MATH 20550 - Calculus III Notes 3
September 15, 2016

13.3 Arc Length and Curvature

1. Given a curve that has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, where f', g', h' are continuous. If the curve is traversed exactly once as t increases from a to b, then its **length** is given by

$$L = \int_{t=b}^{t=a} |\mathbf{r}'(t)|\, dt.$$

   The **arc length function** s is given by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du.$$

   So, $s(t)$ is the length of the part of the curve between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

   Sometimes, we want to **parametrize a curve with respect to arc length** measuring from a point P. To accomplish this parametrization for $\mathbf{r}(t)$, we follow these steps:

   (1) Find $s(t)$. Note that the lower bound of the integral is given by the value of t corresponds to the given initial point P.

   (2) Using the formula of $s(t)$ to solve for t. So, we get $t = t(s)$, which means t is written in terms of s.

   (3) Write the vector-valued function $\mathbf{r}(t(s))$. So, now, $\mathbf{r}$ has parameter s instead of t.

2. At any given point on a smooth space curve $\mathbf{r}(t)$, we have the following vectors:

   (1) The **unit tangent vector T**:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

   (2) The **principal unit normal vector N** (perpendicular to **T**):

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

   (3) The **unit binormal vector** (perpendicular to both **T** and **N**)

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

In general, it is very painful to compute $\mathbf{T}'(t)$ in order to get $\mathbf{N}(t)$. So, in the event that you are asked to compute $\mathbf{N}(t)$, you can compute $\mathbf{B}(t)$ first using this formula

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}.$$

Then, find $\mathbf{N}(t)$ by the formula

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

3. Visually, the **normal plane** of a curve $\mathbf{r}(t)$ at a point P is the plane containing the two vectors $\mathbf{N}$ and $\mathbf{B}$ at P. This means a <u>normal vector of the normal plane</u> is a tangent vector $\mathbf{r}'$ at P.

4. Visually, the **osculating plane** of a curve $\mathbf{r}(t)$ at a point P is the plane containing the two vectors $\mathbf{T}$ and $\mathbf{N}$ at P. This means a <u>normal vector of the osculating plane</u> is a vector pointing in the same direction as the unit binormal vector $\mathbf{B}$. And for computation purpose, you can find a normal vector of the osculating plan by computing $\mathbf{r}' \times \mathbf{r}''$ at P.

5. (a) If a curve C is given by a vector-valued function $\mathbf{r}(t)$, then the formula to compute the **curvature** at any point on C is given by

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

(b) If we have a plane curve C with equation $y = f(x)$, then the formula for curvature above comes down to this:

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}.$$

13.4 Motion in Space: Velocity and Acceleration

1. Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Then, the **velocity vector** $\mathbf{v}(t)$ at time t is

$$\mathbf{v}(t) = \mathbf{r}'(t).$$

The **speed** of the particle at time t is $|\mathbf{v}(t)|$, which is the magnitude of the vector $\mathbf{v}(t)$.

The **acceleration** of the particle at time t is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

2. **Newton's Second Law of Motion:** If, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\,\mathbf{a}(t).$$

So, if we know the force that acts on a particle, then we can find the acceleration by the equation above, assuming we also know the mass of the particle.

3. It is often useful to write the acceleration into two components, one in the direction of the unit tangent vector $\mathbf{T}$ and the other in the direction of the unit normal vector $\mathbf{N}$:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N},$$

where a_T is the **tangential component** of acceleration and a_N is the **normal component** of acceleration. They are given by

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}, \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$