There are two parts to a change-of-coordinates problem. The technical part starts with an integral, say $\iint_R f(x,y)\,dA$ and a change of coordinates function, say $T(u,w) = \big(x(u,w), y(u,w)\big)$ and then says that the number $\iint_R f(x,y)\,dA$ is equal to the number $\iint_S g(u,w)\,dA$ where $T(S) = R$ and $g(u,w) = f\big(x(u,w), y(u,w)\big) \cdot \mathbf{J}(T)$.

There is the additional requirement that $T(S)$ covers the region R once and only once (except for points along the boundary).

The book tends to write the double integral over S as $\iint_S f\big(x(u,w), y(u,w)\big) \cdot \mathbf{J}(T)\,du\,dv$ which is a little too much like an iterated integral for my taste. In change-of-coordinate problems I like the notation $\iint_R f(x,y)\,dA_{xy}$ for the first integral and $\iint_S f\big(x(u,w), y(u,w)\big) \cdot \mathbf{J}(T)\,dA_{uw}$ for the second. This way I can keep track of which variables go with which integrals without committing myself to an iterated integral too soon. I would write the double integral formula as

$$\iint_R f(x,y)\,dA_{xy} = \iint_{T^{-1}(R)} f\big(x(u,w), y(u,w)\big) \cdot \mathbf{J}(T)\,dA_{uw}$$

The formula looks deceptive. You start with the left hand side which looks simple as we wrote it and work out the right hand side which looks more complicated. *In practice, there is something about the left hand side which makes direct evaluation difficult, but, when you calculate the more complicated looking formula on the right, a "miracle" occurs and the end result is simpler.*

The artistic part of a change-of-coordinates problem is to pick the "right" T. Of course in a calculus class there are some problems in which the T is given so you can be sure you understand the technical part. In the wild, change of coordinates can be used to accomplish two rather different objectives.

(1) Improve the region from R to S.
(2) Improve your chances of doing an iterated integral by using $f\big(x(u,w), y(u,w)\big) \cdot \mathbf{J}(T)$ instead of $f(x,y)$.

In first year calculus you learned the technique of substitution, which is essentially a type (2) change of coordinates. Since one interval is pretty much the same as any other, you never saw (1) in a one variable problem.

The "right" change of coordinates, just like the "right" substitution in first year calculus, is any change of coordinates for which the integral $\iint_S g(u,w)\,dA_{uw}$ is easier to do than the original $\iint_R f(x,y)\,dA_{xy}$.

On the technical side, the new region is an additional piece to the problem. If you can write the boundary of R as a bunch of graphs, or more generally level curves $b_i(x,y) = c_i$, write these equations in terms of u and w and you will get level curves in uw space. If you are lucky this set of level curves is the boundary of a unique closed, bounded region S and it is a theorem that this must be the S you want.

1. Examples

Here is an example of choosing the change of coordinates to improve your chances of doing the integral. The book worked out an example of integrating $e^{\frac{x+y}{x-y}}$ over a trapezoid (Example 3, p. 1069), by using $u = x + y$, $w = x - y$. You might wonder why not take $u = x + y$ and $w = \dfrac{1}{x-y}$ so you would be trying to integrate e^{uw} which looks safer than integrating $e^{\frac{u}{w}}$.

Here is $\displaystyle\iint_R e^{\frac{(x+y)}{(x-y)^k}}\, dA$ using $u = x + y$, $w = \dfrac{1}{x-y}$ over the trapezoid $(1, 0)$, $(2, 0)$, $(0, -2)$ and $(0, -1)$ for $k = 1$.

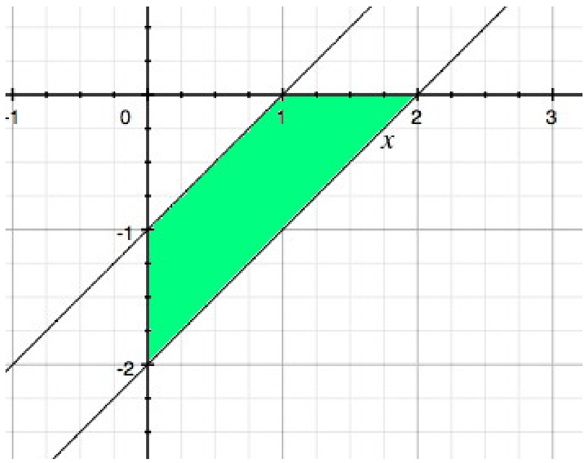

Figure 1

The sloped lines are $x - y = 1$ and $x - y = 2$ and the other pieces of boundary are a piece of the x-axis and a piece of the y-axis. In uw land these are $w = 1$, $w = \frac{1}{2}$, $uw = -1$ and $uw = 1$. Notice if $uw = -1$, $\dfrac{x+y}{x-y} = -1$ so $x = 0$ which is the y-axis and if $uw = 1$, $\dfrac{x+y}{x-y} = 1$ so $y = 0$ which is the x-axis.

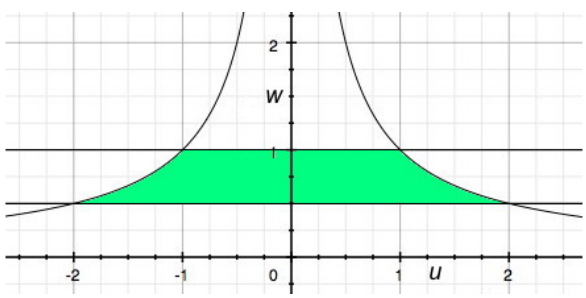

Figure 2

Notice the green region in Figure 2 is the only piece bounded by the four curves and so it must go to green region in Figure 1.

Let us compute

$$\mathbf{J}(T^{-1}) = \det \begin{vmatrix} \dfrac{\partial\,(x+y)}{\partial\, x} & \dfrac{\partial\,(x+y)}{\partial\, y} \\ \dfrac{\partial\,(x-y)^{-1}}{\partial\, x} & \dfrac{\partial\,(x-y)^{-1}}{\partial\, y} \end{vmatrix} = \det \begin{vmatrix} 1 & 1 \\ -(x-y)^{-2} & (x-y)^{-2} \end{vmatrix} = 2(x-y)^{-2}$$

We have calculated $\mathbf{J}(T^{-1})$ and it is easy to rewrite in terms of u and w: $\mathbf{J}(T^{-1}) = 2w^2$ so $\dfrac{f(x,y)}{\mathbf{J}(T^{-1})} = \dfrac{e^{uw^k}}{2w^2}$ and

$$\iint_R e^{\frac{(x+y)}{(x-y)^k}}\,dA_{xy} = \iint_S \frac{e^{uw^k}}{2w^2}\,dA_{uw}$$

This is the end of the change of coordinates part of the problem and in this problem we proceed to setup and evaluate an iterated integral.

$$\frac{1}{2}\iint_S w^{-2}e^{uw^k}\,dA_{uw} = \frac{1}{2}\int_{\frac{1}{2}}^{1}\int_{-\frac{1}{w}}^{\frac{1}{w}} w^{-2}e^{uw^k}\,du\,dw = \frac{1}{2}\int_{\frac{1}{2}}^{1}\frac{1}{w^{k+2}}e^{uw^k}\Big|_{u=-\frac{1}{w}}^{u=\frac{1}{w}}\,dw =$$

$$\frac{1}{2}\int_{\frac{1}{2}}^{1} w^{-(k+2)}e^{w^{k-1}} - w^{-(k+2)}e^{-w^{k-1}}\,dw$$

If $k = 1$ this is

$$\frac{1}{2}\int_{\frac{1}{2}}^{1} w^{-3}e^{1} - w^{-3}e^{-1}\,dw = \frac{e-e^{-1}}{2}\int_{\frac{1}{2}}^{1} w^{-3}dw = \frac{e-e^{-1}}{2}\left(\frac{w^{-2}}{-2}\Big|_{\frac{1}{2}}^{1}\right) =$$

$$\frac{e-e^{-1}}{2}\left(\frac{1}{-2} - \frac{(1/2)^{-2}}{-2}\right) = \frac{e-e^{-1}}{2}\left(2-\frac{1}{2}\right) = \frac{3}{4}(e-e^{-1})$$

as in the book.

We can do this integral for other values of k if we can do $\int w^{-(k+2)}e^{\pm w^{k-1}}\,dw$. If $k \neq 1$ we substitute $v = w^{k-1}$, $dv = (k-1)w^{k-2}\,dw$ so

$$\int w^{-(k+2)}e^{\pm w^{k-1}}\,dw = \frac{1}{k-1}\int w^{-2k}e^{\pm v}\,dv = \frac{1}{k-1}\int v^{\frac{-2k}{k-1}}e^{\pm v}\,dv$$

If $\frac{-2k}{k-1}$ is a positive integer, e.g. $k = \frac{n}{n+1}$, this can be integrated by parts.

You needed to be lucky in this problem since if the region is different you may end up with a S which you will find impossible to work with. Notice that the function blows up along the line $x = y$ so if R includes some of this line you will have an improper integral and in this case you can see by looking at Riemann sums that the answer will be $+\infty$.

Even then simple R may be unmanageable. If R is the yellow square in Figure 3

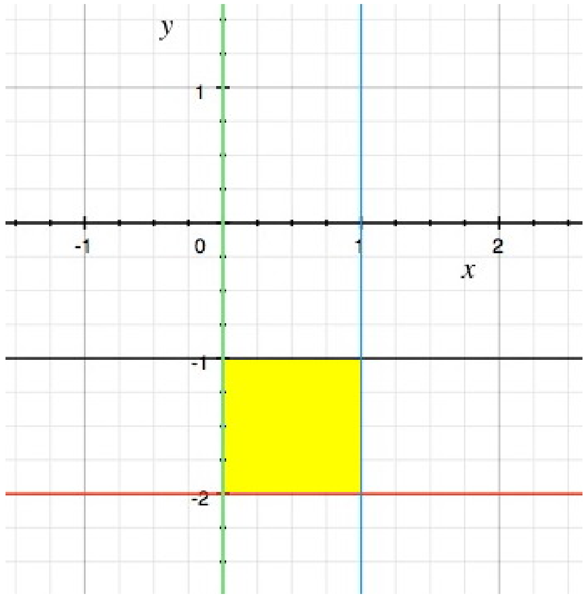

Figure 3.

the corresponding region in uw space is the yellow region in Figure 4.

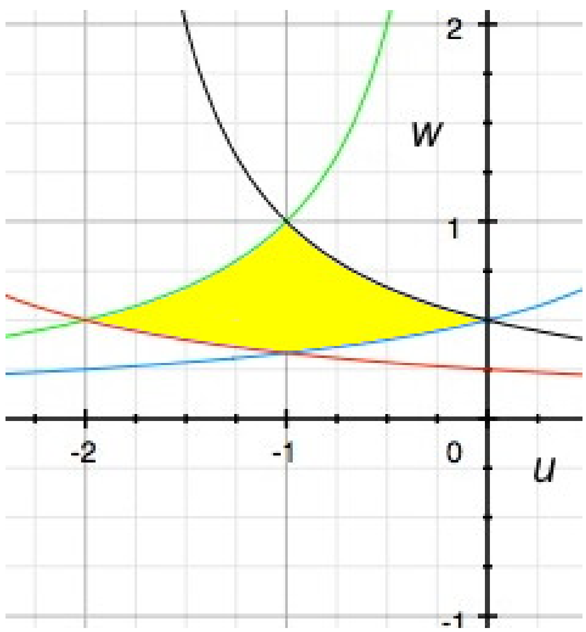

Figure 4.

The curve of a given color in Figure 4 goes to a curve of the same color in Figure 3. The transformation is $x = \frac{1}{2}\left(u + \frac{1}{w}\right)$, $y = \frac{1}{2}\left(u - \frac{1}{w}\right)$ so the red curve is $uw + 4w = 1$; $uw + 2y = 1$ is black; $uw - 2w = -1$ is blue and $uw = -1$ is green. The determinant of the Jacobian is $\frac{1}{4}w^{-2}$ so you need to integrate $\frac{4e^{uw}}{w^2}$ over the yellow region in Figure 4. This leads to integrals we don't know to do.

Example: A common situation occurs when R is the region between two pairs of level curves. In the parallelogram example above, the region can be described as the region between the level curves of $2x+4y$ and $x-3y$: specifically between $2x+4y=4$ & $2x+4y=-2$ and between $x-3y=0$ & $x-3y=2$.

In general, if R is the region between $\ell_1(x,y)=c_1$ & $\ell_1(x,y)=c_2$ and $\ell_2(x,y)=d_1$ & $\ell_2(x,y)=d_2$ then setting $u=\ell_1(x,y)$ an $w=\ell_2(x,y)$ is often a useful choice for T since $S=T^{-1}(R)$ is the rectangle $c_1 \leqslant u \leqslant c_2$, $d_1 \leqslant w \leqslant d_2$.

If you can solve $u=\ell_1(x,y)$ and $w=\ell_2(x,y)$ for x and y as functions of u and v you will be able to finish.

Problems 11, 14, and 22 (at least) are of this sort.

22. An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy=a$, $xy=b$ and two adiabatic curves $xy^{1.4}=c$, $xy^{1.4}=d$, where $0<a<b$ and $0<c<d$. Compute the work done by determining the area of R.

Let $u=xy$ and $w=xy^{1.4}$ so the region in uw space is the rectangle $a \leqslant u \leqslant b$, $c \leqslant w \leqslant d$.

$$\frac{\partial(u,w)}{\partial(x,y)} = \det\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y}\end{vmatrix} = \det\begin{vmatrix} y & x \\ y^{1.4} & 1.4xy^{0.4}\end{vmatrix} = 1.4xy^{1.4} - xy^{1.4} = 0.4xy^{1.4} = 0.4w$$

$$\frac{\partial(x,y)}{\partial(u,w)} = \frac{1}{0.4w}$$

Then

$$\iint_R 1\,dA = \iint_S \frac{1}{0.4w}\,dA = \frac{1}{0.4}\int_a^b \int_c^d \frac{dw}{w}\,dv = \frac{b-a}{0.4}\ln\left(\frac{d}{c}\right)$$

Finally there are problems where you start with an iterated integral which you don't know how to do and first rewrite the iterated integral as a double integral over some region and then think about how to do the double integral. You can try iterating in the other order; try polar coordinates; or now try some other change of coordinates.