DIFFERENTIATION OF CURVES
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LAURENCE R. TAYLOR

If $\mathbf{r}(t)$ is a curve then

$$\mathbf{r}'(a) = \lim_{t\to a} \frac{\mathbf{r}(t) - \mathbf{r}(a)}{t - a}$$

If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$. The pieces, x', y', z' admit the usual interpretations. If $x'(a) > 0$ then the x-coordinate of $\mathbf{r}$ is increasing at a. Similar remarks apply to the other inequality and to the other components.

In the 2-dimensional case, $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Suppose near $t = a$, the curve is the graph of a function $y = F(x)$. Then $y(t) = F\big(x(t)\big)$ and we can differentiate with respect to t

$$y'(t) = F'\big(x(t)\big)x'(t) \quad \text{or} \quad F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}$$

The second formula only holds if $x'(t) \neq 0$. There is a theorem called the Implicit Function Theorem which says that if $\mathbf{r}$ is differentiable in a neighborhood of a and if $x'(t) \neq 0$ in that neighborhood, then F exists and is differentiable. Note F is increasing in the first-year calculus sense if and only if x' and y' have the same sign.

1. Six rules

(1) $(\mathbf{r}_1 + \mathbf{r}_2)'(t) = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)$
(2) $(c\,\mathbf{r})'(t) = c\,\mathbf{r}'(t)$
(3) $\dfrac{d\ c(t)\mathbf{r}(t)}{d\,t} = c'(t)\mathbf{r} + c(t)\mathbf{r}'(t)$
(4) $(\mathbf{r}_1 \bullet \mathbf{r}_2)'(t) = \mathbf{r}_1'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}_2'(t)$
(5) $(\mathbf{r}_1 \times \mathbf{r}_2)'(t) = \mathbf{r}_1'(t) \times \mathbf{r}(t) + \mathbf{r}(t) \times \mathbf{r}_2'(t)$
(6) $\dfrac{d\ \mathbf{r}\big(f(t)\big)}{d\,t} = \mathbf{r}'\big(f(t)\big)f'(t)$

Additional formulas:

$$(1)\quad \frac{d\,|\mathbf{r}(t)|}{dt} = \frac{\mathbf{r}\bullet\mathbf{r}'}{|\mathbf{r}|}$$

$$(2)\quad \frac{d\,\mathbf{r}_1\bullet\big(\mathbf{r}_2\times\mathbf{r}_3\big)}{dt} = \mathbf{r}_1'\bullet\big(\mathbf{r}_2\times\mathbf{r}_3\big)+\mathbf{r}_1\bullet\big(\mathbf{r}_2'\times\mathbf{r}_3\big)+\mathbf{r}_1\bullet\big(\mathbf{r}_2\times\mathbf{r}_3'\big)$$

2. Tangent line

Tangent line to $\mathbf{r}(t)$ at a:

$$\mathbf{r}(a) + t\mathbf{r}'(a)$$

Which way are you moving along a curve $\mathbf{r}$? If $\mathbf{r}'(a) \neq \mathbf{0}$ then at the point $\mathbf{r}(a)$ you are moving in one direction or the other along the curve. At least one of the components of $\mathbf{r}'(a)$ must be non-zero. If it is positive, move in the direction which increases that coordinate; if negative move in the other direction.

Example. One way to parametrize the unit circle is $\mathbf{r}(t) = \langle\cos(t), \sin(t)\rangle$, $0 \leqslant t \leqslant 2\pi$. At the point $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, which way are we moving as t is increasing? Well $\mathbf{r}'(t) = \langle-\sin(t), \cos(t)\rangle$ so $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$. Hence the x-coordinate is decreasing and the y-coordinate is increasing and we are moving counterclockwise.

Example. Another way to parametrize the unit circle is $\mathbf{r}(t) = \langle\sin(t), \cos(t)\rangle$, $0 \leqslant t \leqslant 2\pi$. At the point $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, which way are we moving as t is increasing? Well $\mathbf{r}'(t) = \langle\cos(t), -\sin(t)\rangle$ so $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$. Hence the x-coordinate is increasing and the y-coordinate is decreasing and we are moving clockwise.

Example. A way to parametrize the unit circle minus $(-1, 0)$ is $\mathbf{r}(t) = \left\langle\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right\rangle$, $-\infty < t < \infty$. When $t = \frac{1}{1+\sqrt{2}}$, $1 + t^2 = \frac{2\sqrt{2}}{1+\sqrt{2}}$ and $1 - t^2 = \frac{2}{1+\sqrt{2}}$ Hence $\mathbf{r}\left(\frac{2}{1+\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Which way are we moving at this point as t increases? Well $\mathbf{r}'(t) = \left\langle\frac{-4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2}\right\rangle$ so $\mathbf{r}'\left(\frac{1}{1+\sqrt{2}}\right) = \left\langle-\frac{1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right\rangle$. Hence the x-coordinate is decreasing and the y-coordinate is increasing and we are moving counterclockwise. Notice also that $\mathbf{r}\bullet\mathbf{r}' = 0$ as it must since we are on a circle centered at the origin.

In the first two examples, $|\mathbf{r}'(t)| = 1$ whereas in the third $|\mathbf{r}'(t)| = \frac{2}{1+t^2}$. Note in all three examples, $|\mathbf{r}'(t)| \neq 0$ for all t.

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U.S.A.
E-mail address: taylor.2@nd.edu