The goal of these notes is to show how to compute curvature and torsion from a more or less arbitrary parametrization of a curve. We will also discuss the physics of a particle moving along the curve with the given parametrization. We assume that the curve is sufficiently nice that it has an arc length parametrization, $\mathbf{p}(s)$ as well as the parametrization $\mathbf{r}(t)$ that we are given. The goal is to compute $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ using just $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. The curvature can be computed from these two vectors as well. We include a formula for the torsion in terms of these two vectors and $\mathbf{r}'''(t)$

There is a function, the arc length function $s(t)$ defined by $s(t) = \displaystyle\int_0^t |\mathbf{r}'(x)|dx$.

It follows that

$$\mathbf{r}(t) = \mathbf{p}\big(s(t)\big)$$

In what follows you should regard $\mathbf{r}$ and its derivatives as something you can compute, but neither $\mathbf{p}$ nor $s(t)$ will usually be computable.

However, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

1. Some useful and some less useful formulas

Using the chain rule and patience, one can write out the derivatives of $\mathbf{r}$ in terms of the derivatives of s, and curvature, torsion, unit tangent vector, normal and binormal.

$$\mathbf{r}'(t) = s'(t)\mathbf{p}'\big(s(t)\big) = s'(t)\mathbf{T}\big(s(t)\big) \tag{1}$$

Let us agree to do our calculations around a point where $\mathbf{r}' \neq \mathbf{0}$. Then we can take $\mathbf{T}$ to point in the direction of $\mathbf{r}'$ or in other words, our preferred direction is the direction in which the particle is moving. Then

$$s'(t) = |\mathbf{r}'(t)|$$

When we are thinking of particle moving along the curve with parametrization $\mathbf{r}(t)$, $\dfrac{ds}{dt}$ measures how fast the particle is moving and so is called the *speed*.

Differentiating (1) using the product and chain rules

$$\mathbf{r}''(t) = s''(t)\mathbf{T}\big(s(t)\big) + s'(t)^2\mathbf{T}'\big(s(t)\big) = s''(t)\mathbf{T}\big(s(t)\big) + s'(t)^2\kappa\big(s(t)\big)\mathbf{N}\big(s(t)\big) \tag{2}$$

When we are thinking of particle moving along the curve with parametrization $\mathbf{r}(t)$, $\dfrac{d^2s}{dt^2}$ measures the force pushing you back into your seat or trying to send you through the windshield when you put on the brakes (or hit a tree).

The quantity $f''(t) = v'(t)$ is sometimes referred to as the acceleration, but this is not quite correct. It is what you feel when your step on the gas (or brakes) and the velocity changes, but there is another component in the normal direction. You feel this even if you keep your speed constant but go around a curve. We've all felt something pushing us to the side in this case and the sharper the turn the more noticeable is the push. Part of that force is the curvature and the other part is the (speed)^2.

This means that if you go through a curve twice as fast, you feel four times the force. More importantly, you need four times the force to keep you going through the turn. Most of this force is supplied by the tires refusing to slide sideways but once they start, say due to snow or water or oil slick or what ever, the car will head off down the tangent line which is rarely good.

In the next section we will use (1) and (2) to compute κ. To get τ we will need $\mathbf{r}'''$. The formula is a mess, but here we go.

$$\begin{aligned}\mathbf{r}'''(t) = &\Big(s'''(t)\mathbf{T}\big(s(t)\big) + s''(t)s'(t)\mathbf{T}'\big(s(t)\big)\Big)+\\ &\Big(2s'(t)s''(t)\kappa\big(s(t)\big)\mathbf{N}\big(s(t)\big) + s'(t)^3\kappa'\big(s(t)\big)\mathbf{N}\big(s(t)\big) + s'(t)^3\kappa\big(s(t)\big)\mathbf{N}'\big(s(t)\big)\Big)\\ = &\Big(s'''(t)\mathbf{T}\big(s(t)\big) + s''(t)s'(t)\kappa\big(s(t)\big)\mathbf{N}\big(s(t)\big)\Big)+\\ &\Big(2s'(t)s''(t)\kappa\big(s(t)\big)\mathbf{N}\big(s(t)\big) + s'(t)^3\kappa'\big(s(t)\big)\mathbf{N}\big(s(t)\big)+\\ &- s'(t)^3\kappa^2\big(s(t)\big)\mathbf{T}\big(s(t)\big) + s'(t)^3\kappa\big(s(t)\big)\tau\big(s(t)\big)\mathbf{B}\big(s(t)\big)\Big)\end{aligned}$$

OR

$$\begin{aligned}\mathbf{r}'''(t) = &\Big(s'''(t) - s'(t)^3\kappa^2\big(s(t)\big)\Big)\mathbf{T}+\\ &\Big(s''(t)s'(t)\kappa\big(s(t)\big) + 2s'(t)s''(t)\kappa\big(s(t)\big) + s'(t)^3\kappa'\big(s(t)\big)\Big)\mathbf{N}\\ &\Big(s'(t)^3\kappa\big(s(t)\big)\tau\big(s(t)\big)\Big)\mathbf{B}\end{aligned} \tag{3}$$

For those of you who can read pdf files via split screens we collect the three formula above together. We will also shorten (3) because we will not care what the coefficients of $\mathbf{T}$ and $\mathbf{N}$ are.

$$\mathbf{r}'(t) = s'(t)\mathbf{T}\big(s(t)\big) \tag{1}$$

$$\mathbf{r}''(t) = s''(t)\mathbf{T}\big(s(t)\big) + s'(t)^2\kappa\big(s(t)\big)\mathbf{N}\big(s(t)\big) \tag{2}$$

$$\mathbf{r}'''(t) = (\cdots)\mathbf{T}\big(s(t)\big) + (\cdots)\mathbf{N}\big(s(t)\big) + \Big(s'(t)^3\kappa\big(s(t)\big)\tau\big(s(t)\big)\Big)\mathbf{B}\big(s(t)\big) \tag{3}$$

Under our assumption that $\mathbf{r}'$ and $\mathbf{T}$ point in the same direction,

$$s'(t) = |\mathbf{r}'(t)| \tag{4}$$

Using (4), it is easy to compute s' without knowing s. The book writes (1) as $\mathbf{r}' = \mathbf{v} = v\mathbf{T}$ and (2) as $\mathbf{r}'' = \mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and observes that

$$\begin{aligned}v &= |\mathbf{v}|\\ a_T &= \frac{\mathbf{v}\bullet\mathbf{a}}{v}\\ a_N &= \frac{|\mathbf{v}\times\mathbf{a}|}{v} = v^2\kappa\end{aligned}$$

We can also use similar equations to calculate curvature, torsion and the unit vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ at the point $p = \mathbf{r}(t)$.

$$\begin{aligned} \mathbf{r}'(t) &= |\mathbf{r}'(t)|\mathbf{T} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= |\mathbf{r}'(t)|^3 \kappa \mathbf{B} \\ \big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \times \mathbf{r}'(t) &= |\mathbf{r}'(t)|^4 \kappa \mathbf{N} \end{aligned} \tag{5}$$

The torsion may be computed from

$$\big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \bullet \mathbf{r}'''(t) = |\mathbf{r}'(t)|^6 \kappa^2 \tau$$

Equations (5) can be remembered as follows.

- $\mathbf{r}'$ points in the same direction as $\mathbf{T}$.
- $\mathbf{r}' \times \mathbf{r}''$ points in the same direction as $\mathbf{B}$.
- $(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}'$ points in the same direction as $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

If you need $\mathbf{T}$ and/or $\mathbf{B}$, compute $\mathbf{r}'$ and/or $\mathbf{r}' \times \mathbf{r}''$ and make them unit length. If you need $\mathbf{N}$ compute $(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}'$ and make it unit length.

If you need the curvature, compute $v = |\mathbf{r}'|$, $\mathbf{r}' \times \mathbf{r}''$ and then

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{v^3}$$

Notice that the power of v in the denominator is the number of primes in the terms in the numerator. This also holds in the last two formulas in (5).

The most efficient way to compute $\mathbf{T}$, $\mathbf{B}$ and $\mathbf{N}$ is to compute

$$\mathbf{r}'(t) \tag{6}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) \tag{7}$$

$$\big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \times \mathbf{r}'(t) \tag{8}$$

and make them unit vectors. To compute curvature and torsion the easiest way to go is

$$\begin{aligned} \kappa &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\ \tau &= \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \bullet \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} \end{aligned} \tag{9}$$

Example: (The helix reprised.) Let $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$. As before, let $c = \sqrt{a^2+b^2}$ and assume $a > 0$, $b > 0$.

$$\begin{aligned} \mathbf{r}'(t) &= \langle -a\sin t, a\cos t, b\rangle \\ \mathbf{r}''(t) &= \langle -a\cos t, -a\sin t, 0\rangle \\ \mathbf{r}'''(t) &= \langle a\sin t, -a\cos t, 0\rangle \end{aligned}$$

Hence $|\mathbf{r}'(t)| = c$ and

$$\mathbf{T} = \left\langle -\frac{a}{c}\sin t, \frac{a}{c}\cos t, \frac{b}{c}\right\rangle$$

Now $\mathbf{r}'(t)\times\mathbf{r}''(t) = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \langle ab\sin t, -ab\cos t, a^2\rangle$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{a^2b^2+a^4} = ac \quad (\text{uses } a > 0)$$

Hence $\mathbf{B}$ at the point $\mathbf{r}(t)$ is given by

$$\mathbf{B} = \left\langle \frac{ab}{ac}\sin t, -\frac{ab}{ac}\cos t, \frac{a^2}{ac}\right\rangle = \left\langle \frac{b}{c}\sin t, -\frac{b}{c}\cos t, \frac{a}{c}\right\rangle$$

Compute $\kappa = \dfrac{a\,c}{c^3}$ and so

$$\kappa = \frac{a}{c^2}$$

Next compute

$$\big(\mathbf{r}'(t)\times\mathbf{r}''(t)\big)\times\mathbf{r}'(t) = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ab\sin t & -ab\cos t & a^2 \\ -a\sin t & a\cos t & b \end{vmatrix} = \big\langle (-ab^2-a^3)\cos t, -(a^3+ab^2)\sin t, 0\big\rangle$$

so

$$\Big|\big(\mathbf{r}'(t)\times\mathbf{r}''(t)\big)\times\mathbf{r}'(t)\Big| = \sqrt{(-ab^2-a^3)^2\cos^2 t + (a^3+ab^2)^2\sin^2 t} = \sqrt{(ab^2+a^3)^2} = ac^2$$

$$\mathbf{N} = \left\langle -\frac{ac^2}{c^2a}\cos t, -\frac{ac^2}{c^2a}\sin t, 0\right\rangle = \langle -\cos t, -\sin t, 0\rangle$$

Finally compute the triple product

$$\big(\mathbf{r}'(t)\times\mathbf{r}''(t)\big)\bullet\mathbf{r}'''(t) = \big\langle ab\sin t, -ab\cos t, a^2\big\rangle \bullet \langle a\sin t, -a\cos t, 0\rangle = a^2b$$

Hence

$$\tau = \frac{a^2b}{\big(|a|\,c\big)^2} = \frac{b}{c^2}$$

Since κ and τ are constant for the helix, it is easy to see we got the same answer as before. That we have the right Frenet-Serret frame is a bit more work. To get to the same point on the helix, we need to plug different values into the two parametrization. Last time we noted $\mathbf{r}(t) = \mathbf{p}(ct)$. After this observation it is easy to see that we have the same vectors at each point on the helix.

2. The twisted cubic.

Recall the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. I certainly do not know how to work out an arc length parametrization for this curve. The curve is smooth because $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ is never $\mathbf{0}$ so if we start at $t = 0$ (equivalently at the point $\langle 0, 0, 0 \rangle$)

$$s(t) = \int_0^t \sqrt{1 + 4x^2 + 9x^4}\, dx$$

I have no idea how to do this integral.

Still

$$\begin{aligned} \mathbf{r}' &= \langle 1, 2t, 3t^2 \rangle \\ \mathbf{r}'' &= \langle 0, 2, 6t \rangle \\ \mathbf{r}''' &= \langle 0, 0, 6 \rangle \end{aligned}$$

$$v = \sqrt{1 + 4t^2 + 9t^4}$$

Hence

$$\mathbf{T} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \left\langle 1, 2t, 3t^2 \right\rangle$$

Compute

$$\mathbf{r}' \times \mathbf{r}'' = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \left\langle 12t^2, -6t, 2 \right\rangle = 2 \left\langle 6t^2, -3t, 1 \right\rangle$$

$$|\mathbf{r}' \times \mathbf{r}''| = 2\sqrt{1 + 9t^2 + 36t^4}$$

so

$$\kappa = \frac{2\sqrt{1 + 9t^2 + 36t^4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

and

$$\mathbf{B} = \frac{1}{\sqrt{1 + 9t^2 + 36t^4}} \left\langle 6t^2, -3t, 1 \right\rangle$$

To find $\mathbf{N}$ compute

$$(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}' = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12t^2 & -6t & 2 \\ 1 & 2t & 3t^2 \end{vmatrix} = \left\langle -18t^3 - 4t, -6t, 2 \right\rangle = 2 \left\langle -9t^3 - 2t, -3t, 1 \right\rangle$$

$$|(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}'| = 2\sqrt{1 + 9t^2 + (9t^3 + 2t)^2} = 2\sqrt{1 + 9t^2 + 81t^6 + 36t^4 + 4t^2} = 2\sqrt{1 + 13t^2 + 36t^4 + 81t^6}$$

so

$$\mathbf{N} = \frac{1}{\sqrt{1 + 13t^2 + 36t^4 + 81t^6}} \left\langle -9t^3 - 2t, -3t, 1 \right\rangle$$

Finally

$$(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = \left\langle 12t^2, -6t, 2 \right\rangle \cdot \langle 0, 0, 6 \rangle = 12$$

so

$$\tau = \frac{12}{4(1 + 9t^2 + 36t^4)} = \frac{3}{1 + 9t^2 + 36t^4}$$

If you just need data at one point you can proceed as follows. Suppose you want the curvature, the torsion and the frame $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ for the twisted cubic at the point $(2,4,8)$. With the parametrization $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ you get to the desired point at $t = 2$.

Then at $t = 2$,

$$\begin{aligned}
\mathbf{r}'(2) &= \langle 1, 2t, 3t^2\rangle_{t=2} = \langle 1, 4, 12\rangle \\
\mathbf{r}''(2) &= \langle 0, 2, 6t\rangle_{t=2} = \langle 0, 2, 12\rangle \\
\mathbf{r}'''(2) &= \langle 0, 0, 6\rangle
\end{aligned}$$

After this you can work strictly with constant vectors.

First $v = \sqrt{1+16+144} = \sqrt{161}$ and

$$\mathbf{T}(2) = \frac{1}{\sqrt{161}}\langle 1, 4, 12\rangle$$

Then $\mathbf{r}'(2) \times \mathbf{r}''(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} = \langle 48-24, -(12-0), 2-0\rangle = \langle 24, -12, 2\rangle = v^3\kappa(2)\mathbf{B}(2)$.

$$|\mathbf{r}'(2) \times \mathbf{r}''(2)| = \sqrt{576+144+4} = \sqrt{724} = 2\sqrt{181}$$

$$\kappa(2) = \frac{2\sqrt{181}}{\left(\sqrt{161}\right)^3} = \frac{2\sqrt{181}}{161\sqrt{161}} \qquad \mathbf{B}(2) = \frac{1}{\sqrt{181}}\langle 12, -6, 1\rangle$$

$$\mathbf{N}(2) = \frac{1}{\sqrt{161}\sqrt{181}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 12 \\ 24 & -12 & 2 \end{vmatrix} = \frac{1}{\sqrt{161}\sqrt{181}}\langle 8+144, -(2-12\cdot 24), -12-4\cdot 24\rangle =$$

$$\frac{1}{\sqrt{161}\sqrt{181}}\langle 152, 286, -108\rangle$$

$$\tau(2) = \frac{\big((\mathbf{r}'(2) \times \mathbf{r}''(2)\big) \bullet \mathbf{r}'''}{|\mathbf{r}'(2) \times \mathbf{r}''(2)|^2} = \frac{\langle 24, -12, 2\rangle < 0, 0, 6 >}{(2\sqrt{181})^2} = \frac{12}{4\cdot 181} = \frac{3}{181}$$

3. The Normal and Osculating Planes

The ideas in section 2 are useful for computing equations for the normal and osculating planes.

By definition the normal plane to a curve $\mathbf{r}(t)$ at time $t = a$ has normal vector the tangent vector to the curve, $\mathbf{T}(a)$, or any vector parallel to it. It goes through the point $\mathbf{r}(a)$. Hence

$$\mathbf{r}'(a) \bullet \langle x, y, z\rangle = \mathbf{r}'(a) \bullet \mathbf{r}(a)$$

Example: The normal plane to the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at $(2, 4, 8)$. The curve is at the point $(2, 4, 8)$ when $t = 2$. $\mathbf{r}'(2) = \langle 1, 2t, 3t^2\rangle_{t=2} = \langle 1, 4, 12\rangle$. Hence an equation for the plane is

$$\langle 1, 4, 12\rangle \bullet \langle x, y, z\rangle = \langle 1, 4, 12\rangle \bullet \langle 2, 4, 8\rangle = 114$$

By definition, the osculating plane is the plane spanned by the tangent and normal vectors through the point. Hence the binormal to the curve at the point is a normal vector to the plane at the point.

$$(\mathbf{r}'(a) \times \mathbf{r}''(a)) \bullet \langle x, y, z\rangle = (\mathbf{r}'(a) \times \mathbf{r}''(a)) \bullet \mathbf{r}(a)$$

Example (continued): The osculating plane to the twisted cubic. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at $(2, 4, 8)$.

$$\mathbf{r}'(2) = \left\langle 1, 2t, 3t^2\right\rangle_{t=2} = \langle 1, 4, 12\rangle$$
$$\mathbf{r}''(2) = \langle 0, 2, 6t\rangle_{t=2} = \langle 0, 2, 12\rangle$$

$$\langle 1, 4, 12\rangle \times \langle 0, 2, 12\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} = \langle 48 - 24, -(12 - 0), 2 - 0\rangle = \langle 24, -12, 2\rangle = 2\,\langle 12, -6, 1\rangle$$

so we may use $\langle 12, -6, 1\rangle$. Hence

$$\langle 12, -6, 1\rangle \bullet \langle x, y, z\rangle = \langle 12, -6, 1\rangle \bullet \langle 2, 4, 8\rangle = 8$$

In summary, if you are asked to do calculations proceed as follows.

(1) A normal vector to the normal plane at $t = a$ is $\mathbf{r}'(a)$.
(2) A normal vector to the osculating plane at $t = a$ is $\mathbf{r}'(a) \times \mathbf{r}''(a)$.

Go figure, there is no named plane with a normal vector which is the normal vector $\mathbf{N}$.