1. One variable substitution

Usual version: $u = t(x)$ and find $h(u)$ such that $f(x) = h\big(t(x)\big)t'(x)$ and then

$$\int_a^b f(x)\,dx = \int_{t(a)}^{t(b)} h(u)\,du$$

Version used for trig substitution: $x = s(u)$ and $s(c) = a$, $s(d) = b$

$$\int_a^b f(x)\,dx = \int_c^d f\big(s(u)\big)s'(u)\,du$$

2. Coordinates

A coordinate system is just a special type of vector function of a vector variable. There are coordinate systems in all dimensions and they take as input a k-vector and return as value a k-vector. A coordinate system should also *preserve dimension* in a sense we will make precise as we go along. We call any function $T\colon S \to \mathbb{R}^k$ a C^1*-transformation* where all the partials of the functions in T exist and are continuous, $S \subset \mathbb{R}^k$ and S is a k-dimensional region in the following sense. For any $\mathbf{x} \in \mathbb{R}^k$, let $B_{\mathbf{x}}(r)$ be the ball of radius r centered at $\mathbf{x}$. The *interior* of S consists of all the points in $\mathbf{v} \in S$ for which there exists an $r > 0$ with $B_{\mathbf{v}}(r) \subset S$. Then S is a k*-dimensional region* provided every point in S is as close to an interior point as you like. The part of S not in the interior is called the *boundary.*

- $\mathbb{R}^k$ is a k-dimensional region.
- A closed interval is a 1-dimension region with boundary the two end-points.
- A disk is a 2-dimension region with boundary a circle.
- A ball is 3-dimension region with boundary a sphere.

Here are some 2-dimensional examples of transformations.

$$\begin{aligned} R_1(r,\theta) =& (r\cos\theta, r\sin\theta) \\ R_2(x,y) =& \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right) \quad x \neq 0 \\ R_3(u,w) =& (u^2 - w^2, uw) \end{aligned}$$

Here are some 3-dimensional examples:

$$\begin{aligned} T_1(r,\theta,z) =& (r\cos\theta, r\sin\theta, z) \\ T_2(\rho,\theta,\phi) =& (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi,) \\ T_3(\alpha,\beta,\gamma) =& (\alpha^2 - \gamma\beta^2, \alpha\beta, \alpha+\beta+\gamma) \end{aligned}$$

If S is a k-dimensional region and if $T\colon S \to \mathbb{R}^k$ is a transformation, let

$$T(S) = \big\{\mathbf{w} \in \mathbb{R}^k \quad \text{whenever there exists} \quad \mathbf{v} \in S \quad \text{with} \quad T(\mathbf{v}) = \mathbf{w}\big\}$$

We say T is a *change of coordinates* from S to $R = T(S)$ provided T is a C^1-transformation, R is a k-dimensional region and whenever $T(\mathbf{x}_1) = T(\mathbf{x}_2)$ for $\mathbf{x}_1$, $\mathbf{x}_2 \in S$, then either $\mathbf{x}_1 = \mathbf{x}_2$ or at least one of them is on the boundary of S. We call this last condition the *one-to-one condition.* It can also be stated as

$$T \textit{ is one-to-one on the interior of } S$$

which means

Given $\mathbf{x}_1$ and $\mathbf{x}_2$ in the interior of S, $T(\mathbf{x}_1) = T(\mathbf{x}_2)$ if and only if $\mathbf{x}_1 = \mathbf{x}_2$.

The goal of this section is to describe how to evaluate a multiple integral over R in terms of a multiple integral over S.

$$\iint_R f\,dA = \iint_S f\cdot|\mathbf{J}(T)|\;dA$$
$$\iiint_R f\,dV = \iiint_S f\cdot|\mathbf{J}(T)|\;dV$$

where T is a change of coordinates with $T(S)=R$ and $\mathbf{J}(T)$ is a function, called the *determinant of the Jacobian* of T. (The Jacobian of T is the matrix of which you are taking the determinant.)

There is one additional issue. The book usually writes something like

$$\iint_R f\,dA = \iint_S f\cdot|\mathbf{J}(T)| du\,dv$$

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f\big(x(u,v),y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

I don't like this because it seems to commit you to evaluating your integral as $du\,dv$ when it might be easier to do it as $dv\,du$, (or even convert it to polar coordinates) which is why we write dA in a double integral. On the other hand on one side of the equation we are working in xy space and on the other side in uw space. Hence I like to write the change of coordinate equations when $T\colon S\to R$ as

$$\iint_R f\,dA_{xy} = \iint_S f\cdot|\mathbf{J}(T)|\;dA_{uw} \tag{1}$$
$$\iiint_R f\,dV_{xyz} = \iiint_S f\cdot|\mathbf{J}(T)|\;dV_{uvw} \tag{2}$$

which helps me keep track of which coordinates and region belong on which side.

Remark: You don't care what type of regions you have. For example you want to use polar coordinates to integrate over an annulus R in xy space even though an annulus is of neither type. The corresponding region S is a rectangle.

Remark: The book leaves the interpretation of "one-to-one except possibly on the boundary" for you to figure out. We defined it as one-to-one on the interior of S. This does not mean that there is an inverse transformation T^{-1} from the interior of R back into S.

As an example, in polar coordinates, if R is the unit disk, S is $0\leqslant r\leqslant 1$, $0\leqslant\theta\leqslant 2\pi$ which is a rectangle in $r\theta$ space. The image of the interior of S under polar coordinates is the part of the unit disk minus the boundary circle minus that part of the x-axis with $0\leqslant x\leqslant 1$. Points which are close to the positive x-axis but on opposite sides coms from points in $r\theta$ space which are nearly a distance 2π apart. Hence there is no way to extend the inverse function for polar coordinates to all of the interior of the disk.

3. The Jacobian

To define the Jacobian, suppose the transformation is

$$R(u, w) = \big(x(u, w), y(u, w)\big)$$
or
$$P(\alpha, \beta, \gamma) = \big(x(\alpha, \beta, \gamma), y(\alpha, \beta, \gamma), z(\alpha, \beta, \gamma)\big)$$

Then

$$\mathbf{J}(R) = \det \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial w} \end{vmatrix}$$

$$\mathbf{J}(P) = \det \begin{vmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} & \dfrac{\partial x}{\partial \gamma} \\ \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial y}{\partial \gamma} \\ \dfrac{\partial z}{\partial \alpha} & \dfrac{\partial z}{\partial \beta} & \dfrac{\partial z}{\partial \gamma} \end{vmatrix}$$

With variables as above we will also write

$$\frac{\partial(x, y)}{\partial(u, w)} = \mathbf{J}(R)$$
$$\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)} = \mathbf{J}(P)$$

4. Change of coordinates

Recall that for a transformation T to be a change of coordinates, $T\colon S \to R$, several things must happen. Typically the function in T will have all the continuous partials you like. You may think that the condition that $R = T(S)$ must be a region is going to be difficult to check but a theorem guarantees that if T is a C^1 transformation with $\mathbf{J}(T) \neq 0$ on S, then R will be a region.

Typically the condition $\mathbf{J}(T) \neq 0$ is easy enough to check. The one-to-one condition may be harder to check. For classical coordinate systems we have worked out the answers below and illustrated some techniques for verifying the condition in other cases. In general one-to-one can be a difficult problem.

5. Examples

Example. Consider polar coordinates $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Then

$$\mathbf{J}(T) = \det \begin{vmatrix} \dfrac{\partial\, r\cos\theta}{\partial\, r} & \dfrac{\partial\, r\cos\theta}{\partial\, \theta} \\ \dfrac{\partial\, r\sin\theta}{\partial\, r} & \dfrac{\partial\, r\sin\theta}{\partial\, \theta} \end{vmatrix} = \det \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Note $\mathbf{J}(T) > 0$ if $r > 0$. If $r_i > 0$ with $T(r_1, \theta_1) = T(r_2, \theta_2)$, then $r_1 = \mathbf{J}\big(T(_1, \theta_1)\big) = \mathbf{J}\big(T(_2, \theta_2)\big) = r_2$, $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$, then $\theta_1 = \theta_2 + 2k\pi$ for any integer k. Hence one choice for S is $[0, \infty) \times [0, 2\pi]$ and $T(S) = \mathbb{R}^2$. If it is convenient you can use any other interval for θ as long as it has length $\leqslant 2\pi$.

Example. Consider spherical coordinates $T(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. Then

$$\mathbf{J}(S) = \det\begin{vmatrix} \dfrac{\partial\,\rho\cos\theta\sin\phi}{\partial\,\rho} & \dfrac{\partial\,\rho\cos\theta\sin\phi}{\partial\,\theta} & \dfrac{\partial\,\rho\cos\theta\sin\phi}{\partial\,\phi} \\ \dfrac{\partial\,\rho\sin\theta\sin\phi}{\partial\,\rho} & \dfrac{\partial\,\rho\sin\theta\sin\phi}{\partial\,\theta} & \dfrac{\partial\,\rho\sin\theta\sin\phi}{\partial\,\phi} \\ \dfrac{\partial\,\rho\cos\phi}{\partial\,\rho} & \dfrac{\partial\,\rho\cos\phi}{\partial\,\theta} & \dfrac{\partial\,\rho\cos\phi}{\partial\,\phi} \end{vmatrix} = \det\begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} =$$

$$\cos\phi\det\begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{vmatrix} - 0(?) - \rho\sin\phi\det\begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix} =$$

$$\cos\phi(-\rho^2\cos\phi\sin\phi) - 0 - \rho\sin\phi(\rho\sin^2\phi) = -\rho^2(\cos^2\phi\sin\phi + \sin^3\phi) = -\rho^2\sin\phi$$

Check that if S is the set $0 \leqslant \rho \leqslant \infty$, $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$, the interior is $0 < \rho < \infty$, $0 < \phi < \pi$, $0 < \theta < 2\pi$, and on the interior, R is one to one and $\mathbf{J}(R) > 0$ so S is a solid and $P(S) = \mathbb{R}^3$.

Example. Consider $T(u,w) = (u^2 - w^2, uw)$. Then

$$\mathbf{J}(T) = \det\begin{vmatrix} \dfrac{\partial\,(u^2-w^2)}{\partial\,u} & \dfrac{\partial\,(u^2-w^2)}{\partial\,w} \\ \dfrac{\partial\,(uw)}{\partial\,u} & \dfrac{\partial\,(uw)}{\partial\,w} \end{vmatrix} = \det\begin{vmatrix} 2u & -2w \\ w & u \end{vmatrix} = 2(u^2+w^2)$$

One way to deal with the one to one requirement is to give an explicit formula for the solution to $T(u,w) = (a,b)$. In this case $u^2 - w^2 = a$ and $uw = b$. Since $T(u,w) = T(-u,-w)$ it makes sense to restrict to $u > 0$. Then $w = \dfrac{b}{u}$ and then $u^2 - \dfrac{b^2}{u^2} = a$ or $u^4 - au^2 - b^2 = 0$ so $u^2 = \dfrac{a \pm \sqrt{a^2+4b^2}}{2}$. Since $u^2 > 0$, $u^2 = \dfrac{a + \sqrt{a^2+4b^2}}{2}$ and since $u > 0$, $u = \dfrac{\sqrt{a+\sqrt{a^2+4b^2}}}{\sqrt{2}}$. Then $w = \dfrac{\sqrt{2}\,b}{\sqrt{a+\sqrt{a^2+4b^2}}}$.

It follows that T is one to one when T is restricted to the set $\big\{(u,v) \in (0,\infty)\times(-\infty,\infty)\big\}$ and $\mathbf{J} > 0$ there. Hence on the closure $S = [0,\infty)\times(-\infty,\infty)$, $T\colon [0,\infty)\times(-\infty,\infty) \to \mathbb{R}^2$ is one to one and $\mathbf{J} > 0$ on the interior.

To work out the image of $T(S)$ note that the function

$$P(x,y) = \left(\frac{\sqrt{x+\sqrt{x^2+4y^2}}}{\sqrt{2}}, \frac{\sqrt{2}\,y}{\sqrt{x+\sqrt{x^2+4y^2}}}\right)$$

has positive first coordinate whenever the second coordinate is defined and it satisfies $T\big(P(x,y)\big) = (x,y)$. As long as $y \neq 0$, $\sqrt{x+\sqrt{x^2+4y^2}} > 0$. If $y = 0$, $\sqrt{x+\sqrt{x^2+4y^2}}$ is still positive unless $x \leqslant 0$.

Hence $R = T(S)$ contains $\mathbb{R}^2$ minus the non-positive x-axis. But for $(x,0)$ with $x \leqslant 0$, $T\left(0,\sqrt{-x}\right) = (x,0)$ so $R = \mathbb{R}^2$.

Example. Consider $T(u,w) = (\alpha u + \beta w, \gamma u + \delta w)$ for four numbers α, β, γ and δ. Then

$$\mathbf{J}(T) = \det \begin{vmatrix} \dfrac{\partial\,(\alpha u + \beta w)}{\partial\, u} & \dfrac{\partial\,(\alpha u + \beta w)}{\partial\, w} \\ \dfrac{\partial\,(\gamma u + \delta w)}{\partial\, u} & \dfrac{\partial\,(\gamma u + \delta w)}{\partial\, w} \end{vmatrix} = \det \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$$

Hence $\mathbf{J}(T)$ is non-zero if and only if $\det \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$. Solving two equations in two unknowns shows that

$$T\left(\frac{\delta a - \gamma b}{\ell}, \frac{-\beta a + \alpha b}{\ell}\right) = (a,b)$$

where $\ell = \det \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$. Hence $T(\mathbb{R}^2) = \mathbb{R}^2$.

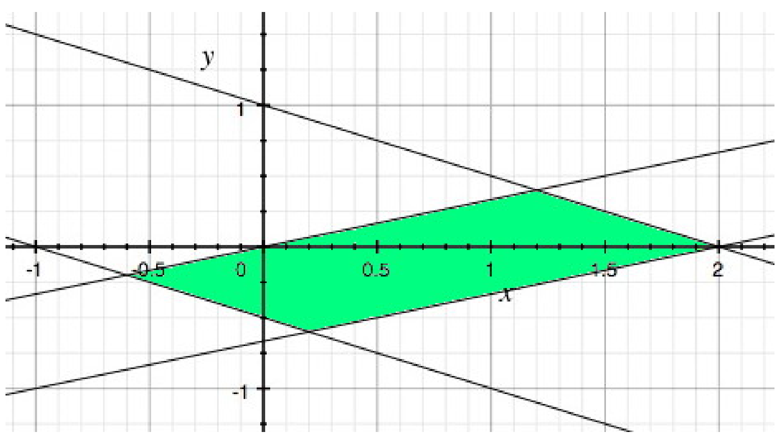
The first two examples, cylindrical and spherical, and the last, any linear example, will get you a long way through this course once you add the following theorem.

Suppose $T\colon S \to R$ is a change of coordinates. Suppose $S_1 \subset S$ and S_1 is a k-dimensional region. Then $T\colon S_1 \to R_1 = T(S_1)$ is also a change of coordinates.

6. Solving for regions

The hardest part of these problems is that you start with a region R and a coordinate change T and you need to find a region S so that $T(S) = R$. In the polar, cylindrical and spherical cases we have discussed this. Here is an example of the linear case.

Example: For example, let R be the parallelogram



The lines are $2x + 4y = 4$, $2x + 4y = -2$, $x - 3y = 0$ and $x - 3y = 2$ and YOU should be able to figure out which is which without any trouble.

Instead of starting with T, let us write $T^{-1}(x,y) = (2x+4y, x-3y)$ or $u = 2x+4y$ and $w = x-3y$.

If S is the rectangle $u = 2$, $u = -2$, $w = 0$, $w = 2$ then $T^{-1}(R) = S$. Then $T(S) = R$ by the definition of the inverse transformation.

To compute the determinant of the Jacobian we seem to need to find a formula for T (which we can do) but it is easier to use equation (3) below. Suppose

$$T^{-1}(x,y) = \big(\ell_1(x,y), \ell_2(x,y)\big)$$

In general you will need to solve equations in order to work out $f(x,y)$ in terms of u and w and to compute $\mathbf{J}(T)$ which is some function of u and w.

However, given T^{-1}, you can work out $\mathbf{J}(T^{-1})$ which will be a function of x and y. It is a theorem generalizing the formula $\dfrac{df^{-1}}{dx}(x) = \dfrac{1}{f'\big(f^{-1}(x)\big)}$ that

$$\mathbf{J}(T) = \frac{1}{\mathbf{J}(T^{-1}) \circ T} \tag{3}$$

In our example

$$\frac{\partial(u,w)}{\partial(x,y)} = \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = -10$$

Hence

$$\frac{\partial(x,y)}{\partial(u,w)} = -0.1$$

We need to find T such that T^{-1} is the inverse transformation to T or more explicitly solve $u = 2x + 4y$ and $w = x - 3y$ for x and y as functions of u and w.

In this case $x = \dfrac{1}{10}(3u + 4w)$ and $y = \dfrac{1}{10}(u - 2w)$ so

$$T^{-1}(u,w) = \left(\frac{1}{10}(3u+4w), \frac{1}{10}(u-2w)\right)$$

Check

$$T\big(T^{-1}(x,y)\big) = T(2x{+}4y, x{-}3y) = \left(\frac{1}{10}\big(3(2x+4y)+4(x-3y)\big), \frac{1}{10}\big((2x+4y)-2(x-3y)\big)\right) = (x,y)$$

$$\mathbf{J}(T) = \det \begin{vmatrix} \dfrac{\partial\, \frac{1}{10}(3u+4w)}{\partial\, u} & \dfrac{\partial\, \frac{1}{10}(3u+4w)}{\partial\, w} \\[2ex] \dfrac{\partial\, \frac{1}{10}(u-2w)}{\partial\, u} & \dfrac{\partial\, \frac{1}{10}(u-2w)}{\partial\, w} \end{vmatrix} = \det \begin{vmatrix} 0.3 & 0.4 \\ 0.1 & -0.2 \end{vmatrix} = -0.06 - 0.04 = -0.1$$

as computed above.

Hence

$$\iint_R f(x,y)\, dA = \iint_S 0.1 \cdot f\left(\frac{1}{10}(3u+4w), \frac{1}{10}(u-2w)\right)\, dA$$

Example: Here is an example where the transformation is so simple it is hard to believe that it could be useful.

$$T(u,w) = (au, bw)$$

for positive constants a and b. Suppose R is the inside of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The inverse of T is $T^{-1}(x,y) = \left(\dfrac{u}{a}, \dfrac{w}{b}\right)$.

If we write $u = \dfrac{x}{a}$ and $w = \dfrac{y}{b}$ then if $u^2 + w^2 = 1$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ so if S is $u^2 + w^2 \leqslant 1$ then $T(S) = R$.

Compute $\mathbf{J}(T) = ab$ so

$$\iint_R f(x,y)\,dA = \iint_S abf(x,y)\,dA = ab\iint_S f\left(\frac{u}{a},\frac{w}{b}\right)\,dA$$

Since S is the unit disk, you may very well want to use polar coordinates on $\displaystyle\iint_{u^2+w^2\leqslant 1} f\left(\frac{u}{a},\frac{w}{b}\right)\,dA$.

Example: A common situation occurs when R is the region between two pairs of level curves. In the parallelogram example above, the region can be described as the region between the level curves of $2x+4y$ and $x-3y$: specifically between $2x+4y=4$ & $2x+4y=-2$ and between $x-3y=0$ & $x-3y=2$.

In general, if R is the region between $\ell_1(x,y)=c_1$ & $\ell_1(x,y)=c_2$ and $\ell_2(x,y)=d_1$ & $\ell_2(x,y)=d_2$ then setting $u=\ell_1(x,y)$ an $w=\ell_2(x,y)$ is often a useful choice for T since $S=T^{-1}(R)$ is the rectangle $c_1\leqslant u\leqslant c_2$, $d_1\leqslant w\leqslant d_2$.

If you can solve $u=\ell_1(x,y)$ and $w=\ell_2(x,y)$ for x and y as functions of u and v you will be able to finish.

Problems 11, 14, and 22 (at least) are of this sort.

22. An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy=a$, $xy=b$ and two adiabatic curves $xy^{1.4}=c$, $xy^{1.4}=d$, where $0<a<b$ and $0<c<d$. Compute the work done by determining the area of R.

Let $u=xy$ and $w=xy^{1.4}$ so the region in uw space is the rectangle $a\leqslant u\leqslant b$, $c\leqslant w\leqslant d$.

$$\frac{\partial(u,w)}{\partial(x,y)} = \det\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y}\end{vmatrix} = \det\begin{vmatrix} y & x\\ y^{1.4} & 1.4xy^{0.4}\end{vmatrix} = 1.4xy^{1.4}-xy^{1.4} = 0.4xy^{1.4} = 0.4w$$

$$\frac{\partial(x,y)}{\partial(u,w)} = \frac{1}{0.4w}$$

Then

$$\iint_R 1\,dA = \iint_S \frac{1}{0.4w}\,dA = \frac{1}{0.4}\int_a^b\int_c^d \frac{dw}{w}\,dv = \frac{b-a}{0.4}\ln\left(\frac{d}{c}\right)$$

7. "Substitution"

Sometimes you need help with the function you are trying to integrate. Examples of this sort are things like $f\left(\dfrac{x+y}{x-y}\right)$ or more generally $f\left(\dfrac{\ell_1(x,y)}{\ell_2(x,y)}\right)$. Here you can try the "substitution" $u=\ell_1(x,y)$, $w=\ell_2(x,y)$.

You have written down the transformation $T^{-1}(x,y) = \big(\ell_1(x,y),\ell_2(x,y)\big)$ and it is easy to write $f(x,y)$ is terms of u and w. Then solve for u and w as functions of x and y and work out $S=T^{-1}(R)$.