we can use the geometric results of §5 to get a well-defined homomorphism θ as follows.

According to Lemma 5.6 shows that we may represent each element of $R_n(\gamma)$ for $n\geq 5$ by a pair (M,h) such that the map $f\colon M\longrightarrow X$ which is part of the γ-structure on M is a 2-equivalence. Hence we may restrict to such pairs in the definition of θ. The Existence Theorem 5.4(1) shows that if (M,h) is zero bordant, then h extends to a positive scalar curvature metric H on M. It follows that the metric g on $\widehat{M}$ can be chosen to be (a,b)-adapted for some $a>0$ and hence $\mathrm{ind}(D(\widehat{M},g))=0$ by Theorem 9.2.

For the proof of Theorem 9.2 we will describe $KO(A)$ in terms of Kasparov modules. For the convenience of the reader we repeat some basic definitions which can be found in [Ka], [Co] or [Schr].

9.4 Adjointable and compact operators on Hilbert C^*-modules. Let A be a C^*-algebra and let H be a Hilbert A-module. An A-linear bounded operator $T\colon H\longrightarrow H$ is *adjointable* if there is an operator $T^*\colon H\longrightarrow H$ such that $<Tx,y>=<x,T^*y>$ for all $x,y\in H$ (unlike the classical case of Hilbert spaces, the existence of T^* is in general not automatic for bounded operators). Then $\mathcal{L}(H)$, the set of adjointable operators, is a C^*-algebra. It contains as an ideal the set of compact operators $\mathcal{K}(H)$, which is defined as the norm closure of the linear span of operators of the type $h\mapsto x<y,h>$ for $x,y\in H$.

9.5 Kasparov modules and $KO_n(A)$. We recall that for a $\mathbb{Z}/2$-graded real C^*-algebra A the group $KO_n(A)$ can be identified with $KK_n(\mathbb{R},A)=KK(\mathbb{R},C\ell_n\widehat{\otimes}A)$, a special case of the KK-groups introduced by Kasparov. We recall that elements of $KK(\mathbb{R},A)$ are equivalence classes of pairs (H,F), called $(\mathbb{R},A)$-*Kasparov modules*, where

(1) H is a $\mathbb{Z}/2$-graded Hilbert A-module and
(2) $F\colon H\longrightarrow H$ is an adjointable operator of degree 1 such that F^2-1 and $F-F^*$ are compact operators.

A Kasparov module (H,F) is called *degenerate* if $F^2=1$ and $F=F^*$. Degenerate Kasparov modules represent the trivial element in $KK(\mathbb{R},A)$. In fact, $KK(\mathbb{R},A)$ is obtained from the set of $(\mathbb{R},A)$-Kasparov modules by identifying Kasparov modules obtained by adding a degenerate module or replacing F by an operator homotopic to F (the homotopy F_t is required to be norm continuous).

9.6 Bunke's construction of $\mathrm{ind}(D)\in KO(A)$. Let D be the A-linear Dirac operator associated to an A-Dirac bundle S over a complete manifold N. Assume that the curvature endomorphism $\mathcal{R}_x$ satisfies $\mathcal{R}_x\geq c$ for some $c>0$ and all x outside a compact subset of N. With these data Bunke constructs a Kasparov module representing an element in $KO(A)$ as follows.

We note that $D: C_c^\infty(S) \to C_c^\infty(S)$ is A-linear, but it does it not extend to a *bounded* operator on $L^2(S)$ (even in the classical case where N is compact, and $A = \mathbb{R}$). The usual fix is to replace D by $F = k(D)$, where k is the function $k(x) = x/(x^2+1)^{-1/2}$. The operator $k(D)$ is defined by functional calculus and is always an element of $\mathcal{L}(L^2(S))$. However, for non-compact M the property $F^2 - 1 \in \mathcal{K}(L^2(S))$ typically fails (even for $A = \mathbb{R}$), due to the possibly non-empty essential spectrum of D. Bunke overcomes this difficulty by considering the operator $F = D(D^2+f)^{-1/2}$ (defined by a suitable integral), where $f: M \to \mathbb{R}^+$ is a function with compact support such that the operator $D^2 + f$ is invertible (as a bounded operator from the Sobolev space H^2 to the Sobolev space H^0). The operator F has the property that $F^2 - 1$ and $F^* - F$ are compact operators, and hence $(L^2(S), F)$ represents an element of $KO(A)$ [Bu, Prop. 1.13]. Moreover, replacing f by f', the corresponding operator $F' = D(D^2+f')^{-1/2}$ differs from F by a compact operator [Bu, Lemma 1.10]; hence $(L^2(S), F)$ and $(L^2(S), F')$ represent the *same* element in $KO(A)$. This way Bunke gets a well-defined index defined by

$$\operatorname{ind}(D) \stackrel{\text{def}}{=} [L^2(S), D(D^2+f)^{-1/2}] \in KO(A).$$

The author has found the operator $D(D^2+f)^{-1/2}$ difficult to work with. In particular, he has not managed to prove the multiplicative properties of the index working with this operator (the technical difficulty is that D and $D^2 + f$ commute only up to compact operators). Following a suggestion of Ulrich Bunke, we replace $D(D^2+f)^{-1/2}$ by $\chi(D)$, where $\chi: \mathbb{R} \to \mathbb{R}$ is a suitable bounded function (cf. Definition 9.). To make sense of $\chi(D)$ we use the "bounded functional calculus" for regular selfadjoint operators described e.g. in the book by Lance [La, Chapters 9 & 10].

We recall that if E and F are Hilbert A-modules, a *regular* operator from E to F is a densely defined A-linear map $D: E \to F$ such that its adjoint $D^*: F \to E$ is densely defined and $1 + D^*D$ has dense range.

9.7 Lemma. *Let $S \to M$ be an A-Dirac bundle over a complete manifold M. Then the closure of the associated A-linear Dirac operator $D: C_c^\infty(S) \to C_c^\infty(S)$ is a self-adjoint regular operator $D: L^2(S) \to L^2(S)$. More generally, if f is a section of the endomorphism bundle of S, and ϵ is the grading involution on S, then the closure of $D + f\epsilon$ is a regular self-adjoint operator on $L^2(S)$.*

Proof. The ususal proof that the closure of a symmetric differential operator on a complete manifold is a selfadjoint operator in the classical case $A = \mathbb{R}$ generalizes upon replacing the $\mathbb{C}$-valued inner product by the A-valued inner product.

Bunke shows that the image of $D^2 + 1$ is dense (see [Bu], Proof of Lemma 1.18); the same idea can be used to show that the image of $(D + f\epsilon)^2 + 1$ is dense. □

9.8 Functional Calculus for self-adjoint regular operators. A functional calculus for self-adjoint regular operators is developed e.g. in Lance's book [La]. We will need this calculus only for a much more restricted class of functions than considered in the book (see e.g. Theorem 10.9 in [La]); however, we will need some properties of this functional calculus not mentioned in Lance's book, which can be extracted from the book as explained below.

As explained below, each selfadjoint regular operator $D: H \to H$ has a *spectrum* $spec(D) \subset \mathbb{R} \cup \{\pm\infty\}$ (the obvious 2-point compactification of $\mathbb{R}$), and there is a $*$-homomorphism (and hence automatically isometric)

$$\alpha: C(spec(D)) \to \mathcal{L}(H) \qquad h \mapsto h(D).$$

Here $C(spec(D))$ is the C^*-algebra of continuous real valued functions on $spec(D)$ furnished with the sup-norm. In particular, if a sequence $h_i \in C(spec(D))$ converges in the sup-norm to $h \in C(spec(D))$, then $h_i(D)$ converges in the norm topology to $h(D)$.

The first step in the construction of α is to make sense of $F = D(1+D^2)^{-1/2}$ as a bounded operator on H. It is not hard to show that $(1+D^2)^{-1}: \operatorname{ran}(1+D^2) \to H$ extends to a self-adjoint positive element of $\mathcal{L}(H)$ (cf. [La, proof of Lemma 9.1]). If $(1+D^2)^{-1/2}$ is the square root of $(1+D^2)^{-1} \in \mathcal{L}(H)$ (defined by the functional calculus for self-adjoint elements of C^*-algebras as explained below), a similar argument shows that $D(1+D^2)^{-1/2}$ extends to a selfadjoint operator $F \in \mathcal{L}(H)$ (cf. [La], proof of Lemma 9.2 and Theorem 10.4). By Gelfand's Theorem the Banach subalgebra of $\mathcal{L}(H)$ generated by F and the identity is $*$-isomorphic to the C^*-algebra of continuous functions on $spec(F) \subset \mathbb{R}$. In particular, one obtains an isometric $*$-homomorphism

$$\alpha_F: C(spec(F)) \to \mathcal{L}(H) \qquad g \mapsto g(F)$$

refered to as "functional calculus" (for the self-adjoint element F of the C^*-algebra $\mathcal{L}(H)$).

We note that $||F|| \le 1$ by [La, Theorem 10.4], and hence $spec(F) \subset [-1,+1]$. We define $spec(D)$ as $k^{-1}(spec(F))$, where $k: \mathbb{R} \cup \{\pm\infty\} \xrightarrow{\approx} [-1,+1]$ is the homeomorphism given by $k(x) = x(1+x^2)^{-1/2}$ for $x \in \mathbb{R}$. Furthermore, we define α to be the composition

$$C(spec(D)) \xrightarrow{k^*} C(spec(F)) \xrightarrow{\alpha_F} \mathcal{L}(H),$$

where the first map is the isomorphism of C^*-algebras given by composition with k.

9.9 Lemma. *If $h_i \in C(spec(D))$ is a sequence converging to $h \in C(spec(D))$ uniformly on compact subsets of $spec(D) \cap \mathbb{R}$ (but not necessarily at $-\infty$ or $+\infty$), then $h_i(D)(x)$ converges to $h(D)(x)$ for all $x \in H$.*

Proof. For the proof of the lemma, it suffices to show that if $g_i \in C(spec(F))$ is a sequence converging to $g \in C(spec(F))$ uniformly on compact subsets of $spec(F) \cap [-1,+1]$, then $g_i(F)(x)$ converges to $g(F)(x)$ for all $x \in H$.

We note that $1 - F^2 \ge 0$, since $||F|| \le 1$, and that the range of $1 - F^2$ is dense in H by [La, Thm. 10.4]. It follows by Lemma 10.1 of [La] that $(1-F^2)^{1/n}(x)$ converges to x for every $x \in H$. Writing e_n for the function $e_n(x) = (1-x^2)^{1/n}$, we hence obtain

$$\begin{aligned}
&||g(F)(x) - g_i(F)(x)|| \le ||g(F)(x) - g(F)e_n(F)(x)|| \\
&+ ||g(F)e_n(F)(x) - g_i(F)e_n(F)(x)|| + ||g_i(F)e_n(F)(x) - g_i(F)(x)|| \\
&\le (||g|| + ||g_i||)||e_n(F)(x) - x|| + ||ge_n - g_ie_n||\,||x||.
\end{aligned}$$

We note that the first summand can be made as small as desired by choosing n large enough, since $||g|| + ||g_i||$ is bounded; the second summand can be made as small as desired by choosing i large enough, since for fixed n the sequence $g_i e_n$ converges in the sup norm to ge_n. $\square$

The main technical result needed for the definition of the $KO(A)$-valued index and the proof of Theorem 9.2 is the following.

9.10 Theorem. *Let A be a C^*-algebra, and let D be an (a,b)-adapted A-linear Dirac operator. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. If $a > 0$ and* $\operatorname{supp}(h) \subset [-\sqrt{a}/2, \sqrt{a}/2]$, *then $h(D) = 0$. If $b > 0$ and* $\operatorname{supp}(h) \subset \left[-\sqrt{b}/2, \sqrt{b}/2\right]$, *then $h(D)$ is a compact operator.*

For the definition of the index of a Dirac operator the following terminology will be useful.

9.11 Definition. Let b be a positive real number. A smooth function $\chi\colon \mathbb{R} \to [-1, +1]$ is called a *b-normalizing function* if χ is odd, $\chi(x) = +1$ if $x \geq \sqrt{b}/2$, and $\chi(x) = -1$ if $x \leq -\sqrt{b}/2$.

Proof of Theorem 9.2. Let D be an A-linear Dirac operator which is (a,b)-adapted with $b > 0$. We define

$$\operatorname{ind}(D) \stackrel{\text{def}}{=} [L^2(S), \chi(D)] \in KO(A), \tag{9.12}$$

where $\chi \in C_b(\mathbb{R})$ is a b-normalizing function. We note that $\chi(D)^2 - 1 = (\chi^2 - 1)(D)$ is compact by Theorem 9.10 since $\chi^2 - 1$ vanishes outside the interval $\left[-\sqrt{b}/2, \sqrt{b}/2\right]$. Moreover, $\chi(D)$ is self-adjoint, and has degree 1 since D is self-adjoint and χ is odd. Hence $(L^2(S), \chi(D))$ is a Kasparov module. If $\chi' \in C_b(\mathbb{R})$ is another function satisfying the conditions above, then $\chi(D) - \chi'(D) = (\chi - \chi')(D)$ is compact again by Theorem 9.10 and hence $[L^2(S), \chi(D)] \in KO(A)$ is independent of the choice of χ.

If $a > 0$, we can choose χ to be an a-normalizing function. Then Theorem 9.10 implies $\chi(D)^2 - 1 = (\chi^2 - 1)(D) = 0$; hence $(L^2(S), \chi(D))$ is a degenerate Kasparov module representing the trivial element of $KO(A)$. $\square$

The main tool in the proof of Theorem 9.10 will be Sobolev spaces and the analogue of the Rellich lemma proved by Miščenko and Fomenko for spaces of sections of vector bundles furnished with the action of a C^*-algebra.

9.13 Sobolev spaces. Let S be an A-Dirac bundle over a complete Riemannian manifold M and let D be the associated Dirac operator. For $s \in \mathbb{Z}^+$, we define an A-valued scalar product $\langle\ ,\ \rangle_s$ on the space $C_c^\infty(S)$ of smooth sections with compact support by

$$\langle \phi, \psi \rangle_s = \sum_{k=0}^{s} \langle D^k \phi, D^k \psi \rangle \qquad \text{for} \qquad \phi, \psi \in C_c^\infty(S),$$

where $\langle\ ,\ \rangle$ is the L^2 inner product on $C_c^\infty(M)$ defined in 5.3. The Sobolev space $H^s(S)$ is the Hilbert A-module obtained by completing the pre-Hilbert A-module $C_c^\infty(S)$ with respect to the norm $||\phi||_s^2 = ||\langle \phi, \phi \rangle_s||$.

We will need the following analog of the Rellich Lemma.

9.14 Theorem (Miščenko-Fomenko [MiFo, Lemma 3.3]**,** [Bu, Lemma 1.3]**).** *If $f: M \to \mathbb{R}$ is a smooth function with compact support then the multiplication operator $f: H^l(S) \to H^k(S)$ is compact for $k < l$.*

The key point in the proof of this result is to show the statement in the special case where M is a torus. Then using partitions of unity the statement follows for general M (in [MF] the result is stated in the slightly restricted form, where M rather than the support of f is required to be compact).

The proof of Theorem 9.10 will be based on the following three lemmas; the first of these lemmas implies the first statement of Theorem 9.10.

9.15 Lemma. *If c is a positive real number such that $\mathcal{R}_x \geq c^2$ for all $x \in M$ and $\operatorname{supp}(h) \subset [-c, +c]$, then $h(D) = 0$.*

Proof. The Weitzenböck formula (see 7.8) implies for all $\phi \in C_c^\infty(S)$

$$||\langle (D^2+1)\phi, \phi\rangle|| = ||\langle (\nabla^*\nabla + \mathcal{R} + 1)\phi, \phi\rangle|| \geq (c^2+1)||\phi||^2.$$

Combined with the Cauchy-Schwartz inequality $||(D^2+1)\phi||\,||\phi|| \geq ||\langle (D^2+1)\phi, \phi\rangle||$ we obtain

$$||(D^2+1)\phi|| \geq (c^2+1)||\phi||.$$

If ψ is in the range of D^2+1, say $\psi = (D^2+1)\phi$, then

$$||(D^2+1)^{-1}\psi|| = ||\phi|| \leq (c^2+1)^{-1}||\psi||.$$

Since the range of D^2+1 is dense by Lemma 9.7, we conclude $||(D^2+1)^{-1}|| \leq (c^2+1)^{-1}$. In terms of $F = D(D^2+1)^{-1/2}$, the operator $(D^2+1)^{-1}$ can be expressed as $(D^2+1)^{-1} = 1 - F^2$, which implies $(D^2+1)^{-1} = g(D)$, where $g(x) = (x^2+1)^{-1}$. Hence

$$\sup\{g(x) \mid x \in spec(D)\} = ||g(D)|| \leq (c^2+1)^{-1} = g(c),$$

where the first equality holds, since $g(D)$ is self-adjoint. It follows that $spec(D) \cap (-c, +c) = \emptyset$, which implies $||h(D)|| = \sup\{h(x) \mid x \in spec(D)\} = 0$, and hence $h(D) = 0$. $\square$

The previous lemma can be generalized as follows.

9.16 Lemma. *Let c be a positive real number and assume that $f: M \to \mathbb{R}$ is a smooth function such that $\mathcal{R}_x + f^2(x) - |\operatorname{grad} f(x)| \geq c^2$ for all $x \in M$. If $h: \mathbb{R} \to \mathbb{R}$ is a continuous function with $\operatorname{supp}(h) \subset [-c, +c]$, then $h(D + f\epsilon) = 0$, where $\epsilon: S \to S$ is the grading homomorphism.*

Proof. We recall that the commutator $[D, f]$ is Clifford multiplication by the gradient $\operatorname{grad} f$. It follows that

$$(D+f\epsilon)^2 = D^2 + Df\epsilon + f\epsilon D + f^2 = D^2 + Df\epsilon - fD\epsilon + f^2 = D^2 + \operatorname{grad} f\epsilon + f^2. \tag{9.17}$$

Hence the Weitzenböck formula implies

$$\begin{aligned} ||\langle ((D+f\epsilon)^2+1)\phi, \phi\rangle|| &= ||\langle (D^2 + \operatorname{grad} f\epsilon + f^2 + 1)\phi, \phi\rangle|| \\ &= ||\langle (\nabla^*\nabla + \mathcal{R} + \operatorname{grad} f\epsilon + f^2 + 1)\phi, \phi\rangle|| \geq (c^2+1)||\phi||^2 \end{aligned}$$

for all $\phi \in C_c^\infty(S)$. Repeating the arguments in the proof of Lemma 9.15 with D replaced by $D + f\epsilon$ shows $h(D + f\epsilon) = 0$. $\square$

9.18 Lemma. *If $f\colon M \longrightarrow \mathbb{R}$ is a smooth function with compact support and $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ vanishes at infinity, then $h(D + f\epsilon) - h(D)$ is compact.*

Proof. We claim that it suffices to prove that $K \overset{\text{def}}{=} k(D + f\epsilon) - k(D)$ is compact, where $k(x) = x(x^2 + 1)^{-1/2}$. To see this we note that if p is any polynomial, then $p(k(D+f\epsilon)) - p(k(D)) = p(k(D)+K) - p(k(D))$ is in the two-sided ideal generated by K and hence compact if K is. So if p_n is a sequence of polynomials converging uniformly to $h \circ k^{-1}$ on $[-1, 1]$, then $p_n(D + f\epsilon)$ and $p_n(D)$ converge to $h(D + f\epsilon)$ respectively $h(D)$ in the norm topology. It follows that $h(D+f\epsilon)-h(D)$ is compact.

To prove that K is compact we note that the Riemann sums for the integral

$$\frac{2}{\pi} \int_0^\infty (x^2 + 1 + \lambda^2)^{-1}\, d\lambda$$

are uniformly bounded and converge uniformly on compact sets to $k(x)$. Hence by Lemma 9.9

$$k(D) = \frac{2}{\pi} \int_0^\infty D(D^2 + 1 + \lambda^2)^{-1}\, d\lambda,$$

where the integral converges in the strong topology. Replacing D by $D + f\epsilon$ and using formula (9.17) we obtain the following integral expression for $k(D + f\epsilon)$.

$$k(D + f\epsilon) = \frac{2}{\pi} \int_0^\infty (D + f\epsilon)(D^2 + \operatorname{grad} f\epsilon + f^2 + 1 + \lambda^2)^{-1}\, d\lambda.$$

This implies $k(D + f\epsilon) - k(D) = \frac{2}{\pi} \int_0^\infty \alpha(\lambda)\, d\lambda$, where

$$\begin{aligned}
\alpha(\lambda) &= (D + f\epsilon)(D^2 + \operatorname{grad} f\epsilon + f^2 + 1 + \lambda^2)^{-1} - D(D^2 + 1 + \lambda^2)^{-1} \\
&= (D + f\epsilon)\big((D^2 + \operatorname{grad} f\epsilon + f^2 + 1 + \lambda^2)^{-1} - (D^2 + 1 + \lambda^2)^{-1}\big) \\
&\quad + f\epsilon(D^2 + 1 + \lambda^2)^{-1} \\
&= -(D + f\epsilon)(D^2 + \operatorname{grad} f\epsilon + f^2 + 1 + \lambda^2)^{-1}(f^2 + \operatorname{grad} f\epsilon)(D^2 + 1 + \lambda^2)^{-1} \\
&\quad + f\epsilon(D^2 + 1 + \lambda^2)^{-1}.
\end{aligned}$$

We claim that

1. The operator $\alpha(\lambda)$ is compact for all λ.
2. The integral $\int_0^\infty \alpha(\lambda)\, d\lambda$ converges in the *norm* topology (unlike the integrals giving $k(D)$ and $k(D + f\epsilon)$).

These claims imply that $k(D + f\epsilon) - k(D) = 2/\pi \int_0^\infty \alpha(\lambda)\, d\lambda$ is compact, since by definition the set of compact operators is closed with respect to the norm topology.

The first claim follows from the "Rellich" Lemma 9.14 and the following factorizations

$$H^0(S) \xrightarrow{(D^2+1+\lambda^2)^{-1}} H^2(S) \xrightarrow{f\epsilon} H^0(S)$$

$$H^0(S) \xrightarrow{(D^2+1+\lambda^2)^{-1}} H^2(S) \xrightarrow{f^2+\operatorname{grad} f\epsilon} H^0(S) \xrightarrow{-(D+f\epsilon)\left((D+f\epsilon)^2+1+\lambda^2\right)^{-1}} H^0(S)$$

The second claim is a consequence of the norm estimates

$$\begin{aligned} ||(D^2+1+\lambda^2)^{-1}|| &= ||g_\lambda(D)|| \leq ||g_\lambda||_{\sup} = (1+\lambda^2)^{-1} \\ ||(D+f\epsilon)\left((D+f\epsilon)^2+1+\lambda^2\right)^{-1}|| &= ||h_\lambda(D+f\epsilon)|| \\ &\leq ||h_\lambda||_{\sup} \leq 1/2(1+\lambda^2)^{-1/2}, \end{aligned}$$

where $g_\lambda(x) = (x^2+1+\lambda^2)^{-1}$ and $h_\lambda(x) = x(x^2+1+\lambda^2)^{-1}$. □

Proof of Theorem 9.10. Assume that $\mathcal{R}_x \geq c^2$ outside a compact set $K \subset M$. Then we can manufacture a function $f \in C_c^\infty(M)$ such that $\mathcal{R}_x + f^2(x) - |\operatorname{grad} f(x)| \geq c^2$ holds for all $x \in M$ (on K choose f to be a constant large enough such that $\mathcal{R}_x + f^2(x) \geq c^2$ for $x \in K$; with increasing distance from K, let $f(x)$ decrease to 0 slowly enough so that $|\operatorname{grad} f(x)|$ is bounded by $c^2 - \mathcal{R}_x$). Then $h(D)$ is a compact operator by Lemmas 9.16 and 9.18. □

The technique used in the proof of Lemma 9.18 also leads to a proof that Bunke's definition of $\operatorname{ind}(D)$ (cf. 9.6) agrees with our definition (cf. 9.12). That allows us in particular to apply his relative index theorem to show that θ is well-defined (cf. remarks after definition 9.3).

9.19 Lemma. $[L^2(S), D(D^2+f^2)^{-1/2}] = [L^2(S), \chi(D)] \in KO(A)$.

Proof. Bunke shows that the integral

$$\frac{2}{\pi}\int_0^\infty D(D^2+f^2+1+\lambda^2)^{-1}\,d\lambda$$

converges in the strong topology. Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous even function with $h(x) = 1$ and $\operatorname{supp}(h) \subset [-\sqrt{b}/2, +\sqrt{b}/2]$. Then $\chi(x) = x(x^2+h(x))^{-1/2}$ is a b-normalizing function, and $\chi(D)$ is given by the integral

$$\chi(D) = \frac{2}{\pi}\int_0^\infty D(D^2+h(D)+1+\lambda^2)^{-1}\,d\lambda$$

which converges in the strong topology by the same argument as in 9.18. Hence $D(D^2+f^2)^{-1/2} - \chi(D) = \frac{2}{\pi}\int_0^\infty \alpha(\lambda)\,d\lambda$, where

$$\begin{aligned} \alpha(\lambda) &= D\left((D^2+f^2+1+\lambda^2)^{-1} - (D^2+h(D)+1+\lambda^2)^{-1}\right) \\ &= D(D^2+f^2+\lambda^2)^{-1}(h(D)-f^2)(D^2+h(D)+1+\lambda^2)^{-1}. \end{aligned}$$

By a result of Bunke, $D(D^2+f^2+\lambda^2)$ is a bounded operator with $||D(D^2+f^2+\lambda^2)|| \leq C(d+\lambda^2)^{-1/2}$ for some constants $C, d > 0$ independent of λ [Bu, Lemma 1.6]. Moreover, $(D^2+h(D)+1+\lambda^2)^{-1}$ is a bounded operator whose norm is bounded by the sup norm $||x(x^2+h(x)+\lambda^2)^{-1}||_{\sup} \leq C'(d'+\lambda^2)^{-1}$ for suitable constants $C', d' > 0$ independent of λ. This shows that the integral $\int_0^\infty \alpha(\lambda)\,d\lambda$ converges in fact in the norm topology. Moreover, the factorization

$$H^0 \xrightarrow{(D^2+h(D)+1+\lambda^2)^{-1}} H^2 \xrightarrow{h(D)-f^2} H^0 \xrightarrow{D(D^2+f^2+\lambda^2)^{-1}} H^0$$

of $\alpha(\lambda)$ shows that $\alpha(\lambda)$ is a compact operator, since $H^2 \xrightarrow{h(D)-f^2} H^0$ is compact by Theorem 9.10 resp. the "Rellich Lemma" 9.14. It follows that $D(D^2+f^2)^{-1/2} - \chi(D) = \frac{2}{\pi}\int_0^\infty \alpha(\lambda)\,d\lambda$ is compact, which proves the Theorem. □

9.20 Theorem. *For $i = 1, 2$, let A_i be a $\mathbb{Z}/2$-graded real C^*-algebra and let $S_i \longrightarrow M_i$ be an A_i-Dirac bundle whose associated Dirac operator D_i is (a_i, b_i)-adapted. Then the external product $S = pr_1^* S_1 \widehat{\otimes} pr_2^* S_2$ is an A-Dirac bundle over $M = M_1 \times M_2$ with $A = A_1 \widehat{\otimes} A_2$ (here $pr_i\colon M \longrightarrow M_i$ is the projection map). The corresponding Dirac operator D is (a, b)-adapted with $a = a_1 + a_2$, $b = \min\{a_1 + b_2, a_2 + b_1\}$. If $b > 0$, then $\operatorname{ind}(D) \in KO(A_1 \widehat{\otimes} A_2)$ is the Kasparov product of $\operatorname{ind}(D_1) \in KO(A_1)$ and $\operatorname{ind}(D_2) \in KO(A_2)$.*

We recall from §6 that the Cartesian product of manifolds defines an exterior product for the groups $R_n(\gamma; a, b)$, and hence via the isomorphism $R_n(\gamma) \cong R_n(\gamma; a, b)$ for the groups $R_n(\gamma)$. By Lemma 8.13 the A-Dirac bundle S over $M = M_1 \times M_2$ determined by the product γ-structure, $\gamma = \gamma_1 \widehat{\times} \gamma_2$, can be identified with the external product $pr_1^* S_1 \widehat{\otimes} pr_2^* S_2$ of the A_i-Dirac bundles S_i determined by the γ_i-structure on M_i. Hence Theorem 9.20 shows that multiplication in the R_n-groups corresponds to the Kasparov product of the K-theory groups. In particular, the lower part of the diagram of Theorem 1.3 commutes (the commutativity of the upper part follows directly from the construction of the product of the R_n-groups in section 6.

Before proving Theorem 9.20 we recall the characterization of the Kasparov product via connections due to Connes-Skandalis.

9.22 Kasparov product via connections (cf. [Co, Chapt. IV, App. A]**).** The Kasparov product of $[E_1, F_1] \in KO(A_1)$ and $[E_2, F_2] \in KO(A_2)$ is represented by $[E, F] \in KO(A_1 \widehat{\otimes} A_2)$, $E = E_1 \widehat{\otimes} E_2$ if $F \in \mathcal{L}(E)$ satisfies the following properties:

(1) (E, F) is a Kasparov A-module
(2) $[F_1 \widehat{\otimes} 1, F] \geq 1$ modulo $\mathcal{K}(E)$
(3) F is an F_2-connection on E_1, that is, for every $\xi \in E_1$ the (graded) commutator $[\widetilde{T}_\xi, F_2 \oplus F] \in (E_2 \oplus E)$ is compact.

Here $\widetilde{T}_\xi$ is given by the matrix $\begin{bmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{bmatrix}$, where $T_\xi(\eta) = \xi \widehat{\otimes} \eta \in E$ for all $\eta \in E_2$.

Proof of Theorem 9.20. Let D_i (resp. D) be the Dirac operator associated to the Dirac bundle S_i (resp. S). Let $\mathcal{R}_{x_i}$ (resp. $\mathcal{R}_x$ be the curvature endomorphismus in the Weitzenböck formula for D_i^2 (resp. D^2) at the point $x_i \in M_i$ (resp. $x = (x_1, x_2) \in M$. It follows directly from the definition of $\mathcal{R}_x$ (see 7.8) that it can be expressed as the endomorphism

$$S_x = S_{x_1} \widehat{\otimes} S_{x_2} \xrightarrow{\mathcal{R}_{x_1} \otimes 1 + 1 \otimes \mathcal{R}_{x_2}} S_{x_1} \widehat{\otimes} S_{x_2} = S_x.$$

Hence the eigenvalues of $\mathcal{R}_x$ are the sums of eigenvalues of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$. This shows that D is (a, b)-adapted.

Identifying $L^2(S)$ with the tensor product $L^2(S_1) \widehat{\otimes} L^2(S_2)$, the Dirac operator D is given by

$$D = D_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} D_2.$$

Let $\chi_i \in C_b(\mathbb{R})$ (resp. $\chi \in C_b(\mathbb{R})$) be a b_i-normalizing function (resp. b-normalizing function). Let $F_i = \chi_i(D_i) \in \mathcal{L}(L^2(S_i))$ and $F = \chi(D) \in \mathcal{L}(L^2(S))$. We observe that the operators D, $D_1 \widehat{\otimes} 1$ and $1 \widehat{\otimes} D_2$ all commute (in the graded sense);

this implies that the commutators $[F_1\widehat{\otimes}1, F]$ and $[\widetilde{T}_\xi, F_2\widehat{\otimes}F]$ both vanish, and hence $\mathrm{ind}(D) = [L^2(S), F]$ is the Kasparov product of $\mathrm{ind}(D_1) = [L^2(S_1), F_1]$ and $\mathrm{ind}(D_2) = [L^2(S_2), F_2]$. □

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